Implied Volatility of the Constant Product Market Maker in Decentralized Finance

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Consortium for Data Analytics in Risk
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Motivation

Motivation:

- Decentralized finance (DeFi) utilizes blockchain technology to provide financial services
- Utilized in lending, borrowing, trading, and insurance underwriting *without* traditional financial intermediaries
- Total value locked up (in dollar denominated terms) in DeFi grew $90\times$ between January 1, 2020 and January 1, 2022

  Peak market capitalization of nearly \$180B
- With the "crypto winter" it is important to explore the viability and risks associated with DeFi

Motivation:

- Automated market makers [AMMs] are a decentralized approach for creating financial markets
- Key Idea: Create a (liquidity) pool of assets to trade against instead of a (central) order book
- AMMs create markets by balancing reserves according to mathematical formulas to execute swaps
- Any individual can pool resources into an AMM and can earn fees from trading activities

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- Automated market makers [AMMs] are a decentralized approach for creating financial markets
- Key Idea: Create a (liquidity) pool of assets to trade against instead of a (central) order book
- AMMs create markets by balancing reserves according to mathematical formulas to execute swaps
- Any individual can pool resources into an AMM and can earn fees from trading activities
- Goal of this talk: Analyze the fair price, **implied volatility**, and other properties of providing liquidity

Background

Automated Market Makers: Construction:

- AMMs employ a constant function concept
  Accept trades so that a utility function $u(x, y)$ is invariant for the trade
- Consider an AMM with reserves (x, y)
  If Δx units of asset 1 are being sold, then Δy units of asset 2 are being received in exchange so that
  $u(x, y) = u(x + \Delta x, y - \Delta y)$

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  If Δx units of asset 1 are being sold, then Δy units of asset 2 are being received in exchange so that
  $u(x, y) = u(x + \Delta x, y - \Delta y)$
- Marginal prices provide a pricing oracle
  $P(x, y) = u_x(x, y)/u_y(x, y)$ with asset 2 as the numéraire
- Traders are allowed to deposit/withdraw funds from the AMM so long as the price is unaffected
  $P(x, y) = P(x + \Delta x, y + \Delta y)$

Automated Market Makers: Fees:

- When depositing assets into an AMM, an LP token is minted
- Owning the LP token grants the holder rights to a fraction of all fees collected from trading
- Fees collected on a **fraction of incoming assets**
  A constant fraction $\gamma \in (0, 1)$ of the incoming asset of a swap (prior to swapping) is collected and distributed to the LP token holders

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- Other fee structures possible (though not utilized in practice), e.g.,
  - Fraction of outgoing assets
  - Fraction of numéraire transacted
  - Fraction of the risky asset transacted

Blockchain Fundamentals:

- AMMs are constructed as smart contracts directly on the blockchain
- Transactions are only processed (and prices updated) at discrete times
- Modern blockchains have a fixed inter-block time $\Delta t > 0$
  Polygon has $\Delta t = 2$ seconds
- Consider some price process P_t, then the AMM only realizes the prices $\{P_{i\Delta t} \mid i \in \mathbb{N}\}$

Constant Product Market Making

3. Constant Product

Setting:

- Consider the log utility function $u(x, y) = \log(x) + \log(y)$
- Equivalent market made by the function $f(x, y) = xy$
- Employed in Uniswap V2 but replicated in many other AMMs (SushiSwap, PancakeSwap, ...)

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- Equivalent market made by the function $f(x, y) = xy$
- Employed in Uniswap V2 but replicated in many other AMMs (SushiSwap, PancakeSwap, ...)
- Pricing oracle is determined by the ratio of assets $P(x, y) = y/x$
- Assume P_t follows the geometric Brownian motion $dP_t = P_t[rdt + \sigma dW_t]$

$$x_t = LP_t^{-1/2} = L\sqrt{\frac{1}{P_0}} \exp\left(-\frac{1}{2}[r - \frac{\sigma^2}{2}]t - \frac{1}{2}\sigma W_t\right)$$

$$y_t = LP_t^{1/2} = L\sqrt{P_0} \exp\left(\frac{1}{2}[r - \frac{\sigma^2}{2}]t + \frac{1}{2}\sigma W_t\right)$$

for $L := \sqrt{x_0 y_0}$

Constant Product Market Making: Valuation

Derivative Construction:

- Value of fees per LP token at block i:

$$\gamma\left[P_{i\Delta t}(x_{i\Delta t}-x_{(i-1)\Delta t})^+ + (y_{i\Delta t}-y_{(i-1)\Delta t})^+\right]/L$$

- Recall: Fees are distributed proportionally to size of contribution to liquidity
  The total value of the AMM and fees are sufficient for considering individual positions

Derivative Construction:

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$$\gamma\left[P_{i\Delta t}(x_{i\Delta t}-x_{(i-1)\Delta t})^{+}+(y_{i\Delta t}-y_{(i-1)\Delta t})^{+}\right]/L$$

- Recall: Fees are distributed proportionally to size of contribution to liquidity
  The total value of the AMM and fees are sufficient for considering individual positions
- Depositing liquidity buys the investor a Bermudan perpetual option:
  - Can be exercised at any block
  - Fees are paid at each block until exercise
  - Continues indefinitely until exercise
- Cost of an LP token is $V_0 := [P_0 x_0 + y_0]/L = 2\sqrt{P_0}$
- Goal: Determine the value of this option V_i at any block i

Derivative Value:

- DPP formulation for value of the option $V(P)$:

$$V(P) = \max \left\{ \begin{array}{l} 2\sqrt{P}\ , \\ e^{-r\Delta t}\mathbb{E}\left[\begin{array}{l} V\left(Pe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma W_{\Delta t}}\right) + \\ \gamma F\left(P, Pe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma W_{\Delta t}}\right) \end{array} \right] \end{array} \right\}$$

$$F(P_0, P_1) = P_1\left(\frac{1}{\sqrt{P_1}} - \frac{1}{\sqrt{P_0}}\right)^+ + \left(\sqrt{P_1} - \sqrt{P_0}\right)^+$$

Value of Fees:

- Goal: Determine the discounted expected value of the fees in a single time step:

$$\bar{F}(P,\Delta t) := e^{-r\Delta t}\mathbb{E}\left[F\left(P, Pe^{(r-\frac{\sigma^2}{2})\Delta t+\sigma W_{\Delta t}}\right)\right]$$

- Can be calculated directly:

$$\bar{F}(P,\Delta t) = \frac{2(1-e^{-\frac{1}{2}(r+\frac{\sigma^2}{4})\Delta t})\sqrt{P}}{\gamma^*}$$

$$\gamma^* = 2\left[-1+\frac{\Phi\left(\frac{(r+\frac{\sigma^2}{2})\sqrt{\Delta t}}{\sigma}\right)-e^{-r\Delta t}\Phi\left(\frac{(r-\frac{\sigma^2}{2})\sqrt{\Delta t}}{\sigma}\right)}{1-e^{-\frac{1}{2}(r+\frac{\sigma^2}{4})\Delta t}}\right]^{-1}$$

Optimal Stopping:

- Ansatz: Either never invest in the AMM or, once invested, never exercise the option (i.e., perpetually accumulate fees)

$$\begin{aligned}\tilde{V}(P_0) &= \gamma \sum_{i=0}^{\infty} e^{-ri\Delta t}\mathbb{E}[\bar{F}(P_{i\Delta t}, \Delta t)] \\ &= \gamma \sum_{i=0}^{\infty} e^{-\frac{1}{2}(r+\frac{\sigma^2}{2})i\Delta t}\bar{F}(P_0, \Delta t) \\ &= \frac{\gamma \bar{F}(P_0, \Delta t)}{1 - e^{-\frac{1}{2}(r+\frac{\sigma^2}{2})\Delta t}} \\ &= \frac{2\gamma\sqrt{P_0}}{\gamma^*}\end{aligned}$$

- Value $V(P) = \tilde{V}(P)\mathbb{I}_{\{\gamma \geq \gamma^*\}} + 2\sqrt{P}\mathbb{I}_{\{\gamma < \gamma^*\}}$

Optimal Stopping:

- Verify the ansatz $V(P)$

Ansatz Verification

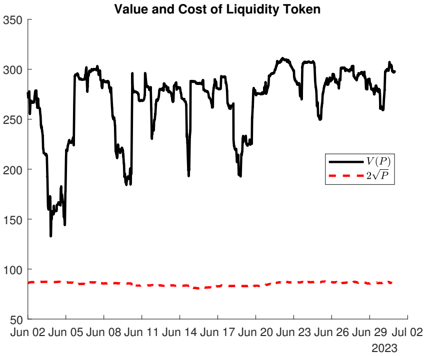
- Assume $\gamma \geq \gamma^*$
  - $\tilde{V}(P)$ coincides with its continuation value
  - $\tilde{V}(P) \geq 2\sqrt{P}$
- Assume $\gamma < \gamma^*$
  - Continuation value

$$2\sqrt{P}\left[e^{-\frac{1}{2}(r+\frac{\sigma^2}{4})\Delta t} + (1 - e^{-\frac{1}{2}(r+\frac{\sigma^2}{4})\Delta t})\frac{\gamma}{\gamma^*}\right]$$

is strictly less than $2\sqrt{P}$

Example USDC/WETH Pool

- Constant product market maker on Polygon blockchain with data from June 1 to June 30, 2023
- $\Delta t = 2$ seconds, $\gamma = 5bps$, $r = 5\%$ (annualized)
- σ as the 1-day historical volatility computed from minute-returns



Constant Product Market Making: Implied Volatility

Implied Volatility:

- Goal: Determine the volatility $\sigma^* > 0$ so that $V(P_0) = 2\sqrt{P_0}$
- Equivalent to equilibrium fees $\gamma^*(\sigma^*) = \gamma$
- Note that the implied volatility is independent of the initial price P_0 and the level of liquidity L

Implied Volatility

- If $r = 0$ then there exists a unique implied volatility $\sigma^* > 0$
- If $r > 0$ then:
  - If $\Delta t \geq \overline{\Delta t} := \sqrt{\frac{8}{\pi} \frac{\gamma}{(2+\gamma)r}} e^{-1/2}$ then *no* implied volatility exists
  - If $\Delta t < \overline{\Delta t}$ then define $\bar{\sigma} := r\sqrt{\frac{\Delta t}{-W\left(-\frac{\pi}{2}\left[\frac{(2+\gamma)r\Delta t}{2\gamma}\right]^2\right)}}$ and
    - there does *not* exist an implied volatility if $\gamma < \gamma^*(\bar{\sigma})$
    - there exists a *unique* implied volatility $\sigma^* = \bar{\sigma}$ if $\gamma = \gamma^*(\bar{\sigma})$
    - there exists exactly *two distinct* implied volatilities $\sigma_1^* < \bar{\sigma} < \sigma_2^*$ if $\gamma > \gamma^*(\bar{\sigma})$

Example:

- $\Delta t = 2$ seconds and $r = 5\%$ (annualized)
- If $\gamma = 1bps$ then:
  - $\overline{\Delta t} \approx 8.48$ hours
  - $\bar{\sigma} \approx 0.3168$
  - $\gamma^*(\bar{\sigma}) \approx 1.4963bps > \gamma$
  - There does *not* exist an implied volatility
  - A risk-neutral investor should *never* deposit liquidity

Example:

- $\Delta t = 2$ seconds and $r = 5\%$ (annualized)
- If $\gamma \approx 1.4116 bps$ then:
  - $\overline{\Delta t} \approx 11.97$ hours
  - $\bar{\sigma} \approx 0.4472$
  - $\gamma^*(\bar{\sigma}) = \gamma$
  - There exists the unique implied volatility $\sigma^* = \bar{\sigma} \approx 0.4472$
  - If $\sigma \neq \sigma^*$ then a risk-neutral investor should *never* deposit liquidity

Example:

- $\Delta t = 2$ seconds and $r = 5\%$ (annualized)
- If $\gamma = 5bps$ then:
  - $\overline{\Delta t} \approx 42.38$ hours
  - $\bar{\sigma} \approx 1.5838$
  - $\gamma^*(\bar{\sigma}) \approx 2.699bps < \gamma$
  - There exists two implied volatilities: $\sigma_1^* \approx 0.0644 < \bar{\sigma} < \sigma_2^* \approx 3.1031$
  - If $\sigma \in (\sigma_1^*, \sigma_2^*)$ then $V(P_0) > 2\sqrt{P_0}$ and a risk-neutral investor should deposit liquidity
  - If $\sigma \in (0, \sigma_1^*) \cup (\sigma_2^*, \infty)$ then $V(P_0) < 2\sqrt{P_0}$ and a risk-neutral investor should *never* deposit liquidity

Constant Product Market Making: Greeks

Greeks:

- Assume $\gamma \geq \gamma^*$
- **Delta**: Sensitivity of the value to the underlying price

$$\frac{\partial}{\partial P} V(P) = \frac{\gamma}{\gamma^* \sqrt{P}} = \frac{V(P)}{2P} > 0$$

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- **Gamma**: Sensitivity of the Delta to the underlying price

$$\frac{\partial^2}{\partial P^2} V(P) = -\frac{\gamma}{2\gamma^* P^{3/2}} = -\frac{V(P)}{4P^2} < 0$$

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- **Vega**: Sensitivity of the value to the realized volatility

$$\frac{\partial}{\partial \sigma} V(P) = \frac{\gamma \sqrt{P} e^{-\frac{1}{2}(r+\frac{\sigma^2}{4})\Delta t}}{1-e^{-\frac{1}{2}(r+\frac{\sigma^2}{4})\Delta t}} \left[\sqrt{\frac{\Delta t}{2\pi}} e^{-\frac{r^2 \Delta t}{2\sigma^2}} - \frac{\sigma \Delta t}{4} \frac{\Phi\left(\frac{(r+\frac{\sigma^2}{2})\sqrt{\Delta t}}{\sigma}\right) - e^{-r\Delta t} \Phi\left(\frac{(r-\frac{\sigma^2}{2})\sqrt{\Delta t}}{\sigma}\right)}{1-e^{-\frac{1}{2}(r+\frac{\sigma^2}{4})\Delta t}} \right]$$

Vega:

- If $r = 0$ then $\frac{\partial}{\partial \sigma} V(P) < 0$
- Assume $r > 0$ and $\Delta t < \overline{\Delta t}$
  - $\frac{\partial}{\partial \sigma} V(P) > 0$ if $\sigma < \bar{\sigma}$
  - $\frac{\partial}{\partial \sigma} V(P) < 0$ if $\sigma > \bar{\sigma}$

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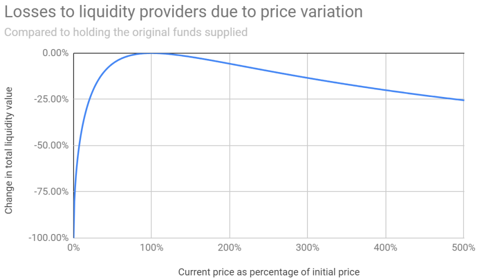
  - $\frac{\partial}{\partial\sigma}V(P) > 0$ if $\sigma < \bar{\sigma}$
  - $\frac{\partial}{\partial\sigma}V(P) < 0$ if $\sigma > \bar{\sigma}$
- **Conventional wisdom**: Providing liquidity to an AMM "behave[s] like a bet on volatility" (Millionis et al. *Automated market making and loss-versus-rebalancing*)
- This conventional view does *not* hold in reality
- Follows due to specific assumptions on the market construction (e.g., neglecting fees)

Constant Product Market Making: Divergence Loss

Divergence Loss:

- Also called impermanent loss
- The opportunity cost quantifying the difference between the buy-and-hold strategy and depositing liquidity in the AMM



https://docs.uniswap.org/contracts/v2/concepts/advanced-topics/understanding-returns

Divergence Loss:

- Assume $\gamma \geq \gamma^*$
- Value of buy-and-hold strategy after i blocks without earning interest

$$(1 + e^{-ri\Delta t})\sqrt{P_0}$$

  or $2\sqrt{P_0}$ with interest
- Value of liquidity position

$$2\sqrt{P_0}\left[e^{-\frac{1}{2}(r+\frac{\sigma^2}{4})i\Delta t} + \left(1 - e^{-\frac{1}{2}(r+\frac{\sigma^2}{4})i\Delta t}\right)\frac{\gamma}{\gamma^*}\right]$$

  is nondecreasing in block number i
- *Expected* divergence loss does not exist if being a liquidity provider is a good deal

Divergence Loss

- Historically: over *half* of all liquidity providers lose money to divergence loss
- Empirically: Distribution of optimal AMM returns is highly skewed under the risk-neutral measure

TIPRANKS
KNOW WHO TO TRUST

Half of Uniswap Liquidity Providers are Losing Money

November 19, 2021 — 08:33 am EST

Written by **Reuben Jackson** for **TipRanks →**

Concentrated Liquidity Constant Product Market Making

4. Concentrated Liquidity

Setting:

- Liquidity is concentrated between upper P^U and lower P^L price bounds
- Implemented within UniSwap V3
- Between these prices, utility follows constant product $u(x, y) = (\alpha + x)(\beta + y)$
- Quoted price is determined by the ratio of assets $P = \frac{\beta+y}{\alpha+x}$
- X, Y are the maximum total liquidity available
- $\alpha = \alpha(P^L, P^U, Y)$ and $\beta = \beta(P^L, P^U, Y)$

Setting:

- Fix price bounds P^L, P^U and liquidity X, Y
- Assume P_t follows the discrete geometric Brownian motion

$$x_t = L\sqrt{\frac{1}{P_0}} \exp\left(-\frac{1}{2}[r - \frac{\sigma^2}{2}]t - \frac{1}{2}\sigma W_t\right) - \alpha$$
$$y_t = L\sqrt{P_0} \exp\left(\frac{1}{2}[r - \frac{\sigma^2}{2}]t + \frac{1}{2}\sigma W_t\right) - \beta$$

for $L = (\alpha + x_0)(\beta + y_0)$

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$$y_t = L\sqrt{P_0} \exp\left(\frac{1}{2}[r - \frac{\sigma^2}{2}]t + \frac{1}{2}\sigma W_t\right) - \beta$$

for $L = (\alpha + x_0)(\beta + y_0)$

- Throughout use $\hat{x}_t = x_t + \alpha$ and $\hat{y}_t = y_t + \beta$

Derivative Construction:

- Fees are only collected when the price crosses through the region of liquidity (P^L, P^U)
- Value of fees per LP token at block i:

$$\gamma \left[\begin{array}{l} P_{i\Delta t} \left(\hat{x}_{i\Delta t} \vee \frac{1}{\sqrt{P^U}} \wedge \frac{1}{\sqrt{P^L}} - \hat{x}_{(i-1)\Delta t} \vee \frac{1}{\sqrt{P^U}} \wedge \frac{1}{\sqrt{P^L}} \right)^+ \\ +(\hat{y}_{i\Delta t} \wedge \sqrt{P^U} \vee \sqrt{P^L} - \hat{y}_{(i-1)\Delta t} \wedge \sqrt{P^U} \vee \sqrt{P^L})^+ \end{array} \right]$$

- Recall: Fees are distributed proportionally to size of contribution to liquidity
  The total value of the AMM and fees are sufficient for considering individual positions

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- Recall: Fees are distributed proportionally to size of contribution to liquidity
  The total value of the AMM and fees are sufficient for considering individual positions
- Depositing liquidity buys the investor a Bermudan perpetual option:
- Cost of an LP token is $V_0 := (P_0[x_0^+ \wedge X] + [y_0^+ \wedge Y])/L$
- Goal: Determine the value of this option V_i at any block i

Derivative Value:

- DPP formulation for value of the option $V(P)$:

$$V(P) = \max \left\{ \begin{array}{l} P\left(\frac{1}{\sqrt{P \wedge P^U \vee P^L}} - \alpha\right) + \left(\sqrt{P \wedge P^U \vee P^L} - \beta\right), \\ e^{-r\Delta t}\mathbb{E}\left[\begin{array}{l} V\left(Pe^{(r-\frac{\sigma^2}{2})\Delta t + \sigma W_{\Delta t}}\right) + \\ \gamma F\left(P, Pe^{(r-\frac{\sigma^2}{2})\Delta t + \sigma W_{\Delta t}}\right) \end{array} \right] \end{array} \right.$$

$$F(P_0, P_1) = \begin{array}{l} P_1\left(\frac{1}{\sqrt{P_1 \wedge P^U \vee P^L}} - \frac{1}{\sqrt{P_0 \wedge P^U \vee P^L}}\right)^+ \\ + \left(\sqrt{P_1 \wedge P^U \vee P^L} - \sqrt{P_0 \wedge P^U \vee P^L}\right)^+ \end{array}$$

Value of Fees:

- Goal: Determine the discounted expected value of the fees in a single time step:

$$\bar{F}(P, \Delta t) := e^{-r\Delta t}\mathbb{E}\left[F\left(P, Pe^{(r-\frac{\sigma^2}{2})\Delta t + \sigma W_{\Delta t}}\right)\right]$$

- Can be calculated directly:

$$\bar{F}(P, \Delta t) = \sqrt{P}\left[\begin{array}{l} \left(F_1(P, P \wedge P^U, P^L) + F_x\right)\mathbb{I}_{\{P \geq P^L\}} \\ + \left(F_1(P, P^U, P \vee P^L) + e^{-r\Delta t}F_y\right)\mathbb{I}_{\{P \leq P^U\}} \end{array}\right]$$

$$F_1(P, \bar{P}^U, \bar{P}^L) = e^{-\frac{1}{2}(r+\frac{\sigma^2}{2})\Delta t}\left[\Phi\left(\frac{\log(\frac{\bar{P}^U}{P}) - r\Delta t}{\sigma\sqrt{\Delta t}}\right) - \Phi\left(\frac{\log(\frac{\bar{P}^L}{P}) - r\Delta t}{\sigma\sqrt{\Delta t}}\right)\right]$$

$$F_x = \sqrt{\frac{P}{P^L}}\Phi\left(\frac{\log(\frac{P^L}{P}) - (r + \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}}\right) - \sqrt{\frac{P}{P \wedge P^U}}\Phi\Bigg(\frac{\log(\frac{P \wedge P^U}{P}) - (r + \frac{\sigma^2}{2})\Delta}{\sigma\sqrt{\Delta t}}$$

$$F_y = \sqrt{\frac{P^U}{P}}\Phi\left(\frac{\log(\frac{P}{P^U}) + (r - \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}}\right) - \sqrt{\frac{P \vee P^L}{P}}\Phi\Bigg(\frac{\log(\frac{P}{P \vee P^L}) + (r - \frac{\sigma^2}{2})\Delta}{\sigma\sqrt{\Delta t}}$$

Optimal Stopping:

- When price $P \leq P^L$ then holdings take value PX (when discounted) is a martingale
- When price $P \in (P^L, P^U)$ then fees are being collected
- When price $P \geq P^U$ then holdings take value Y (when discounted) is a supermartingale
- Ansatz: Stop when the price exceeds some level $P^* \geq P^U$

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$$V(P) = \begin{cases} Y & \text{if } P \geq P^* \\ e^{-r\Delta t}\mathbb{E}[V(Pe^{(r-\frac{\sigma^2}{2})\Delta t + \sigma W_{\Delta t}}] + \gamma \bar{F}(P, \Delta t) & \text{if } P < P^* \end{cases}$$

Implied Volatility:

- Goal: Determine the volatility $\sigma^* > 0$ so that $V(P_0) = P_0[(\frac{1}{\sqrt{P_0}} - X)^+ \wedge X] + [(\sqrt{P_0} - Y)^+ \wedge Y]$
- Note that the implied volatility is independent of the level of liquidity L

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Implied Volatility

- If $P_0 < P^U$ then there does not exist an implied volatility
- If $P_0 \geq P^U$, solve for $P^*(\sigma) = P_0$

Conclusion

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- Depositing in an AMM constructs a path dependent Bermudan option
- Determining the implied volatility can assist in finding whether this is a good investment or not
  In practice, roughly half of all AMM investors lose money by depositing liquidity

Conclusion:

- Depositing in an AMM constructs a path dependent Bermudan option
- Determining the implied volatility can assist in finding whether this is a good investment or not
  In practice, roughly half of all AMM investors lose money by depositing liquidity
  This can happen even if the investment is a good deal in expectation
- The Greeks of this position can be calculated and hedged
- Choice of blockchain on which to operate fundamentally alter the value of an AMM

Future Work:

- Complete the analysis of the concentrated liquidity AMM
- Constant product market maker is only one AMM design. Valuation of more sophisticated AMMs
- Lack of dependence of the implied volatility on the amount of liquidity L is by AMM design that minting/burning LP tokens is invariant to the amount of liquidity that exists Update the constant product $xy = L^2$ to $xy = f(L)$
- Risk analysis of the liquidity provision position
- Optimal liquidity provision under GBM and other price processes

Thank You!

- Bichuch, Feinstein (2024): Valuation, Greeks, and Implied Volatility in Decentralized Finance from Constant Product Market Makers