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Research

Risk Management and the Optimal Combination of Equity Market Factors

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Managing the intertemporal risk of optimally constructed multifactor portfolios adds to performance. The increases in Sharpe ratios are in addition to the utility that investors gain from controlling how much active risk they are exposed to over time. We derive a simple closed-form formula for security weights in optimal multifactor portfolios with an active-risk target. We test the risk control of five well-known factors—value, momentum, small size, low beta, and profitability—and the optimal multifactor portfolio. Our empirical research was carried out on the large-capitalization US equity market for 1966 through 2019. We conclude that for the equity market, more active factors are better than fewer if each subportfolio is "pure" as to factor, anchored to the benchmark, and combined on the basis of forecastable risks. Our portfolio construction methodology allows for transparent performance attribution and replication of the process in other markets and time periods.

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As of the beginning of 2020, factor investing in the public equity markets has fallen on hard times.[1] Performance failures by a number of prominent quantitative portfolio management firms have been attributed to poor implementation of multifactor strategies and negative returns to popular factors such as value (i.e., low P/E) and small size (i.e., small-capitalization stocks). In addition, investors are increasingly concerned about active risk and the source of active returns but hesitant to "time" factors by allowing asset managers to adjust exposures. Concurrently, academic research by Moreira and Muir (2017), among others, has established that timing the market and some active factors based on risk does increase performance, although little has been published about how to combine factors into one integrated multifactor portfolio.

We derive a simple closed-form formula for the mean–variance security weights in optimal multifactor portfolios and test the performance of portfolios constructed from the largest 1,000 US stocks. We examine the optimal combination and dynamic exposure adjustments to five factors: (1) value, (2) momentum, (3) small size, (4) low beta, and (5) profitability. The backtests used calendar-month observations for the past 54 years, 1966–2019, and risk forecasts based on trailing daily return standard deviations. We also show how the risk management process with exposure adjustments based on standard deviation forecasts would have improved the performance of the market factor. When we decompose the sources of value added from optimally combining the active factors, we find that dynamic risk management is empirically similar to one additional factor. Documentation of the long-term risk-adjusted performance of the pure value, momentum, small-size, low-beta, and profitability factors also allows for a comparison of the ebbs and flows in their performance over time.

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An underlying theme of both the analytic formula and our empirical research is that, whereas changes in factor risks over time are persistent and thus partially predictable, changes in realized returns to the various nonmarket factors generally are not. Active factor payoffs may be behaviorally induced or informational anomalies, as opposed to explicit rewards for taking on risk. If risk changes can be predicted in advance that do not alter the realization of average returns, then portfolio performance is enhanced by intertemporal risk management.

Constructing and managing multifactor portfolios is viewed by some investors as complex, with performance and risks coming from various sources. Mean–variance-optimal portfolio mathematics, however, applied to pure benchmark-anchored factors without idiosyncratic risk, leads to an intuitive formula for security weights. This article thus adds to a body of prior research into the combination of factor portfolios in Clarke, de Silva, and Thorley (2016) and the composition of pure-factor portfolios in Clarke, de Silva, and Thorley (2017). We extend our prior work to show how the optimal combination of factor portfolios can be anchored to a benchmark and how security weights can be adjusted to target a specific level of active risk. Furthermore, we show that risk management can be done in a multifactor context by varying the factor exposures over time in response to changes in expected factor risk. Finally, we simulated how various multifactor portfolios would have performed historically in the 1966–2019 period.

Literature Review

The objective of risk-based timing is to increase exposure when the active factors are forecasted to be calm and decrease exposure when the active factors are predicted to be volatile, resulting in a more stable level of realized risk over time than experienced without timing. The academic guidance for performance improvements in such strategies comes from Moreira and Muir (2017) for the market factor and Barroso and Santa-Clara (2015) for the momentum factor. We use portfolio construction methodologies that are more investable, however, by avoiding long–short portfolios, as in Harvey, Hoyle, Korgaonkar, Rattray, Sargaison, and Van Hemert (2018) and Li and Shim (2019). Not all equity characteristics seem to respond to volatility management. The approach used by Cederburg, O'Doherty, Wang, and Yan (forthcoming) to evaluate volatility-managed portfolios produced mixed performance results for risk-managed formulations for more than 100 different factors.

Liu, Tang, and Zhou (2019) questioned the ease and practical implementation of volatility-timing strategies and expressed concerns about look-ahead biases. Many studies have used variance instead of standard deviation to adjust exposures for risk. Leverage and the adjustment of Sharpe ratios in financial economics are based on standard deviations, however, which have the same unit of measure as returns. In more applied, less theoretical, literature, Clarke et al. (2016) discussed how to combine active factors but failed to allow for reasonable levels of active risk, as noted by Ghayur, Heaney, and Platt (2018). Many research studies on combining active factors continue to be based on Fama–French (e.g., 1992, 1996) portfolio construction techniques that have long–short portfolio structures and material secondary exposures. The incremental contribution of this article is to extend recent innovations in multifactor portfolio construction by using optimization mathematics (derived in Appendix A) and to summarize the various issues with a simple formula for security weights based on user-defined parameters, including dynamic factor risk management.

Optimal Portfolio Formula

After some involved matrix algebra derivations, Appendix A shows that the security weights (subscripted by i) of a mean–variance-optimal two-factor portfolio (subscripted by P) at any given point in time are

$$w_{P,i} = w_{M,i} + \frac{\sigma_C}{\sqrt{IR_1^2 + IR_2^2}}\left(\frac{IR_1}{\sigma_1}\Delta w_{1,i} + \frac{IR_2}{\sigma_2}\Delta w_{2,i}\right), \quad (1)$$

where

$w_{M,i}$ is the weight of the security in the benchmark portfolio (subscript M)

IR_k and σ_k are the estimated information ratio and active risk, respectively, for factor k (this generic example has just two active factors, enumerated 1 and 2)

$\Delta w_{k,i}$ are sets of active security weights (subscripted by i) for each factor k

σ_C is a user-specified active-risk parameter—for example, 3%—which the optimization algebra indicates should be divided by the square root of the sum of the expected factor information ratios squared

A key concept in Equation 1 for active-risk management is that the optimal exposure to each factor varies over time with the factor-specific term, IR_k/σ_k. For example, volatility scaling in Harvey et al. (2018) is conceptually expressed as

$$\text{Exposure}_t = \frac{\text{Target risk}}{\text{Predicted risk}_{t-1}}, \qquad (2)$$

where exposure to the factor at each point in time, t, is based on the targeted level of risk divided by risk prediction at the beginning of the period, time $t-1$, which uses prior data or options-based (e.g., VIX) risk-forecasting variables. Although volatility management is more precise if daily changes in exposure are used, we employ monthly rebalancing in our backtests to reduce concerns about transaction costs and the timeliness of information that would have been available to investors in real time. Even with monthly holding periods, reacting to sudden increases in volatility that become evident during any calendar month also leads to significant improvements in volatility management. We maintained calendar-month rebalancing, however, to allow direct comparisons with widely published backtests of other active portfolio strategies.

A variety of *ad hoc* methodologies have evolved over the years for constructing factor portfolios. If only one nonmarket factor is involved, the active security weight, with the symbol Δ indicating the difference between the portfolio and benchmark, is

$$\Delta w_{k,i} = w_{M,i}\, b_{k,i}, \qquad (3)$$

where $w_{M,i}$ is the weight in the benchmark and $b_{k,i}$ is the capitalization-weighted (not equally weighted) standardized exposure of the security to factor k. The raw (before standardization) exposures are any numerical measure across stocks—for example, trailing earnings yield—with outliers winsorized at the 1st and 99th percentiles. Security weights derived from multivariate regressions versus univariate Fama and MacBeth (1973) regressions are similar to Equation 3, but the exposure to the factor of interest is neutralized with respect to the other factors. For example, the security weight with a second nonmarket factor is

$$\Delta w_{k,i} = \frac{w_{M,i}}{1-\rho^2}\left(b_{k,i} - \rho\, b_{2,i}\right), \qquad (4)$$

where $b_{2,i}$ is the security exposure to the second factor and ρ is the cross-sectional correlation between the two sets of factor exposures.

The weight adjustment needed to construct pure versus primary (i.e., univariate) factor portfolios is sometimes material. On the one hand, momentum stocks naturally tilt away from value, so the ρ parameter in Equation 4 is negative and the pure momentum portfolio performance over time is much higher than the primary momentum portfolio defined by Equation 3. On the other hand, if $\rho = 0$, then Equation 4 collapses to Equation 3. Inserting the pure active-factor weights in Equation 4 into Equation 1 completes the full description of an optimal two-factor active-risk-managed portfolio at each point in time.

Equation 1 is a closed-form portfolio construction process under mean–variance optimization. With rare exceptions, the portfolio in Equation 1 is long only because the market benchmark is long only. In contrast, many *ad hoc* sorting methodologies with arbitrary delimiters have undisclosed levels of security shorting and exposure to secondary factors that drive the reported results. The most egregious secondary exposure in many studies is the small-size factor because the studies use equal-weighted statistics in portfolio construction.

Without any consistent process for multifactor portfolio construction, researchers tend to select a set of sorting and delimiter rules that work best *ex post*. As derived in the Appendix A, Equation 1 is the portfolio that investors using the Markowitz (1952) algorithm would have used historically if they had ignored security-specific idiosyncratic risk.

Factor Definitions and Pure-Factor Returns

Table 1 reports the performance of five pure factors (without secondary exposures) from 1966 to 2019, the full time period for our empirical work on the US equity market, with data from CRSP and Compustat. **Figure 1** illustrates the performance of each factor over time. Return and exposure data are derived from the underlying securities, not inferences from other arbitrarily constructed factor portfolios observed in practice or Fama–French style factor portfolios. The factor portfolios are constructed to be pure, with weights specified by a five-factor version of Equation 4 and constant one-standard-deviation exposures over time.

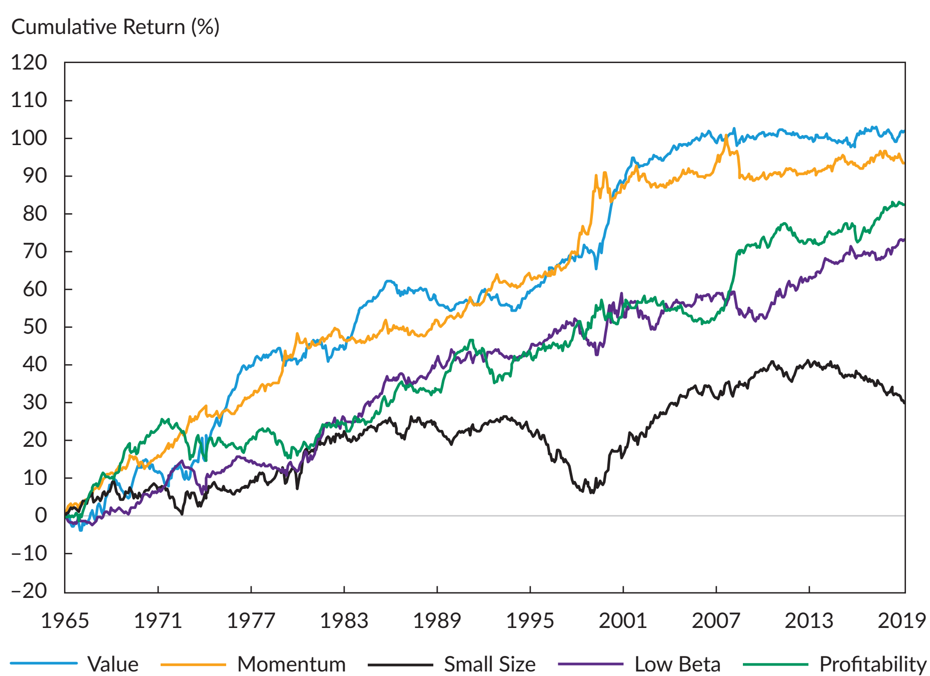
Table 1. Statistics for Pure-Factor Returns, 1966–2019

		Active Return				
Measure	Market Factor	Value Factor	Momentum Factor	Small-Size Factor	Low-Beta Factor	Profitability Factor
Average	6.29%	1.72%	2.72%	0.96%	0.51%	1.54%
Std. dev.	15.07%	3.19%	4.81%	3.62%	5.33%	3.12%
Simple ratio	0.417	0.538	0.565	0.265	0.096	0.492
Market beta	1.000	−0.040	−0.009	0.049	−0.219	−0.008
Market alpha	0.00%	1.97%	2.77%	0.65%	1.89%	1.59%
Active risk		3.13%	4.81%	3.55%	4.19%	3.12%
Information ratio		0.628	0.576	0.184	0.450	0.508

The full (648-month) sample period was chosen so that at least 1,000 publicly traded common stocks existed each month, making the Russell 1000 Index the effective market benchmark. Later in the article, we provide subperiod analyses of the monthly returns by dividing the full 54-year sample into three equal 18-year subsamples. We report empirical research on the large-cap US public equity market for the factors and definitions specified in the following material, but the analysis can be easily replicated and extended to other time periods, markets, and factors by using the formula for portfolio construction in Equation 1. The market returns are in excess of the contemporaneous risk-free rate, and active-factor returns are simple differences from the contemporaneous market return. All returns are reported before transaction costs, which are discussed later.

The average market excess return of 6.29%, divided by the realized return standard deviation of 15.07%, results in a Sharpe ratio of 0.417, which is reported as the "Simple ratio" in Table 1 to align it with the average active returns and risks of the five factors. The information ratios reported in the final row are beta adjusted—meaning that the realized market beta

Figure 1. Cumulative Risk-Adjusted Pure-Factor Active Returns, 1966–2019



was used to calculate the alpha and active risk of each factor. Except for the small-size factor, included in this study because of its historical popularity, the full-sample factor information ratios are similar in magnitude to the market Sharpe ratio—ranging from 0.450 for the low-beta factor to 0.628 for the value factor.

Figure 1 allows a visual comparison of the long-term track record of the value, momentum, small-size, low-beta, and profitability factors. To facilitate the comparison of the pure-factor returns over time, each factor's return has been risk adjusted in two ways. First, the factor portfolios are market-beta adjusted, meaning that minor (except for the low-beta factor) amounts of leverage or cash were included if the realized 54-year *active* beta was different from zero. Second, the exposure to each factor was adjusted *ex post* so that the portfolios have realized active risks of 3%, although other risk targets produce similar relative results. This adjustment is needed because the constant one-standard-deviation exposure to the various factors in Table 1 does not result in equal, let alone time-invariant, levels of active risk. For example, the realized standard deviation of active returns to the value factor of 3.13% is about two-thirds of the 4.81% realized standard deviation of the momentum factor shown in Table 1.

Several equity factors have emerged over time as being major sources of added value in managed equity portfolios. Two classical equity factors, value and small size, were popularized by Fama and French (1992, 1996). Because of distortions over time in US GAAP book values, we measure value exposure in this study by trailing earnings yield (i.e., the inverse P/E), not the original book-to-market ratio of Fama and French. Small-size exposure is the log of the inverse of market cap, although we avoid stocks that were outside the criterion of the largest 1,000 at any time to reduce concerns about transaction costs.

Note that in Figure 1 the pure value factor performed well until the mid-1980s. Then, it had some mediocre years before surging up at the turn of the century, only to show almost no additional performance (i.e., a flat cumulative return plot) from 2009 to 2019. The small-size factor was declared "dead" by some academics and quant managers at the turn of the century, only to show life again in the two decades since then. But it has had deteriorating performance since about 2015. The poor performance of the small-size factor, to which many active managers are exposed because of equal weighting of stocks in their portfolios, together with the recent flat performance of the value factor, have probably reduced the average active return of US equity managers since 2015.

The momentum factor is captured by overweighting, compared with the market benchmark, individual stocks that have recently done well. Specifically, we examined Carhart (1997) momentum, earlier associated with Jegadeesh (1990) and Jegadeesh and Titman (1993), which is defined by cross-sectional standardized scores on the most recent one-year historical return for each stock, excluding the most recent month. The momentum factor has had long-term positive payoffs that few financial economists believe have anything to do with investors intentionally taking on active risk.

The low-beta factor is based on overweighting, compared with the benchmark portfolio, stocks that have historically exhibited lower market betas than 1. The academic interest in low-beta stocks goes back to studies of the empirical failures of the original Sharpe (1964) capital asset pricing model (CAPM) in Jensen, Black, and Scholes (1972) and the applied work of Haugen and Heins (1975). The volatility factor morphed for a while in academic literature to idiosyncratic risk in Ang, Hodrick, Xing, and Zhang (2006) but was reestablished as a "betting against beta" anomaly by Frazzini and Pedersen (2014), among others. We measure exposure to the low-beta factor by the difference of the security beta from 1 (1 minus each stock's prior 36-month beta against the S&P 500 Index), resulting in a factor portfolio that has a realized beta of about 0.8.

To complete the list of five equity factors, we include profitability, defined as gross margin over assets from the most recently available annual financial statements. Aspects of this "quality" factor have been discussed by several analysts, but we used the definition of profitability introduced by Novy-Marx (2013), with an extension for stocks in the financial sector, which have been excluded in prior studies on profitability.[2]

Market-Risk Management with the Use of the VIX

Before turning to the optimal combination and risk management of these five factors, we pause to discuss the value of dynamic risk management for the market itself. Although such results for managing the volatility of the "market factor" are not new, being well documented in academic literature by Moreira and Muir (2017) and Ang (2014), they illustrate the process for intertemporal risk management. The market example also uses widely available data on S&P 500 returns and Cboe Volatility Index (VIX) quotes that can be easily replicated. Note that option-implied volatility from the VIX tends to forecast higher risk in the equity market than is realized over time, an effect called the "the volatility premium" by Eraker (2009), although the targeted level of risk does not affect the resulting Sharpe ratios.[3] The objective of risk-based timing in this example is to increase exposure when the VIX predicts the equity market to be calm and decrease exposure when the market is predicted to be volatile, resulting in a stable level of realized risk over time.

Specifically, consider a strategy of maintaining a *static* 50/50 exposure to the S&P 500 portfolio and cash, compared with a *dynamic* strategy of varying market exposure based on the inverse of predicted volatility, in accordance with Equation 2. The static portfolio has realized returns and risk of, respectively, 3.98% and 7.09% over the 30-year (1990–2019) period for which VIX quotes are available. By construction, the static portfolio has half the average return, risk, and market beta, but the same 0.561 Sharpe ratio as the market. In contrast, the dynamic strategy is based on a target volatility parameter of 10.92%, chosen so that that the *ex post* market beta is 0.50 to match the beta of the static strategy. The dynamic portfolio has a realized return of 4.92% and risk of 7.56%, yielding a Sharpe ratio of 0.651 compared with just 0.561 for the market. Using the Treynor and Black (1973) rule and these Sharpe ratio numbers, the information ratio for the impact of market-risk management is quite material at $(0.651^2 - 0.561^2)^{1/2} = 0.329$. Note that the results are invariant to the specific level of targeted risk. For example, the same Sharpe ratio and IR hold for a leveraged dynamic strategy with a targeted volatility that results in an *ex post* beta of exactly 1 to the market.

Nonpure Factors and Factor Risk Control

In this section, we report returns for reference purposes on single-factor (primary-factor) portfolios, in contrast to the five pure-factor portfolios in Table 1 that will be used in our multifactor portfolio construction. Then, we report statistics on risk-controlled pure-factor portfolios. The primary-factor portfolios were constructed using the same security return data and factor definitions in Table 1, but the single-factor (univariate) formula in Equation 2 was used instead of the appropriate version of Equation 3. The primary-factor portfolios thus have large ancillary, or secondary, exposures to the other factors that in some cases dominate the intended factor exposure.

Table 2 reports on the primary-factor portfolio returns in the same format as the bottom half of Table 1. As previously documented in Clarke et al. (2017), the primary portfolios for all five factors have performed poorly compared with the pure-factor portfolios, as seen by the lower information ratios in the bottom row of Table 2. Conceptually, pure-factor portfolios perform better because their active risk is reduced when the unintended exposures to other factors are removed. Specifically, all five primary factors in Table 2 have active realized risk numbers that are greater than the pure factors in Table 1, even though they have the same constant exposure over time as measured by one standardized score.

As a practical matter, some primary-factor portfolios also suffer in terms of performance because their exposures have been negatively correlated in the cross-section with other factors that have positive realized active returns. As shown in **Table 3**, the realized active returns of the primary-factor portfolios since 1966 are highly correlated in some cases.

Table 2. Primary-Factor Return Statistics, 1966–2019

		Active Return				
Measure	Market	Value	Momentum	Small Size	Low Beta	Profitability
Market beta	1.000	−0.077	−0.010	0.076	−0.262	−0.014
Market alpha (%)	0.00	1.36	2.44	0.62	1.67	1.23
Active risk (%)	—	4.37	6.13	3.98	4.89	3.79
IR	—	0.312	0.398	0.157	0.341	0.324

Table 3. Return Correlations of Primary-Factor Portfolios, 1966–2019

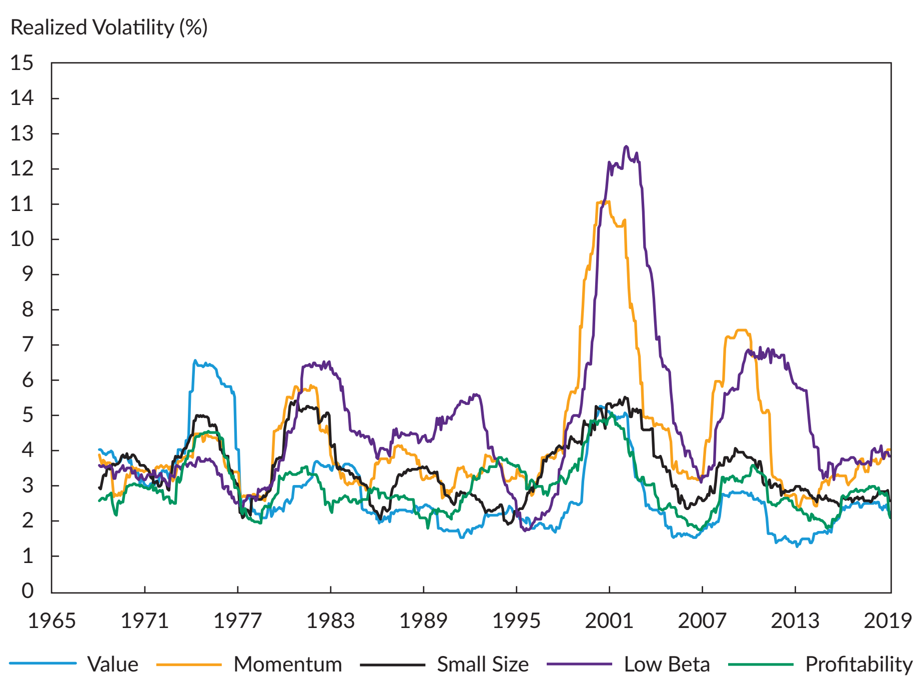
	Value	Momentum	Small Size	Low Beta	Profitability
Value	1.000	−0.436	0.155	0.452	−0.454
Momentum		1.000	−0.117	−0.009	0.256
Small size			1.000	−0.289	−0.298
Low beta				1.000	−0.036
Profitability					1.000

A similar correlation matrix for the pure-factor returns in Table 1 has off-diagonal numbers that are generally close to zero. For example, the information ratio for the pure value factor in Table 1 is 0.628, about double the 0.312 number reported for the value factor in Table 2, because value exposures have tended to be negatively correlated with both momentum and profitability.

In contrast, the returns to the value factor shown in Table 2 are positively correlated with returns to the low-beta factor, as shown by the 0.452 correlation coefficient in Table 3. The reason is that for much of the time frame in this study, value stocks tended to be stocks with low betas. Without the perspective of pure-factor portfolio construction, the performance of the value and low-beta factors can be confounded. Similar secondary exposure problems exist for factor portfolios constructed according to the Fama–French (1992, 1996) approach, except for small-size exposure that was heuristically controlled. Fama–French factor portfolios also tend to be long–short composites with implicit leverage, as opposed to being anchored to the long-only benchmark.[4]

Before examining the dynamic risk control of individual-factor portfolios, we include **Figure 2** to document changes in factor risks over time. Using the pure-factor returns reported in Table 1, Figure 2 plots the rolling 36-month realized risk of each factor return with a constant exposure over time of 1.0. The realized level of active risk is clearly unstable. For example, the risk of the value factor has varied from a high of more than 6% in the mid-1970s to just 1%–3% in 2005–2019. The realized active risk of all five factors jumped around the turn of the century; the highest numbers are for the momentum and low-beta factors. Note that these active-risk numbers are from realized returns on the pure factors, so the correspondence between the levels of active

Figure 2. Realized 36-Month Active Risks from Constant Factor Exposures, 1966–2019



risk for each factor over time is not a result of their secondary (i.e., unintentionally shared) exposures.

Similar to the results from risk management of the market factor, we now report the empirical results of managing the active risk of individual equity factors. As one might expect, more opportunity is available to react to changes in the predicted risk for factors whose risk is less stable over time. Although a variety of sophisticated risk prediction models, such as generalized autoregressive conditional heteroskedasticity (GARCH) models, are available, we use a simple but dynamic prediction process for individual monthly factor risks—namely, trailing daily-return standard deviation. Specifically, we use the square root of the sum of the squared daily active returns for the prior 60 trading days (approximately three calendar months) as the input parameter for the expected active risk of each factor.[5] The factor performance results we report next were generally invariant with the number of prior days used (e.g., 20 instead of 60) and improve only slightly with GARCH-based estimates of forward risk.[6]

Figure 3 plots the realized active risk of the risk-managed factor portfolios over time to illustrate the effectiveness of the *ex ante* risk control. Each portfolio's exposure was adjusted over time according to Equation 2, with a risk target of 3%, although other values could be used. Interestingly, the realized risks are generally higher than the 3% target because we used past daily return volatility to predict monthly risk. Annualized realized risk as measured by daily returns provides a downward bias in predicting the risk of monthly returns, so the desired exposure is too high. The result is similar (but opposite in direction) to the use of the VIX in predicting the realized risk for the market factor.[7] The plots in Figure 3 are generally level compared with Figure 2 and confirm that dynamic risk control of the individual factors works.

Table 4 reports the information ratio for each of the pure factors when exposures varied over time according to Equation 2. Table 4 is reported in the same format as the lower portion of Table 1 to allow a direct comparison. In addition to the objective of controlling and leveling out the sources of active risk, the IRs in the bottom row of Table 4 show how the performance of each pure factor was affected by dynamic risk management. The value factor's IR drops slightly from 0.628 in Table 1 to 0.552 in Table 4, whereas the increase in the IR of the momentum factor from 0.576 to 0.810 is quite large. The small-size factor's IR also drops slightly in Table 4, but the low-beta IR increases from 0.450 to 0.584. Finally, the IR for the profitability factor drops from 0.508 to 0.429. These results are generally consistent with the analysis by Cederburg et al. (forthcoming), who find that the market factor and the momentum factors seem to be more responsive to dynamic risk management than the other factors they tested.

Figure 3. Realized 36-Month Active Risks with *ex Ante* Risk Control, 1966–2019

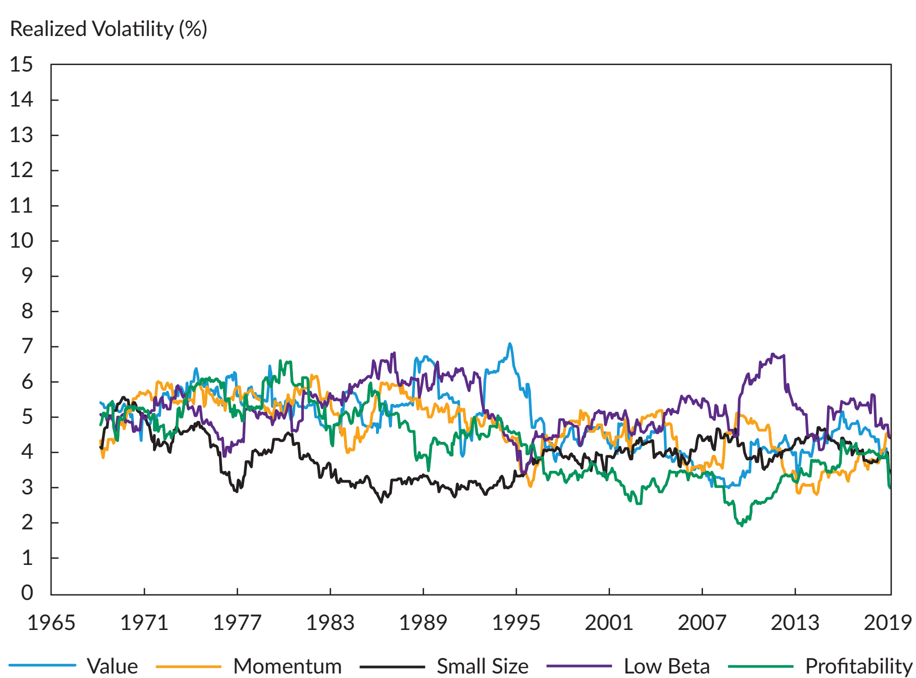


Table 4. Active Return Statistics for Risk-Managed Exposures, 1966–2019

	Value	Momentum	Small Size	Low Beta	Profitability
Market beta	−0.058	−0.017	0.053	−0.220	−0.003
Market alpha (%)	2.78	3.98	0.46	2.44	1.93
Active risk (%)	5.04	4.91	3.92	4.18	4.50
IR	0.552	0.810	0.118	0.584	0.429

The risk-control performance enhancements for the momentum factor are similar to those of Moskowitz, Ooi, and Pedersen (2012), Barroso and Santa-Clara (2015), and Moreira and Muir (2017), although less research has been done on the low-beta factor. From the *ex post* perspective of Figure 2, risk management adds more value for the momentum and low-beta factors because those are the factors that show the greatest changes in constant exposure risk over time.

In general, the adjustment in factor exposures over time to maintain constant levels of active risk appears to do little harm to the performance of three factors while increasing the value added from the other two. The dynamic adjustment of active factor exposures does increase security turnover, however, as shown later in the implementation section. Nevertheless, the impact of transaction costs is small with respect to the changes in information ratios.

Multifactor Portfolio Combination and Risk Control

We now examine the long-term realized information ratios of various multifactor portfolios, with a final portfolio constructed according to a version of Equation 1 with five factors. The improvement in performance of the final multifactor portfolio over the passive benchmark comes from three sources: (1) the inclusion of five (as opposed to only one or two) nonmarket factors that have been important historically, (2) the use of pure-factor portfolios that have zero secondary exposures to the other factors under consideration, and (3) risk management of each of the five factor exposures using the persistent component of their active risks.

We did not examine the added value that comes from a potential fourth source in Equation 1—namely, adjusting the relative magnitudes of the individual-factor information ratios—because we do not know what investors might have expected starting back in the 1960s. In other words, in **Table 5**, we do not show the result of optimally combining the various active factors, although we comment later on improvements in performance by emphasizing some factors over others in sensitivity and subperiod analyses. For example, the recent historical performance of the small-size factor and the value factor may motivate investors now to assign zeros to their *ex ante* information ratios. The IRs that might have seemed reasonable to investors several decades ago, however, are unknown, and adjusting the IRs

Table 5. Multifactor Portfolio Return Statistics, 1966–2019

Measure	Market	Equal Mix of Primary	Equal Mix of Pure	Risk-Based Timing
Mean (%)	6.28	7.39	7.77	10.79
Std. dev. (%)	15.07	14.32	14.47	14.10
Sharpe ratio	0.417	0.516	0.537	0.765
Market beta	1.000	0.943	0.954	0.891
Market alpha (%)	0.00	1.46	1.77	5.18
Active risk (%)	—	1.80	1.58	4.32
IR	—	0.811	1.122	1.200

to account for the relative realized performance of the factors is a clear *ex post* bias, although some bias already exists in selecting these five factors. Active factors were only gradually introduced by the academic work of Fama and French (1992) and the active portfolio management frameworks of Grinold (1989) and Black and Litterman (1992).

Table 5 reports statistics on the market portfolio's return and the return on three multifactor portfolios for the 54 years of the study. Each of the three actively managed portfolios illustrates the incremental impact from the various sources of multifactor portfolio construction. The second column of data reports results for an equal mix of the five *primary* factors with a constant exposure to each factor over time of 1/5 = 0.200. Because the primary-factor portfolios were constructed on the basis of the security weights in Equation 2, they sometimes have substantial secondary exposures to the other factors, so the total (i.e., intended and unintended) exposure to any given factor may be quite different from 0.200. Nevertheless, this equal mix of the five primary factors provides a Sharpe ratio of 0.516 and a beta-adjusted information ratio of 0.811 compared with zero, by definition, for the market.

The calculation of the Treynor–Black (1973) rule for this portfolio's Sharpe ratio, based on the market Sharpe ratio and its information ratio, is $(0.417^2 + 0.811^2) = 0.912$, higher than the 0.516 ratio given in the third row of Table 5. The reason is that IRs at the bottom of Table 5 take the low market betas into account. Specifically, the equal-mix portfolios have betas that are about one-fifth of the way between 1.0 for the market portfolio and the Table 1 calculation of the pure low-beta market beta of 1.000 – 0.219 = 0.781. Although realized market betas over the full 54-year sample were used for the IR calculations in Table 5, the pure low-beta portfolio has a predictably lower beta, as shown later in the subperiod analysis, so the *ex post* bias is immaterial.

The "Equal Mix of Pure" column of Table 5 reports results for an equal combination of the five *pure* factors. The exposure to each factor is now exactly 0.200, rather than being potentially offset or amplified by the secondary exposures in the other factor portfolios. The information ratio of this multifactor portfolio increases from the 0.811 shown for the primary-factor portfolio to 1.122, a large improvement. To be more precise, the calculation of the impact on IR compared with the mix of primary-factor portfolios using the Treynor–Black (1973) rule is $(1.122^2 - 0.811^2)^{1/2} = 0.775$.

The last column of Table 5 reports results for a risk-managed multifactor portfolio with month-by-month predictions of each factor's active risk. As calculated later in Equation 5, the exposure without the risk management parameters goes from 0.200 to 0.447 for each factor, so the active risk more than doubles, from 1.58% to 4.32%, compared with the prior column. Similarly, the magnitude of the reduction in market beta below 1.000 to 0.891 is about twice that of the equal-mix portfolios. The adjustment of factor exposures based on dynamic risk management increases the multifactor portfolio's information ratio from the equal mix's IR of 1.122 to 1.200, giving a Treynor–Black calculation for the incremental impact of $(1.200^2 - 1.122^2)^{1/2} = 0.426$.

The use of the Treynor–Black (1973) rule is particularly important in assessing the incremental economic impact of the already high information ratios. The 0.426 impact of risk management is similar to the addition of a completely new factor—that is, numerically similar to the entries in the bottom row of Table 1. The realized information ratio of 1.200 compared with zero by definition for the market portfolio is invariant with the level of active risk chosen by the investor. For example, when a 2% (rather than 3%) active-risk target is used in portfolio construction, the realized alpha in the last column of Table 5 is 3.46% and the realized active risk is 2.88%, resulting in the same 346/288 = 1.200 realized IR.

Regression Analysis

We also use a regression of realized active returns for the entire 54 years on the primary-factor and pure-factor active returns to measure the value added by the factors in the final dynamic multifactor portfolio. The results capture both the value added from individual-factor risk management and the balance between factor risks over time. Panel A of **Table 6** shows a regression of the active returns for the final risk-managed portfolio on the primary-factor active returns reported in Table 2. As discussed in Cederburg et al. (forthcoming), caution should be exercised in interpreting the alpha in some full-sample regressions, although the volatility-managed portfolios and subsequent regressions by those authors were structured differently from those here. The intercepts (alphas) in Table 6 represent the difference in mean return between the risk-managed portfolio and a portfolio with a constant mix of factor exposures.

The 232 basis point (bp) alpha in Panel A of Table 6 is the annualized (multiplied by 12) intercept term from

Table 6. Full-Sample Regressions of Risk-Managed Portfolio Returns on Factor Returns

	Alpha	Value	Momentum	Small Size	Low Beta	Profitability
A. Primary-factor returns						
Coefficient (%)	2.32	0.510	0.320	0.348	0.211	0.512
t-Statistic	5.1	12.4	13.4	10.0	8.2	12.9
B. Pure-factor returns						
Coefficient (%)	0.67	0.691	0.422	0.415	0.394	0.583
t-Statistic	2.1	24.2	23.0	17.0	23.1	20.5

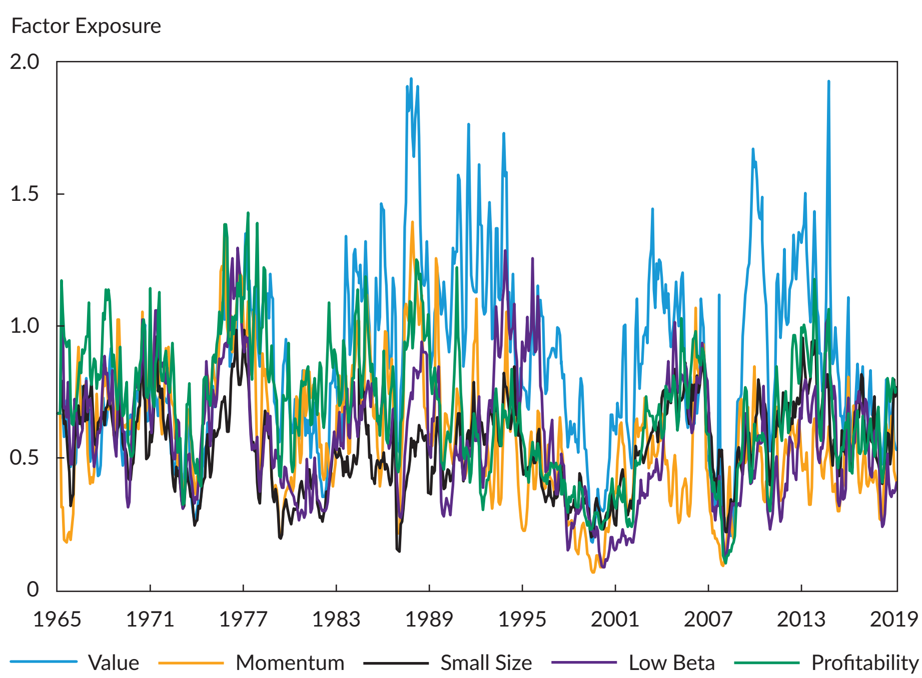
the regression, with a highly significant *t*-statistic of 5.1. The alpha is related to the improvement in information ratios between the portfolio in the last column of Table 5 and the equal mix of primary factors in the second column of Table 5. The regression coefficient on each factor return represents the average exposure over time to that factor. Given that the multifactor portfolio is constructed from the individual-factor portfolios, the slope coefficient *t*-statistics all indicate large statistical significance, and the R^2 for the regression in Panel A of Table 6 is 0.507.

Panel B of Table 6 reports results for a regression of the final risk-managed portfolio active returns on the pure-factor active returns shown in Table 1. Thus, the 67 bp alpha is related to the improvement in information ratios between the portfolio in the final column of Table 5 and the mix of pure factors in the prior column. Based on the reported standard errors, the slope coefficients in Panel B of Table 6 are even more statistically significant than in Panel A, and the regression R^2 increases from 0.507 to 0.771.

Subperiod Analysis and Factor IR Parameters

The exposures of the risk-managed multifactor portfolio reported in the final column of Table 5 change over time according to the estimated risk of each pure factor, as shown in **Figure 4**. For example, all five factor exposures dipped to about 0.25 at the turn of the century because the estimated active risks from prior 60-day returns were high at the time. This conclusion is verified by the realized 36-month

Figure 4. Risk-Managed Factor Exposures over Time, 1966–2019



rolling risks shown in Figure 2. The exposure does not vary over time or across factors on the basis of differences in the forecasted information ratios, however, in the five-factor version of Equation 1. In fact, based on a comparison of each factor IR with the square root of the sum of the five IR parameters squared, the formula-driven exposure to each factor without changes in risk is 0.447:

$$\text{Exposure}_k = \frac{IR_k}{\sqrt{IR_1^2 + IR_2^2 + IR_3^2 + IR_4^2 + IR_5^2}}. \qquad (5)$$

Equation 5 follows from Equation 1 with the risk terms removed and with five, rather than two, factors. The constant 0.447 exposure is driven by the fact that there are five factors—no matter what individual-factor IR parameters are used, as long as they are equal. The approach here is similar to the risk-parity work done by Asness, Frazzini, and Pedersen (2012), which links risk estimates to asset class weights in a portfolio, in contrast to our linking of individual security weights to equity factor risk.

Table 7 reports results for three equal 18-year (216-month) subperiods, in order to examine the changes in pure-factor performance over time. Table 7 also illustrates how investor beliefs about factor selection and performance affect multifactor portfolio construction.

We focus on information ratios, so we report only those, like the bottom row of Table 1, consistent with the visual impressions in the risk-adjusted cumulative performance shown in Figure 1. Several notable changes occurred in performance over time, as was shown in Figure 1—a decline in IR for the value factor in the last subperiod, a decline in the momentum factor in each subperiod, and the almost zero IR for the small-size factor in the middle period. In fact, the small-size IR of 0.236 in the last period masks the decline in performance in the previous subperiod that is shown in Figure 1.

In contrast to the pattern for the small-size factor, the realized information ratios of the low-beta factor, which had an even lower realized market beta of about 0.75 in recent periods, has been fairly stable. The profitability factor, with a realized information ratio of 0.689, was the clear winner in the last subperiod. Table 7 also reports the risk-managed information ratios of each factor. Notably, risk management added to the performance of the value factor in the first subperiod. The value IR increases from 0.675 to 0.743, which is consistent with the fact that the value factor's riskiness was unstable, as shown in Figure 3, as a result of constant exposure in those early years.

Optimally combining factors on the basis of their relative information ratios in Equation 1 was described previously as a possible fourth source of added value, but using different IR parameters for individual factors in Equation 1 presents the investor with the challenge of how to form expectations about the future. The approach might also have *ex post* biases, as did the selection and definition of the (now popular) five factors in this study starting in 1966. To illustrate and measure the potential added value,

Table 7. Pure-Factor Return Statistics by Subperiod

	Active Return				
Measure	Value	Momentum	Small Size	Low Beta	Profitability
A. 1966–1983					
IR	0.675	1.010	0.364	0.537	0.443
Risk-managed IR	0.743	1.176	0.274	0.742	0.444
B. 1984–2001					
IR	0.848	0.537	−0.001	0.502	0.453
Risk-managed IR	0.548	0.809	−0.129	0.568	0.520
C. 2002–2019					
IR	0.239	0.266	0.236	0.468	0.689
Risk-managed IR	0.277	0.362	0.217	0.472	0.290

however, we examined several nonequal pure-factor mixes. For example, when the full-sample *ex post* IR numbers from the bottom of Table 1 were used in a five-factor version of Equation 1, an obvious *ex post* bias, the multifactor portfolio realized IR increased from 1.122 to 1.186 and the risk-managed, or final, portfolio IR increased from 1.200 to 1.231.

When the small-size factor was given no weight, by assigning it an information ratio of zero, the multifactor portfolio IR increased from 1.122 to 1.151 and the risk-managed portfolio IR increased from 1.200 to 1.229. When the small-size and value factors were both eliminated, as might be done in the future on the basis of contemporaneous investor perspectives, the full-sample multifactor portfolio IR declined from 1.122 to 0.916 because three factors do not provide as much factor diversification as five provide. Alternatively, the increase in value added from dynamic risk management in the three-factor case would be measured by comparing the 0.916 number in what would be the third column of Table 5 with a 1.070 number in the fourth column, because risk management historically did not add value over the full sample period to the small-size or value factors. Using the Treynor–Black (1973) rule, the incremental value added by dynamic risk management at the 3% target level for the three-factor case would be $(1.070^2 - 0.916^2)^{1/2} = 0.553$, substantially larger than the 0.426 number reported previously for an equal mix of five pure factors.

In summary, the ability to choose different commitments to the factors by choosing different IR parameters for Equation 1, rather than choosing an implicit equal mix, could significantly improve multifactor portfolio construction going forward, but represents *ex post* biases in the empirical backtest.

Implementation and Turnover

The pure-factor portfolios in Table 1 and the combined multifactor optimal portfolios in Table 5 are directly investable, as opposed to the long–short, or leveraged, portfolios from quintile sorts often used in academic studies of factor returns. The security weights for large-scale (e.g., 1,000-stock) factor-based optimal portfolios are determined by an intuitive closed-form formula in Equation 1. Portfolio weights can be calculated in a spreadsheet without the black-box effects associated with numerical optimizers and constraints. The optimal portfolio is anchored to the benchmark, rather than the implicit anchoring to an equal-weighted portfolio in other mean-variance-optimal solutions. With the addition of a user-specified level of active risk (e.g., 3%), the solution is, by nature, long only or close to long only without a constraint. The investor has complete transparency on how each security weight depends on

- the benchmark weight for that security, $w_{M,i}$, at the beginning of the period,
- the $k = 1$ to K pure-factor portfolio weights, $\Delta w_{k,i}$, calculated from the security exposures to the factors using multivariate regression mathematics, and
- mean–variance-optimal factor allocations, IR_k/σ_k, based on the assumed *IR* and risk of each factor.

To emphasize the practicality of factor investing and the multifactor portfolio construction in the backtests of this study, we examine a year 2020 optimal portfolio using specific parameter choices and then report security weights for the largest 20 out of 1,000 US stocks. **Table 8** shows the input

Table 8. Optimal Multifactor Portfolio Parameters

	Active Risk: 3.00%		Low-Beta Correlation with Market: –0.600			
Factor	IR	Risk	Return	Actual Risk	Actual Return	Optimal Exposure
Market	0.40	15.00%	6.00%			1.000
Value	0.20	2.50	0.50			0.300
Momentum	0.30	5.00	1.50			0.225
Small size	0.10	3.00	0.30			0.125
Low beta	0.50	4.00	2.00	5.00%	0.80%	0.469
Profitability	0.50	4.00	2.00			0.469
Portfolio	0.80					

parameters in the first two columns, including the user-supplied target of 3.00% portfolio active risk, and then the resulting expected returns and optimal factor exposures. We chose different information ratios for the various factors, in contrast to the equal IRs across factors in the backtest, to illustrate the calculations. For example, the investor in this example has assigned an *ex ante* IR of 0.20 to the value factor, with active risk of 2.50%, and a higher *ex ante* IR for the momentum factor, 0.30, with active risk of 5.00%.

As described in Appendix A (see Equation A12 and Equation A13), the assigned information ratio of 0.50 for the low-beta factor in Table 8 is based on *adjusted* active-risk and return numbers and an assumed correlation between the active low-beta factor return and the market return of –0.600. Table 8 shows that the actual active risk of the low-beta factor is 4.00/ $(1 - 0.600^2)^{1/2}$ = 5.00% and the actual active return is just 2.00 – 0.60 × (5.0/15.0) × 6.0 = 0.80%.

The IR for the optimal portfolio, the last row of Table 8, is exactly 0.80, calculated as the square root of the sum of the five factor IRs squared. The user-supplied active-risk parameter of 3.00% and the portfolio IR of 0.80 are then used to calculate the optimal factor exposures, as specified by the five-factor version of Equation 1. For example, the calculation of the optimal exposure to the value factor is (3.00/0.80) × (0.20/2.50) = 0.300, and the optimal exposure to the momentum factor is (3.00/0.80) × (0.30/5.00) = 0.225.

To illustrate how investor expectations in Table 8 translate into the portfolio weights of individual securities, we next describe exposure data for 1,000 stocks in early 2020. **Table 9** reports the benchmark (i.e., market portfolio) weight, the five active weights, and the total managed portfolio weight for the largest 20 (sorted by declining market weight) stocks in the optimal portfolio. The market weight and active weights in each row of Table 9 sum to the total portfolio weight in the final column, where each factor's active weight is the optimal exposure calculation from Table 8 times the pure-factor active weight in the five-factor version of Equation 4. The weights shown in Table 9 cannot be replicated without the 1,000-by-6 exposure matrix designated **B** in Appendix A, too large a table for this article.

As an example, consider the portfolio weight on Facebook (ticker FB) of 2.16% in Table 9. The total portfolio weight is the market weight of 1.70% plus the sum of the active tilts. Most of the active tilt in favor of FB is from exposure to the profitability factor, although this effect is offset by the tilt away from FB on the basis of its negative exposure to the low-beta factor. Ironically, the biggest negative total portfolio weight among the largest 20 stocks in Table 9 is the –1.00% for Berkshire Hathaway (BRK), with negative tilts from all five factors. This negative total weight indicates the investor should sell BRK short in the optimal portfolio, although as indicated in **Table 10**, the total short weight for all stocks is only 7.7% (i.e., effectively a "108/8" long–short portfolio).

Table 10 reports the expected return and risk for the optimal portfolio described in Table 9 based on basic portfolio calculations and all 1,000 security weights. Also shown is the performance of a strictly long portfolio created by setting the few negative security weights to zero and resizing the rest to sum to 100%. The expected annual alpha of the optimal portfolio is 240 bps, and by design, the *ex ante* active risk is exactly 300 bps, giving an information ratio of 240/300 = 0.800. This result confirms the calculation from the underlying portfolio construction parameters in Table 8. The optimal portfolio's Sharpe ratio, which does not account for the *ex ante* market beta of 0.906, is 0.563, compared with the benchmark Sharpe ratio of 0.400.

The third column in Table 10 shows the expected return and risk of a strictly long-only portfolio based on the simple heuristic of setting the weights of the few short-sell stocks to zero and resizing the rest to 100%. Because this portfolio is no longer strictly optimal, the expected information ratio drops slightly, from 0.800 to 0.794. The result is an *ex ante* transfer coefficient (TC) of 0.794/0.800 = 0.992, suggesting that this approach has little impact on performance at the 3% target active-risk level.[8] The long-only portfolio has materially lower risk and market beta than the benchmark, however, so in the final column of Table 10, we report the performance of a levered portfolio with a market beta of exactly 1.0. The leverage factor based on the *ex ante* market beta is 1/0.927 = 1.08, meaning 8% of the notional value of the portfolio is borrowed at the risk-free rate and used to proportionally increase the position sizes of all the positive-weight securities. The portfolio alpha of 195 bps was calculated directly from expected returns of the managed and market portfolios of, respectively, 7.95% and 6.00%. Note that leverage does not alter the long-only Sharpe ratio of 0.523 or the underlying IR of 0.794.

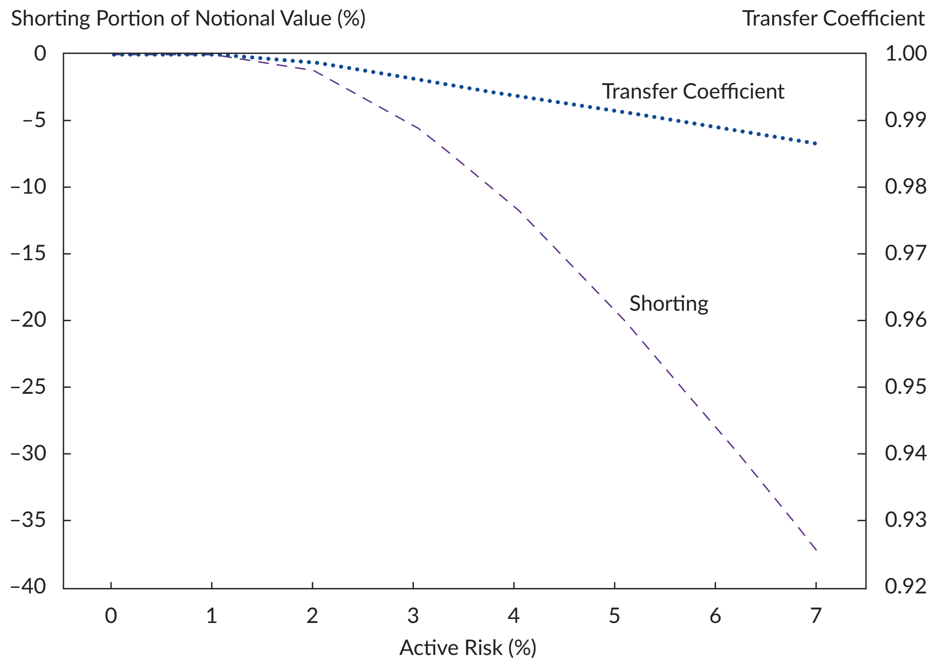
Table 9. Optimal Portfolio Weights of the Largest 20 Market Cap Stocks, 2020

		Optimal Exposure					
		0.300	0.225	0.125	0.469	0.469	
		Active Weights					
	Market Weight	Value	Moment	Small Size	Low Beta	Profit	Total Weight
AAPL	4.49%	0.27%	1.57%	−0.87%	−0.25%	−0.46%	4.75%
MSFT	4.14	−0.30	0.59	−0.82	0.15	−0.42	3.35
AMZN	3.15	−1.10	−0.95	−0.58	−2.56	1.92	−0.11
FB	1.70	0.13	0.26	−0.18	−0.49	0.74	2.16
GOOG	1.58	0.35	−0.08	−0.19	0.06	0.23	1.94
JPM	1.50	0.34	0.10	−0.22	−0.23	−0.88	0.61
GOOGL	1.38	0.43	−0.07	−0.15	0.03	0.24	1.85
JNJ	1.32	−0.08	−0.31	−0.15	0.19	0.25	1.22
WMT	1.16	−0.13	−0.03	−0.08	0.52	0.86	2.31
V	1.11	−0.06	0.09	−0.11	0.20	−0.32	0.91
BAC	1.09	0.35	0.03	−0.14	−0.63	−0.63	0.07
BRK	1.08	−0.49	−0.33	−0.16	−0.15	−0.94	−1.00
PG	1.07	−0.24	0.08	−0.09	0.76	−0.05	1.52
MA	1.03	−0.16	0.21	−0.08	0.16	0.01	1.17
XOM	1.02	0.12	−0.30	−0.13	−0.32	−0.42	−0.03
T	0.98	0.24	0.14	−0.08	0.42	−0.30	1.40
UNH	0.96	−0.02	−0.17	−0.09	0.20	0.09	0.97
INTC	0.90	0.37	0.03	−0.05	0.40	0.32	1.96
VZ	0.87	0.15	−0.10	−0.07	0.55	0.01	1.42
HD	0.82	0.15	−0.01	−0.01	0.12	1.12	2.18

Table 10. Expected Optimal Multifactor Portfolio Characteristics, 2020

	Market	Optimal	Long Only	Levered Long
Expected return (%)	6.00	7.84	7.37	7.95
Total risk (%)	15.00	13.92	14.08	15.20
Sharpe ratio	0.400	0.563	0.523	0.523
Market beta	1.000	0.906	0.927	1.000
Market alpha (%)	0.00	2.40	1.81	1.95
Active risk (%)		3.00	2.28	2.46
IR		0.800	0.794	0.794
Transfer coefficient		1.000	0.992	0.992
Total long (%)		107.7	100.0	107.9

Figure 5. Optimal Portfolio Shorting and Long-Only Portfolio Transfer Coefficients



The illustrative optimal and long-only portfolios in Table 10 were constructed under the assumption of an active-risk target of 3%, the same parameter used in the empirical backtests in this study. **Figure 5** explores the sensitivity of the portfolio's amount of shorting and TC presented in Table 10 to various levels of active risk. For example, at the 3% active-risk level, the optimal portfolio has shorting equal to −7.7% of the portfolio notional value, shown at that risk level in Figure 5. When the short sells are removed, the heuristic long-only portfolio has a TC of 0.992, as reported in Table 10, shown at that risk level in Figure 5.

At lower levels of active risk, shorting and the information ratio loss as measured by the TC reduction are essentially zero, but as the active-risk target parameter increases to 6% and 7%, shorting goes up to more than 30% of the notional value of the portfolio, producing, essentially, a 130/30 long–short portfolio. Even at these relatively high levels of active risk, the distortion in optimal exposures from using the long-only portfolio construction heuristic is relatively minor; the TC is still above 0.98. We obtained similar results for most other choices for the factor information ratios in Table 10. The exception was that high loadings on the small-size factor from using a larger *ex ante* IR resulted in a materially lower TC.

An additional consideration in the implementation of multifactor portfolios is security turnover, which we measured by monthly percentage changes in security weights. Specifically, we calculated turnover as the sum of the absolute differences in the rebalance weight and the weight of each security in the prior month, adjusted for the return on that security during the month, divided by two. For reference purposes, **Table 11** reports the average monthly turnover over the full sample period for the market portfolio, the five pure-factor portfolios, and the equal mix of pure factors reported in the third column of Table 5. Table 11 also reports turnover on the strictly long-only implementation of the equal mix of the pure-factor portfolio and the

Table 11. Average Monthly Portfolio Turnover, 1966–2019

Portfolio	Turnover
Market (largest 1,000 stocks)	0.8%
Pure value	9.9
Pure momentum	19.6
Pure small size	5.7
Pure low beta	11.0
Pure profitability	5.4
Equal mix of pure factors	6.3
Long-only equal mix of pure factors	5.9
Risk-managed optimal portfolio	22.3

dynamic risk-managed portfolio in the final column of Table 5. The turnover numbers may be higher than what would be experienced in practice based on a transaction-cost model as part of the rebalancing process, so these data should be viewed on a relative basis by factor.

The market portfolio, based on changes in the constitution of the index and the reinvestment of individual stock dividends, has a nonzero monthly turnover; the annual rate would be 9.6%. The value portfolio turnover tends to accumulate in April of each year because of the use of calendar-year financial statements with an assumed four-month reporting lag to ensure the information was available to investors in real time. The momentum portfolio turnover is the highest of the five factors in this study because the exposure (the prior 11-month return, lagged 1 month) changes faster over time than the other factors. The small-size portfolio turnover is relatively low because these factor portfolios are anchored to the cap-weighted benchmark.

The low-beta portfolio turnover is high compared with that of the other factor portfolios, except for momentum, because of constant changes in the rolling 36-month market beta for each stock. The pure profitability portfolio has relatively low turnover. As with the value portfolio, turnover in the profitability portfolio occurs mostly in the month of April when the new calendar-year financial statements are incorporated into the signal. Turnover for the profitability-based portfolio is even lower than for the value portfolio, which also has month-to-month changes in the earnings yield signal (i.e., inverse trailing P/E) because of changes in individual stock prices. The turnover results for individual-factor portfolios are generally consistent with the findings of Li and Shim (2019).

Table 11 shows that the multifactor portfolio with an equal mix of pure factors (i.e., constant exposure of 0.200 over time to each factor) has monthly turnover (6.3%) that is much lower than a simple average of the five factors. This illustrates a key advantage in our portfolio construction process that allows for offsets between factors in total security weights. For example, a security that has increased weight in the optimal multifactor portfolio because of an increase in the momentum signal might be offset in that same month because of a decrease in the low-beta signal. The strictly long-only application of the equal-mix portfolio, based on the simple reweighting heuristic, has slightly lower turnover than that of the multifactor portfolio with an equal mix of pure factors because some of the rare short-sell positions in the portfolio were set to zero in consecutive months.

The key statistic from Table 11 for active-risk management of equity market factors is reported in the last row: the 22.3% turnover of the dynamic portfolio. The final portfolio's factor exposures, as plotted in Figure 4, change from month to month on the basis of changes in predicted factor risks, necessitating adjustments to the weights of all the securities that have a large exposure to any given factor. At the 3% active-risk target used as the base case in this study, turnover is about three times higher than for static multifactor portfolios; it would be naturally lower or higher at different levels of active risk.

Based on the multifactor portfolio implementation study of Li and Shim (2019), we applied a simple method of estimating transaction costs that minimized trades to about 10% of annualized portfolio turnover. For example, a rough transaction-cost calculation for implementation of the equal-mix pure-factor portfolio in Table 11 is 10% × 12 × 6.3% = 8 bps per year, and a rough transaction-cost calculation for the dynamic risk-managed multifactor portfolio is 10% × 12% × 22.3% = 27 bps. Rounding up, we infer reductions of approximately 10 bps in the gross annualized returns for our static portfolios and approximately 30 bps for dynamic portfolios. In other words, the gross alpha of 67 bps from the regression analysis in Table 6 that is specific to risk management would be, after costs, about 47 bps.

Conclusion

So, what have we learned? Intertemporal portfolio risk management based on the construction of an optimal multifactor portfolio has worked historically and carried a performance bonus. For example, month-to-month adjustments to the S&P 500 exposure in an equity/cash portfolio using the prior month-end VIX quote not only smooth out the level of market risk over time but also produce the bonus of a higher realized Sharpe ratio. Similar procedures for multifactor optimal portfolios, with the use of factor-specific prior 60-day return standard deviations, smooth out the level of active risk over time and enhance performance as measured by realized information ratios. We find that the combined impact of risk management on five well-known active factors is roughly equivalent to the addition of a new independent factor.

The empirical tests in this article illustrate several recent methodological innovations, including the construction of pure-factor portfolios from cap-weighted Fama–MacBeth (1973) cross-sectional regression mathematics. In addition to using pure-factor portfolios, we develop a formula for the construction of multifactor optimal portfolios that is anchored to the market benchmark, rather than the implicit anchoring to an equal-weighted portfolio that comes from the use of arithmetically constructed scores. We include the ability for investors to specify the level of active risk, rather than using the optimal level that results from maximizing the managed portfolio's Sharpe ratio. With one formula, investors can construct risk-controlled multifactor portfolios in a spreadsheet with complete transparency as to the sources of realized performance.

The examination of five currently popular factors, each over a 54-year time frame, is facilitated by the factors being pure and adjusted for both benchmark beta and active risk. Value and small-size active returns have been flat or declining in the US equity market for so long that investors may question whether they are still beneficial. The low-beta factor in its leveraged form has been a relatively steady performer, and the profitability factor has been consistently positive. We use simple analytic formulas to construct the factor portfolios, so the analysis of these five factors can be readily replicated for other potential factors. Because closed-form formulas are used to create the optimal multifactor portfolio, the empirical research can easily be extended to other country markets or the global equity market.

Appendix A. Optimal Multifactor Portfolios

The factor-based portfolio that maximizes the Sharpe ratio was identified in Clarke et al. (2016) as

$$\mathbf{w}_P = \mathbf{B}\,(\mathbf{B}'\mathbf{B})^{-1}\,\mathbf{V}^{-1}\,\mathbf{U}\left(\frac{\sigma_P}{SR_P}\right), \tag{A1}$$

where

B is an exposure matrix of N securities to $1 + K$ factors

V is a factor-return covariance matrix

U is a vector of expected factor returns

The final scalar term at the end of Equation A1, given as portfolio risk, σ_P, divided by the portfolio Sharpe ratio, SR_P, ensures that the individual security weights sum to 100%. The solution in that article followed from the standard mean–variance objective function, with expected security returns given by the matrix **B U**, and the security covariance matrix given by **B V B′**. A key distinction with respect to other mean–variance solutions is that Equation A1 is factor based, meaning the objective function does not have security-specific alpha parameters and idiosyncratic risk is ignored. The expected alpha of any given security is determined solely by the exposure of that security to the underlying factors. Idiosyncratic risk is left out of the objective function under the assumption that the securities will be part of well-diversified factor portfolios.

Another distinction with respect to other factor-based portfolio solutions is that the first factor in Equation A1 is the benchmark (e.g., cap-weighted market) portfolio and the remaining K factors are defined to be benchmark relative. Specifically, the first column of the exposure matrix **B** is filled with 1s, and each of the next K columns have mean-zero unit-variance scores for the factor exposures. Similarly, the first element of **U** is the expected excess (of the risk-free rate) return on the benchmark portfolio, and the other K elements are the expected active (i.e., market differential) factor returns. Note that Equation A1 is not technically closed form because the scalar multiple at the end uses values that can be determined only after the fact. But given the structure of **B**, that adjustment can be calculated directly (i.e., without reference to the final portfolio solution) as 1.0 over the first element of $\mathbf{V}^{-1}\,\mathbf{U}$, which is hereafter referred to as d for "divisor." For example, with uncorrelated factors we have $d = SR_M/\sigma_M$, the market (benchmark) Sharpe ratio divided by market risk.

The portfolio solution in Equation A1 is not unique, in that alternative sets of security weights also maximize the factor-based Sharpe ratio. Lack of uniqueness is a problem for numerical searches, commonly used in portfolio optimizations involving a large security set, although not in Equation A1 because security weights are calculated directly from the underlying parameters. For example, the cap-weighted version of exposure matrix **B** used in this article also results in an optimal portfolio, and it has the virtue of having larger active (deviation-from-benchmark) weights associated with larger stocks, which reduces turnover and transaction costs in rebalancing. In addition, the optimal portfolio that uses benchmark-weighted exposures is closer to being long only—in contrast to Equation A1, which uses arithmetic scores and thus anchors the solution to an equal-weighted portfolio.

Benchmark Anchoring

To anchor the optimal portfolio to the market benchmark, the scores in each column of exposure matrix **B** are adjusted to have benchmark-weighted means of zero and benchmark-weighted variances of 1.0. The matrix form $\mathbf{w}_M \circ \mathbf{B}$ uses the dot product function so that each element of **B** is multiplied by the corresponding row's element in the *N*-by-1 benchmark weight vector $\mathbf{w}_M$. The resulting 1 + *K* by 1 + *K* exposure correlation matrix $\mathbf{B}'\left(\mathbf{w}_M \circ \mathbf{B}\right)$ is symmetrical, with 1s down the diagonal and 0s in the first column and the first row. The benchmark-anchored version of Equation A1 is thus

$$\mathbf{w}_P = \left(\mathbf{w}_M \circ \mathbf{B}\right)\left[\mathbf{B}'\left(\mathbf{w}_M \circ \mathbf{B}\right)\right]^{-1} \mathbf{V}^{-1}\, \mathbf{U}/d. \qquad \text{(A2)}$$

Proof of factor-based optimality follows from the fact that the portfolio expected return, $\mathbf{w}_P{}'\mathbf{B}\,\mathbf{U}$, and portfolio variance, $\mathbf{w}_P{}'\mathbf{B}\,\mathbf{V}\,\mathbf{B}'\,\mathbf{w}_P$, eliminate the security-specific exposures, leaving only the factor return matrices in the portfolio Sharpe ratio of $\sqrt{\mathbf{U}'\,\mathbf{V}^{-1}\,\mathbf{U}}$.

The first part of Equation A2,

$$\mathbf{w}_F = \left(\mathbf{w}_M \circ \mathbf{B}\right)\left[\mathbf{B}'\left(\mathbf{w}_M \circ \mathbf{B}\right)\right]^{-1}, \qquad \text{(A3)}$$

gives the *pure* portfolio weights of Clarke et al. (2017), where the first column of $\mathbf{w}_F$ contains the benchmark weights and each of the next *K* columns contains the active weights of portfolios that have an exposure of 1 to the factor of interest and an exposure of 0 to each of the other nonmarket factors.

The construction of pure-factor portfolios follows from cap-weighted Fama–MacBeth (1973) cross-sectional regressions of security returns on the cap-weighted scores in matrix **B**. The use of Fama–MacBeth regression mathematics to calculate factor-replicating portfolio returns—and implicitly the weights within such portfolios—has a long tradition in academic research. The methodology of benchmark versus equal weighting of the observations, however, is relatively recent. Defining factor portfolios through regression analysis avoids the use of portfolio construction heuristics such as quintile and double sorting, arbitrarily chosen delimiters that exclude securities with lower exposure, equal weighting of securities included in each factor portfolio, and, most importantly, the secondary or unintended factor exposures of such portfolios.

Optimal Factor Allocation

The second part of Equation A2 is a 1 + *K* vector of Sharpe ratio–maximizing *allocations* to the factors:

$$\mathbf{A} = \mathbf{V}^{-1}\, \mathbf{U}/d. \qquad \text{(A4)}$$

Equation A4 specifies how the benchmark portfolio and the active portion of the various pure-factor portfolios are combined into the optimal portfolio. Specifically, the first element of vector **A** is, by definition, 1.0, and the other *K* elements are scalars that determine how the various sets of active weights in Equation A3 are optimally added to the benchmark weights. The combination of factor portfolios is determined by mean–variance optimization and allows the investor to choose more exposure to some active factors than to others on the basis of either higher expected information ratios or lower expected factor risk.

To provide some intuition for vector **A**, note that the *K* active-factor active returns can be assumed to be uncorrelated because factor correspondence is already modeled by the correlation of factor exposures. In addition, the *K* active-factor returns can be assumed to be uncorrelated with the benchmark return, except for the low-beta factor, for which the definition of security exposures is specifically based on the correlation of the past security returns with the market return. Under the assumption of uncorrelated active-factor returns, the square covariance matrix **V** is diagonal (i.e., off-diagonal elements are zero), and the optimal factor allocation elements are simply

$$a_k = \frac{IR_k/\sigma_k}{SR_M/\sigma_M}, \qquad \text{(A5)}$$

where IR_k/σ_k is the expected information ratio over active risk of factor *k* and SR_M is the market's Sharpe ratio. Intuitively, the mean–variance-optimal active allocation to each factor is the active mean–variance ratio for that factor over the mean–variance ratio of the market.

Taken together, the *pure-factor* portfolios in matrix $\mathbf{w}_F$ of Equation A3 and the *optimal* factor exposures in vector **A** of Equation A4, allow for a closed-form calculation of optimal security weights for large-set portfolios. The individual security weights, subscripted by *i*, are written without matrix notation as

$$w_{P,i} = w_{M,i} + a_1 \Delta w_{1,i} + a_2 \Delta w_{2,i} + \ldots + a_K \Delta w_{K,i}, \qquad \text{(A6)}$$

where the a_k parameters are constants across securities. Equation A6 for any given security starts with the benchmark weight, to which pure active-factor portfolio weights are added after being multiplied by the optimal allocation to each factor.

Controlled Active Risk

The separation of optimal factor allocations (Equation A4) from pure-factor portfolio construction (Equation A3) allows for a *constrained* active-risk portfolio solution. Consider the simple case of one nonmarket factor with an information ratio of 0.20, based on an expected active return of 1.0% divided by active risk of 5.0%. Assume that the market portfolio's Sharpe ratio is 0.4, based on an expected excess (of the risk-free rate) return of 6.0% over market risk of 15.0%. According to the Treynor–Black (1973) rule, $SR_P = (SR_M^2 + IR^2)^{1/2}$, the Sharpe ratio of the optimal portfolio is $(0.40^2 + 0.20^2)^{1/2} = 0.45$, and the optimal factor exposure in Equation A5 is (0.20/0.05)/(0.40/0.15) = 1.50. This relatively high level of exposure is based on the investor taking the *optimal* amount of active risk, defined in Clarke et al. (2016) as

$$\sigma_A = \frac{IR}{SR_M} \beta_P \, \sigma_M. \qquad \text{(A7)}$$

The optimal level of active portfolio risk from Equation A7 associated with a 1.50 exposure to the nonmarket factor is equal to (0.20/0.40) × 1.00 × 0.15 = 7.50%, too high for many investors. Note that the *ex ante* portfolio beta in this example is 1.0 if the single nonmarket factor is not the low-beta factor.

Investors can back off from the optimal level of factor exposure—and thus optimal active risk—but at the sacrifice of the managed portfolio's Sharpe ratio. Suppose the investor specifies the active portfolio risk, defined by

$$\sigma_A = \left(\sigma_P^2 - \beta_P^2 \, \sigma_M^2\right)^{1/2}, \qquad \text{(A8)}$$

to be 3.0%. The ratio of the investor-specified 3.0% active risk to the 7.5% optimal level of active risk, 3.0/7.5 = 0.4, times the optimal factor exposure of 1.50 translates into a more modest factor exposure of 0.40 × 1.50 = 0.60. This level of exposure allows for a portfolio Sharpe ratio of only 0.43, not the unconstrained value of 0.45, because the exposure to the factor is not optimally balanced with the market factor.

Returning to *K* (more than one) nonmarket factors, the definition of total portfolio risk from matrix algebra is $\sigma_P = \sqrt{\mathbf{U}' \, \mathbf{V}^{-1} \mathbf{U}}/d$, and the relationship of the portfolio beta to the benchmark is defined by $\beta_P = SR_M / d\sigma_M$. Plugging these results into the definition of active risk in Equation A8 gives

$$\sigma_A = \frac{\sqrt{\mathbf{U}' \, \mathbf{V}^{-1} \mathbf{U} - SR_M^2}}{d}. \qquad \text{(A9)}$$

Under the maintained assumption that the *k* = 1 to *K* active-factor returns are uncorrelated with the market factor, we have $d = SR_M / \sigma_M$, so the portfolio beta is 1.0. The portfolio active risk in Equation A9 under this assumption is $\sigma_A = \sigma_M IR_P / SR_M$, where the *portfolio* or combined information ratio is

$$IR_P = \sqrt{\sum_{k=1}^{K} IR_k^2}. \qquad \text{(A10)}$$

To set the active risk of the portfolio to a user-specified level, σ_C, we divide each of the optimal factor allocations, a_k, by the scalar value in Equation A10 and insert the user-specified active-risk parameter to derive the nonmatrix algebra solution:

$$w_{P,i} = w_{M,i} + \frac{\sigma_C}{IR_P} \left(\frac{IR_1}{\sigma_1} \Delta w_{1,i} + \frac{IR_2}{\sigma_2} \Delta w_{2,i} + \ldots + \frac{IR_k}{\sigma_k} \Delta w_{K,i} \right), \qquad \text{(A11)}$$

which is equivalent to Equation 1 in the body of this article for two nonmarket factors.

Low-Beta Factor Allocation

For the low-beta factor, the assumed structure of the factor covariance matrix, **V**, is nondiagonal because one active factor has an expected nonzero correlation parameter of ρ_β with the market factor. Note that the ρ_β parameter will typically be negative because it describes the correlation of the returns with the active (market differential) low-beta (below 1.0) factor with the market return. Solving the matrix algebra for the inverse of the factor–return covariance matrix shows that the optimal allocation to the low-beta factor, in balance with other active factors, should correspond to a modified information ratio, $IR_\beta = \alpha_\beta/\sigma_\beta$, in Equation A11 as opposed to the generic IR for factor *k* of $IR_k = \alpha_k/\sigma_k$. Going through the matrix algebra derivations for the optimal multifactor portfolio, where factor covariance matrix **V** has a single nonzero off-diagonal parameter, gives the modified expected market differential return as

$$\mu_\beta = \mu_k - \rho_\beta \frac{\sigma_k}{\sigma_M}\mu_M \tag{A12}$$

and the modified *ex ante* risk as

$$\sigma_\beta = \sigma_k\sqrt{1-\rho_\beta^2}, \tag{A13}$$

where μ_k and σ_k are the raw low-beta factor parameters. Note that with $\rho_\beta = 0$, Equations A12 and A13 revert back to the unadjusted, or generic, factor equations.

Equation A12 shows how an allocation to the low-beta factor can add value, even if the actual expected active return is zero. According to the classic CAPM, the expected active return to the low-beta factor should be $\mu_k = \rho_\beta(\sigma_k/\sigma_M)\,\mu_M$. As a numerical example, suppose that ρ_β is −0.6, the active-factor risk is 5%, and the market risk is 15%. These data translate into a total beta for the low-beta factor portfolio of 1 − 0.6 × (5/15) = 0.8, which is close to the long-term value seen empirically over time. In other words, if the expected excess market return is 6.0%, the active return to the low-beta factor according to the classic CAPM should be −0.6 × (5/15) × 6.0 = −1.2%. In contrast, the observed active return to low-beta factor portfolios over time has been slightly positive.

Pure-Factor Portfolios

Fama–MacBeth (1973) cross-sectional regression math defines the pure-factor portfolio weights in Equation A3. For the market plus one active factor, designated Factor 1, the weighted cross-sectional regression is univariate and the factor portfolio weight for each security is $w_{k,i} = w_{M,k}(1 + b_{k,i})$. The factor portfolio weights start with the benchmark weight and then proportionally scale up or down (rather than arithmetically deviating up or down) depending on the exposure of each security to the factor of interest. The *active* weight on the security, $\Delta w_{k,i} = w_{k,i} - w_{M,i}$, is a simple multiplicative product of the benchmark weight and the standardized factor exposure of the security, as given in Equation 2 of the body of the article. For two nonmarket factors, Factor 1 and Factor 2, the individual securities' active weights, derived from the math that underlies a multivariate Fama–MacBeth cross-sectional regression, are, respectively,

$$\Delta w_{1,i} = w_{M,i}\left[b_{1,i}\left(\frac{1}{1-\rho^2}\right) - b_{2,i}\left(\frac{\rho}{1-\rho^2}\right)\right] \tag{A14}$$

and

$$\Delta w_{2,i} = w_{M,i}\left[b_{2,i}\left(\frac{1}{1-\rho^2}\right) - b_{1,i}\left(\frac{\rho}{1-\rho^2}\right)\right], \tag{A15}$$

where $\rho = \sum_{i=1}^{N} w_{M,i}\, b_{1,i}\, b_{2,i}$ is used in Equation 4 in the body of this article.

More complex expressions can be derived for three or more factors, but the investor would typically simply calculate active-factor weights from the matrix algebra in Equation A3 that describes the cap-weighted Fama–MacBeth (1973) regression. In multifactor or combined portfolios, some active-factor weights for any given security are negative and some are positive and thus cancel each other out. In addition, given typical values for a user-specified active-risk parameter, σ_C, few securities have enough negative active weight on each of the factors to lead to a negative total weight or short selling. Thus, although the optimal multifactor portfolio solution in Equation A11 can in theory lead to short selling of some securities at given times, the formal imposition of a no-shorting constraint has little impact on portfolio construction or realized returns.

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Notes

1. "Most quantitative mutual funds are failing to beat their benchmarks as US stocks are on track for their best annual performance since 2013, according to research from Bank of America Corp. Large-cap quant funds lagged the Russell 1000 index by an average 3.2 percentage points this year through November" (Idzelis 2019).

2. The Novy-Marx (2013) definition of profitability is revenue minus cost of goods sold, all divided by total assets—(REVT – COGS)/AT—using Compustat account codes. For financial sector stocks—for example, commercial banks—we subtracted customer deposits from total assets, so our definition of profitability is (REVT – COGS)/(AT – DPTC). Deposits, which are technically a liability rather than an asset of a bank, represent a source of capital that is neither equity nor long-term debt.

3. Earning positive returns from the volatility premium requires positions in the derivatives market—for example, delta-neutral writing of at-the-money index put and call options—together with a long position in the market portfolio or, recently, derivative contracts known as "volatility swaps."

4. We use the phrase "anchoring to the benchmark" to describe the impact of using scores with cross-sectional cap-weighted means of zero and variances of 1.0, as described in Clarke et al. (2016), in contrast to arithmetic mean-zero unit-variance cross-sectional scores. The benchmark-anchored pure factor returns used in this study are available at www.deepcreekm.com.

5. Daily factor returns in this study were calculated from beginning-of-month factor exposures and the total daily return to each security in CRSP. In other words, the factor portfolios are not rebalanced during the month, although such rebalancing does not materially change the results.

6. The Bollerslev (1986) GARCH(1, 1) model estimates for the daily pure-active-factor returns over the entire 54-year sample are similar to market returns, with highly significant ARCH and GARCH terms. Specifically, the ARCH parameter estimate for the excess market return was 0.088 (*t*-statistic of 39.0), and the GARCH parameter estimate was 0.898 (*t*-statistic of 283.8). The ARCH and GARCH estimates for the active factor returns are as follows:

Factor	ARCH	GARCH
Value	0.084	0.902
Momentum	0.098	0.884
Small size	0.104	0.879
Low beta	0.095	0.887
Profitability	0.086	0.898

Similar to the market return, the active returns to all five factors have large and significant "theta" coefficients in the EGARCH ("E" for exponential) specification of Nelson and Cao (1992), indicating that negative returns have a larger impact on subsequent volatility than do positive returns.

7. Daily returns tend to be positively autocorrelated, so sample variance is less than it would be for more independent observations, such as monthly returns. This issue was first noted by Lo and MacKinlay (1988) in their "volatility ratio" tests.

8. The transfer coefficient, introduced in Clarke, de Silva, and Thorley (2002), has various interpretations, including the cross-sectional correlation coefficient between security positions that are constrained to be long only and those positions in an unconstrained optimization. In this context, the TC is defined by the quotient of the long-only portfolio IR and the optimal IR.

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