# Endogenous Network Valuation Adjustment and the Systemic Term Structure in a Dynamic Interbank Model

Zachary Feinstein* Andreas Søjmark†

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**Abstract**

In this work we introduce an interbank network with stochastic dynamics in order to study the yield curve of bank debt under an endogenous network valuation adjustment. This entails a forward-backward approach in which the future probability of default is required to determine the present value of debt. As a consequence, the systemic model presented herein provides the network valuation adjustment to the term structure for free without additional steps required. We present this problem in two parts: (i) a single maturity setting that closely matches the traditional interbank network literature and (ii) a multiple maturity setting to consider the full term structure. Numerical case studies are presented throughout to demonstrate the financial implications of this systemic risk model.

**Keywords:** systemic risk, default contagion, dynamic network model, valuation adjustment, yield curve

## 1 Introduction

Systemic risk can cause outsized losses within the financial system due to feedback mechanisms such as direct default contagion, or, as we study in this paper, expectations of future defaults. The severity of this was evidenced by the excessive losses during the 2008 financial crisis. In

*Stevens Institute of Technology, School of Business, Hoboken, NJ 07030, USA. zfeinste@stevens.edu. This material is based upon work supported by the OeNB anniversary fund, project number 17793. Part of the manuscript was written when the author visited Vienna University of Economics and Business.

†London School of Economics, Department of Statistics, London, WC2A 2AE, UK. a.sojmark@lse.ac.uk. The author gratefully acknowledges the hospitality of the Institute for Mathematical and Statistical Innovation at the University of Chicago, where he was a visitor while working on parts of this project.

this work we present a model for debt interlinkages in a dynamic financial system that allows us to study such contagious events, arising from the appropriate (equilibrium) pricing of the interbank obligations.

The network of interbank obligations, and the resulting default contagion, has been studied in the seminal works of Eisenberg and Noe [2001], Rogers and Veraart [2013], Gai and Kapadia [2010] in deterministic one-period systems. These static systems have been extended in a number of directions and remains an active area of research; we refer the interested reader to Glasserman and Young [2016], Weber and Weske [2017] for surveys of this literature. More recent extensions have been presented in, e.g., Paddrik et al. [2020] which studies the potential for contagion through margin requirements, Klages-Mundt and Minca [2020] which analyzes (re)insurance networks, and Ghamami et al. [2022] which considers collateralization in interbank networks.

Within this work, we base much of our analysis on a system with fixed recovery of liabilities, i.e., in which institutions either make all debt payments in full or a fixed fraction of liabilities are returned. We extend this modeling framework to allow for system dynamics between the origination date and maturities of the obligations. Dynamic default contagion extensions of the works of Eisenberg and Noe [2001], Rogers and Veraart [2013], Gai and Kapadia [2010] have previously been studied in Capponi and Chen [2015], Ferrara et al. [2016], Kusnetsov and Veraart [2019] in discrete-time settings and Banerjee et al. [2022], Sonin and Sonin [2020], Feinstein and Søjmark [2021] in continuous-time frameworks. All of these cited works, however, use (implicitly) a historical price accounting rule for marking interbank assets. That is, all interbank assets are either marked as if the payment will be made in full (prior to maturity or a default) or marked based on the realized payment (after debt maturity or the default event). This is in contrast to the mark-to-market accounting used for tradable assets where the value of interbank assets would depend on the expected payments (prior to maturity or a default).

Consequently, we want to study a dynamic default contagion model with forward-looking probabilities of default so as to mark interbank assets to a market within the balance sheet of all firms. This construction is a network valuation adjustment to the single-firm Black–Cox model (Black and Cox [1976]) for pricing obligations in a network. We thus extend the prior literature on network valuation adjustments (e.g., Cossin and Schellhorn [2007], Fischer [2014], Barucca et al. [2020], Banerjee and Feinstein [2022]) which consider obligations in systems with a single maturity and without the possibility for early declarations of default (i.e., default can only be declared at the maturity of the obligations). Such static systems are comparable to the single firm model of Merton [1974] for pricing debt and equity. Herein we relax this assumption to allow for, e.g., safety covenants as in Black and Cox [1976], Leland [1994] to permit for early declarations of bankruptcy.

As already emphasized, we introduce a dynamic network model with either a single or multiple maturity dates. In constructing and analyzing this systemic risk model, we make the following four primary innovations and contributions.

(i) First, we construct a dynamic network model with a forward-looking marking of interbank assets based on their (risk-neutral) probabilities of default. This mark-to-market accounting of interbank assets is in contrast to prior works on dynamic networks which (implicitly or explicitly) consider historical price accounting, i.e., all debts are assumed to be paid in full until a default is realized (at which time a downward jump in equity occurs). Notably, by accounting for the possibility of future defaults, contagion can occur without a default having occurred but merely that one may happen in the future. With abuse of terminology, we find that default contagion can occur without any realized default.

(ii) Second, using these equilibrium default probabilities, we are able to construct a term structure for debts that includes the possibility of contagious events. As such, the proposed modeling framework naturally constructs a network valuation adjustment for the entire yield curve for every bank within the financial system.

(iii) Third, we study the mathematical properties of the clearing solutions of this dynamic network. Notably, though the system is presented as a fixed point problem at all times simultaneously, we present an equivalent formulation satisfying the dynamic programming principle which, in particular, allows us to prove that the (maximal) equilibrium solution in an extended state space satisfies the Markov property.

(iv) Fourth, as far as the authors are aware, this is the first dynamic interbank network model that directly considers the manner in which banks may rebalance their portfolio over time as asset prices fluctuate and defaults are realized. Specifically, we present simple rebalancing strategies as well as an optimal strategy that incorporates regulatory constraints.

Throughout this work, we demonstrate the financial implications of this model with numerical case studies. In particular, we wish to highlight the nonlinear and non-monotonic behavior of the probability of default with respect to the correlations between bank assets (see Section 3.5.2 and Figure 3). Furthermore, holding everything else fixed, we find that the shape of the yield curve for one firm can depend on the riskiness of other banks in the system; specifically, if the volatility of core institutions grow, the yield curve for all institutions can transform from a normal to inverted shape (see Section 4.4.3 and Figure 6).

The remainder of this paper is as follows. In Section 2, background information is provided. Specifically, in Section 2.1, the static version of our interbank system is presented so as to illuminate notation and the general structure considered for the dynamic models presented later. The tree-based probability spaces used throughout this work are presented in Section 2.2. In Section 3, a dynamic system is constructed in which all obligations (interbank and external) are due at the same *future* maturity. In constructing this system, the value of interbank assets is weighted by the (risk-neutral) probability of the payment being made in full, i.e., the risk-neutral expectation of the asset value at maturity. Existence of maximal and minimal clearing solutions is presented along with properties of these extremal solutions. In Section 4, this system is extended to permit multiple maturity dates for obligations. In doing so, defaults can occur due to either *insolvency* or *illiquidity*. As with the single maturity setting, the existence of clearing solutions and their properties are presented. Within this construction, the composition of assets and the manner in which banks rebalance their portfolios are directly considered. Finally, in Section 5, we provide some simple policy implications of our model as well as directions for future research.

# 2 Setting

## 2.1 Interbank networks

Within this work we will consider a *dynamic* financial system akin to the static model of Gai and Kapadia [2010] for default contagion. To briefly provide the financial context and some basic notation, we will summarize the static setting which we will extend in the subsequent sections.

Consider a financial system comprised of $n$ banks or other financial institutions labeled $1, 2, ..., n$. The balance sheet of each bank is made up of both interbank and external assets and liabilities. Specifically, on the asset side of the balance sheet, bank $i$ holds external assets $x_i \geq 0$ and interbank assets $L_{ji} \geq 0$ for each potential counterparty $j = 1, 2, ..., n$ (with $L_{ii} = 0$ so as to avoid self-dealing). On the side of the balance sheet, bank $i$ has liabilities $\bar{p}_i := \sum_{j=1}^n L_{ij} + L_{i0}$ with external liabilities $L_{i0} \geq 0$. We will often denote $\mathbf{x} := (x_1, x_2, ..., x_n)^\top \in \mathbb{R}^n_+$, $\mathbf{L} := (L_{ij})_{i,j=1,2,...,n} \in \mathbb{R}^{n\times n}_+$, $\mathbf{L}_0 := (L_{ij})_{i=1,2,...,n;\; j=0,1,...,n} \in \mathbb{R}^{n\times(n+1)}_+$, and $\bar{\mathbf{p}} := \mathbf{L}_0 \mathbf{1} \in \mathbb{R}^n_+$.

With these assets and liabilities, we can consider the default contagion problem. Consider $P_i \in \{0, 1\}$ to be the indicator of whether bank $i$ is solvent ($P_i = 1$) or in default on its obligations ($P_i = 0$). Following a notion of recovery of liabilities, if a bank is in default then it will repay a fraction $\beta \in [0, 1]$ of its obligations; the Rogers–Veraart model (Rogers and Veraart [2013]) is comparable but with recovery of assets instead. Mathematically, $\mathbf{P} \in \{0,1\}^n$ solves the fixed point problem

$$\mathbf{P} = \psi(\mathbf{P}) := \mathbb{1}_{\{\mathbf{x}+\mathbf{L}^\top[\mathbf{P}+\beta(1-\mathbf{P})]\geq\bar{\mathbf{p}}\}} = \mathbb{1}_{\{\mathbf{x}+\mathbf{L}^\top[\beta+(1-\beta)\mathbf{P}]\geq\bar{\mathbf{p}}\}}. \tag{1}$$

**Proposition 2.1.** *The set of clearing solutions to* (1)*, i.e.,* $\{\mathbf{P}^* \in \{0,1\}^n \mid \mathbf{P}^* = \psi(\mathbf{P}^*)\}$*, forms a lattice in* $\{0,1\}^n$ *with greatest and least solutions* $\mathbf{P}^\uparrow \geq \mathbf{P}^\downarrow$*.*

*Proof.* This follows from a direct application of Tarski's fixed point theorem. □

To conclude this discussion of the static system, let $\mathbf{P}^*$ be an arbitrary clearing solution of (1). The resulting net worths $\mathbf{K} = \mathbf{x}+\mathbf{L}^\top[\mathbf{P}^*+\beta(1-\mathbf{P}^*)]-\bar{\mathbf{p}}$ provide the difference between realized assets and liabilities. The cash account $\mathbf{V} = \mathbf{K}^+$ provides the assets-on-hand for each institution immediately after liabilities are paid; notably the cash account equals the net worths if, and only if, the bank is solvent, otherwise it is zero.

**Remark 2.2.** As mentioned above, throughout this work we consider the case of recovery of liabilities. This notion corresponds to the "recovery of face value" accounting rule in the corporate bond literature (see, e.g., Guo et al. [2008], Hilscher et al. [2021]). The recovery of assets case, as in Rogers and Veraart [2013], could be considered instead. We restrict ourselves to the recovery of liabilities because this formulation has been shown to "provide a better approximation to realized recovery rates" (Hilscher et al. [2021]) than the recovery of assets formulation.

## 2.2 Tree model

For the remainder of this work, we will consider a *stochastic* financial system. For mathematical simplicity, we will focus entirely on tree models for the randomness in the system. That is, throughout this work, we consider a *finite* filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t_l})_{l=0}^{\ell}, \mathbb{P})$ with times $0 =: t_0 < t_1 < ... < t_\ell := T$, $\mathcal{F}_T = \mathcal{F} = 2^\Omega$, and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Following the notation from Feinstein and Rudloff [2017], we will define $\Omega_t$ to be the set of atoms of $\mathcal{F}_t$. For any $\omega_t \in \Omega_t$ ($t < T$), we denote the successor nodes by

$$\mathcal{S}(\omega_{t_l}) = \{\omega_{t_{l+1}} \in \Omega_{t_{l+1}} \mid \omega_{t_{l+1}} \subseteq \omega_{t_l}\}.$$

To simplify notation, let $\mathcal{L}_t := L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}) = \mathbb{R}^{|\Omega_t|}$ denote the space of $\mathcal{F}_t$-measurable random variables. We use the convention that for an $\mathcal{F}_t$-measurable random variable $x \in \mathcal{L}_t$, we denote by $x(\omega_t)$ the value of $x$ at node $\omega_t \in \Omega_t$, that is $x(\omega_t) := x(\omega)$ for some $\omega \in \omega_t$ chosen arbitrarily.

**Assumption 2.3.** *Throughout this work we consider an (arbitrage-free) market of $n$ external assets $\mathbf{x} \in \prod_{l=0}^{\ell} \mathcal{L}_t^n$ on the tree. This is made more explicit in Sections 3.1 and 4.1 below. To simplify the discussion, we consider the probability measure $\mathbb{P}$ to provide a pricing measure consistent with $\mathbf{x}$.*

Often, for the case studies, we consider a specific tree model from He [1990] to construct a geometric random walk for a financial system with $n$ banks. This multinomial tree permits us to encode a correlation structure on our processes from $n+1$ branches at each node. Let $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_{l\Delta t}^n)_{l=0}^{T/\Delta t}, \mathbb{P}^n)$ denote the filtered probability space for this multinomial tree with constant time steps $\Delta t > 0$. (For simplicity, we assume throughout that $T$ is divisible by the time step $\Delta t$.) As there are $n+1$ branches at each node within this tree, $|\Omega_t^n| = (n+1)^{t/\Delta t}$ at each time $t = 0, \Delta t, ..., T$ with equal probability $\mathbb{P}^n(\omega_t^n) = (n+1)^{-t/\Delta t}$ for any $\omega_t^n \in \Omega_t^n$. Because of the regularity of this system, we will index the atoms at time $t$ as $\omega_{t,i}^n \in \Omega_t^n$ for $i = 1, 2, ..., (n+1)^{t/\Delta t}$. Similarly, we can encode the successor nodes automatically as $\mathcal{S}(\omega_{t,i}^n) = \{\omega_{t+\Delta t,j}^n \mid j \in (n+1)(i-1) + \{1, ..., n+1\}\}$ for any time $t$ and atom $i$.

Within the tree $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_{l\Delta t}^n)_{l=0}^{T/\Delta t}, \mathbb{P}^n)$, we are interested in a discrete analog $\mathbf{x} := (\mathbf{x}(0), \mathbf{x}(\Delta t), ..., \mathbf{x}(T)) \in \prod_{l=0}^{T/\Delta t} \mathcal{L}_{l\Delta t}^n$ of the vector-valued geometric Brownian motion (such that $\mathbf{x}(t) = (x_1(t), x_2(t), ..., x_n(t))^\top$). Following the construction in He [1990], let $\sigma = (\sigma_1, ..., \sigma_n) \in \mathbb{R}^{n \times n}$ be a nondegenerate matrix encoding the desired covariance structure $C = \sigma^2$. The $k^{\text{th}}$ element $x_k$ can be defined recursively by

$$x_k(t + \Delta t, \omega_{t+\Delta t,(n+1)(i-1)+j}^n) = x_k(t, \omega_{t,i}^n) \exp\left( (r - \frac{\sigma_{kk}^2}{2})\Delta t + \sigma_k^\top \tilde{\epsilon}_j \sqrt{\Delta t} \right) \tag{2}$$

for some initial point $x_k(0, \Omega^n) \in \mathbb{R}_{++}^n$ and such that $\tilde{\epsilon} = (\tilde{\epsilon}_1, ..., \tilde{\epsilon}_{n+1}) \in \mathbb{R}^{n \times (n+1)}$ is generated from an $(n+1) \times (n+1)$ orthogonal matrix as in He [1990].

# 3 Single maturity setting

To simplify the presentation and to separate out those effects that arise due to dynamic networks (see Section 4 and also Banerjee et al. [2022]) from those arising due to the pricing and endogenous default notions considered in this section, we will begin by assuming that all interbank and external obligations $L_{ij}, L_{i0}$ (for $i, j \in \{1, 2, ..., n\}$) have the same maturity $T$. We start our discussion of this single maturity setting by presenting the basic balance sheet notions in Section 3.1. With this balance sheet construction, we then derive the clearing problem and deduce the existence of (minimal and maximal) equilibria in Section 3.2. Next, we consider two useful reformulations of the resulting mathematical model for the clearing problem: the first relies on Bayes' rule to provide a forward-backward equation (Section 3.3) that is tractable computationally; the second provides a dynamic programming principle formulation (Section 3.4) for the maximal clearing solution which, in particular, provides a Markovian property for this clearing solution. Finally, we conclude the section with case studies in Section 3.5.

### 3.1 Balance sheet construction

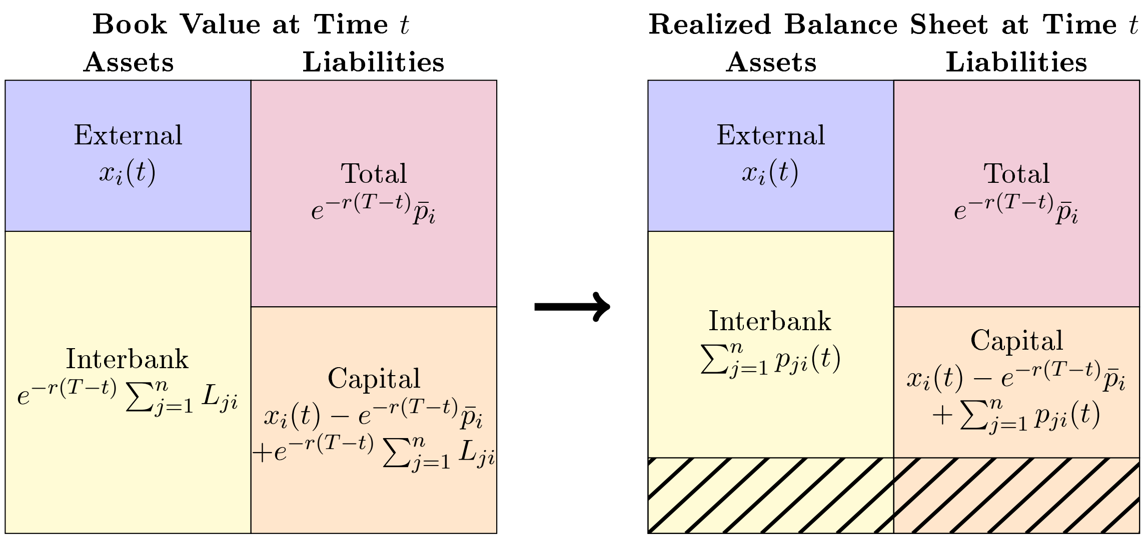



Figure 1: Stylized book and balance sheet for a firm at time $t$ before maturity of interbank claims.

In this work we are focused on the valuing of interbank assets with endogenous defaults in a networked system of $n$ banks in a way that extends the static model presented in Section 2.1. We refer to Figure 1 as a visual depiction of both the book value and realized balance sheet for arbitrary bank $i$ in the financial system. Within this system, we assume a constant risk-free rate $r \geq 0$ used for discounting all obligations.

Consider, first, the banking book for bank $i$ as depicted in Figure 1. The bank holds two types of assets at time $t$: *external assets* $x_i(t) \in \mathcal{L}_t$ and *interbank assets* $\sum_{j=1}^n L_{ji}$ where $L_{ji} \geq 0$ is the total obliged from bank $j$ to $i$ ($L_{ii} = 0$ so as to avoid self-dealing). The bank has liabilities $\bar{p}_i := \sum_{j=1}^n L_{ij} + L_{i0}$ where $L_{i0} \geq 0$ denotes the *external obligations* of bank $i$. The external assets are held in liquid and marketable assets so that their value fluctuates over time and are adapted to the filtration. The book value of the capital, discounted appropriately, is given by

$$x_i(t) + e^{-r(T-t)} \sum_{j=1}^n L_{ji} - e^{-r(T-t)}\bar{p}_i.$$

However, depending on the probability of default (see below for more details), the interbank assets will not be valued at their face value. That is, the obligation from bank $j$ to $i$ is valued as $p_{ji}(t) := e^{-r(T-t)} L_{ji}(\beta + (1-\beta)\mathbb{E}[P_j(T) \mid \mathcal{F}_t]) \in \mathcal{L}_t$ that takes value (almost surely) in the interval $e^{-r(T-t)} L_{ji} \times [\beta, 1]$ where $P_j(T, \omega)$ is the realized indicator of solvency for bank $j$ at the maturity time $T$ and $\beta \in [0,1]$ is the recovery rate. Defining $P_j(t) := \mathbb{E}[P_j(T) \mid \mathcal{F}_t] \in \mathcal{L}_t$ as the (conditional) probability of solvency at maturity as measured at time $t$, we can write the discounted payments as $p_{ji}(t) = e^{-r(T-t)} L_{ji}(\beta + (1-\beta)P_j(t))$. Thus, the realized balance sheet for bank $i$ has, possible, write-downs in the value of assets and, therefore, the realized capital at time $t$ is given by

$$K_i(t) = x_i(t) + e^{-r(T-t)} \sum_{j=1}^{n} L_{ji}(\beta + (1-\beta)P_j(t)) - e^{-r(T-t)} \bar{p}_i.$$

The default determination of banks is fundamental for determining the value of interbank assets. In this work, as in the static setting, we will assume that bank $i$ will default on their obligations at the first time their realized capital drops below 0, i.e., at the stopping time $\tau_i$ given by

$$\tau_i := \inf\{t \geq 0 \mid K_i(t) < 0\}. \tag{3}$$

In the context of endogenous defaults, the shareholders of bank $i$ choose to default at time $t$ when expected bank assets are worth less than the liabilities; by declaring bankruptcy, these shareholders are able to increase their risk-neutral utility as zero capital is preferred to negative expected return. Alternatively, these defaults can be triggered by a safety covenant as in Black and Cox [1976], Leland [1994]. This default condition was studied in Feinstein and Søjmark [2021] in a different dynamic network context. By convention, and without loss of generality, we will assume $\tau_i(\omega) = T + 1$ if bank $i$ does not default on $\omega \in \Omega$.

With banks defaulting when their capital drops below 0, this network valuation problem can be viewed as a fixed point problem in pricing digital (down-and-out) barrier options. That is, $P_i(t)$ is the value of the digital barrier option with maturity $T$ with a payoff of \$1 if the barrier that the capital level $K_i$ never drops below 0 and payoff of \$0 otherwise. This is a fixed point due to the dependence of the capital $K_i$ on the value of the barrier options $P_j$ for banks $j \neq i$. As such, though we do not compute the valuations as digital barrier options, we view this system as being inextricably linked to questions in derivatives pricing.

**Remark 3.1.** (i) The default rule considered herein implicitly studies the liquidity problem as well because bank $i$ has sufficient assets to cover its liabilities at the maturity $T$ if and only if the realized capital at $T$ is nonnegative. If only the liquidity question is desired, then the default time can be reformulated such that $\tau_i = T + \mathbb{1}_{\{K_i(T)\geq 0\}}$, i.e., bank $i$ is in default if and only if it has insufficient capital at maturity to cover its obligations. This illiquidity default rule is akin to the financial setting of Banerjee and Feinstein [2022].

(ii) The condition that a bank will default when its net worth is negative is similarly considered in Banerjee et al. [2022], Feinstein and Søjmark [2021]. However, fundamentally different from those works (which consider a backward-looking historical price accounting rule), herein we consider a forward-looking accounting mark-to-market accounting rule which values interbank assets based on the *future* probability of default.

**Remark 3.2.** Throughout this work we assume that the interbank network is fixed even as the value of these obligations can fluctuate. As we assume all valuations are taken w.r.t. the risk-neutral measure $\mathbb{P}$, and under the assumption that banks are risk-neutral themselves, in equilibrium the banks have no (expected) gains by altering the network structure by buying or selling interbank obligations. Moreover, in reality, these interbank markets may not be liquid; therefore, transacting to buy or sell interbank debt could be accompanied by transaction costs and price slippage which discourage any such modifications to the network.

## 3.2 Mathematical model

As highlighted within the balance sheet modeling, we can immediately define an equilibrium model for default contagion that is jointly on the net worths ($\mathbf{K} = (K_1, ..., K_n)^\top$), survival probabilities ($\mathbf{P} = (P_1, ..., P_n)^\top$), and default times ($\boldsymbol{\tau} = (\tau_1, ..., \tau_n)^\top$). We can, thus, define the domain for this default contagion model as the complete lattice

$$\mathbb{D}^T := \left\{ (\mathbf{K}, \mathbf{P}, \boldsymbol{\tau}) \in \left( \prod_{l=0}^{\ell} \mathcal{L}_{t_l}^n \right)^2 \times \{t_0, t_1, ..., t_\ell, T+1\}^{|\Omega| \times n} \;\middle|\; \begin{array}{l} \forall t = t_0, t_1, ..., t_\ell : \\ \mathbf{K}(t) \in \mathbf{x}(t) + e^{-r(T-t)} \left( [\mathbf{0} \;,\; \mathbf{L}^\top \mathbf{1}] - \bar{\mathbf{p}} \right) \\ \mathbf{P}(t) \in [\mathbf{0} \;,\; \mathbf{1}] \end{array} \right\}.$$

(We wish to recall for the reader that throughout this work we focus entirely on the tree model structure provided within Section 2.2, which is, in particular, a finite probability space.) This clearing system $\Psi^T : \mathbb{D}^T \to \mathbb{D}^T$ is mathematically constructed in (4):

$$(\mathbf{K}, \mathbf{P}, \boldsymbol{\tau}) = \Psi^T(\mathbf{K}, \mathbf{P}, \boldsymbol{\tau}) := \left( \Psi^T_{\mathbf{K}}(t_l, \mathbf{P}(t_l)) \;,\; \Psi^T_{\mathbf{P}}(t_l, \boldsymbol{\tau}) \;,\; \Psi^T_{\boldsymbol{\tau}}(\mathbf{K}) \right)_{l=0}^{\ell} \tag{4}$$

$$\begin{cases} \Psi^T_{\mathbf{K},i}(t,\tilde{\mathbf{P}}) = x_i(t) + e^{-r(T-t)}\sum_{j=1}^n L_{ji}(\beta + (1-\beta)\tilde{P}_j) - e^{-r(T-t)}\bar{p}_i \\ \Psi^T_{\mathbf{P},i}(t,\boldsymbol{\tau}) = \mathbb{P}(\tau_i > T \mid \mathcal{F}_t) & \forall i = 1,...,n. \\ \Psi^T_{\boldsymbol{\tau},i}(\mathbf{K}) = \inf\{t \geq 0 \mid K_i(t) < 0\} \end{cases}$$

**Remark 3.3.** The clearing problem $(\mathbf{K},\mathbf{P},\boldsymbol{\tau}) = \Psi^T(\mathbf{K},\mathbf{P},\boldsymbol{\tau})$ can also be viewed as a discrete-time McKean–Vlasov problem for the process $\mathbf{K}$ and the conditional law of its first hitting time of zero, thus emphasizing a link to works on continuous-time McKean–Vlasov problems involving hitting times (Hambly et al. [2019], Nadtochiy and Shkolnikov [2019]). However, the situation here is very different, as we are now concerned with the law of the hitting time at the maturity $T$ conditional on the information available at time $t$, rather than just the law at time $t$. Whether in discrete- or continuous-time, we are not aware of any works in the literature on McKean–Vlasov problems encompassing problems of this type, but it turns out that the discrete-time setting and monotonicity yields an easy way of obtaining a solution. A more general treatment of this class of problems could pose interesting challenges for future research.

**Theorem 3.4.** *The set of clearing solutions to* (4)*, i.e.,* $\{(\mathbf{K}^*,\mathbf{P}^*,\boldsymbol{\tau}^*) \in \mathbb{D}^T \mid (\mathbf{K}^*,\mathbf{P}^*,\boldsymbol{\tau}^*) = \Psi^T(\mathbf{K}^*,\mathbf{P}^*,\boldsymbol{\tau}^*)\}$*, forms a lattice in* $\mathbb{D}^T$ *with greatest and least solutions* $(\mathbf{K}^\uparrow,\mathbf{P}^\uparrow,\boldsymbol{\tau}^\uparrow) \geq (\mathbf{K}^\downarrow,\mathbf{P}^\downarrow,\boldsymbol{\tau}^\downarrow)$.

*Proof.* As with Proposition 2.1, this result follows from a direct application of Tarski's fixed point theorem since $\Psi^T$ is monotonic in the complete lattice $\mathbb{D}^T$. □

**Remark 3.5.** Within Theorem 3.4, we only prove the existence of a clearing solution. Generically, there may not be a unique solution. This can clearly be seen by noting that the static model needs to be satisfied by $(\mathbf{K}(T), \mathbb{1}_{\{\mathbf{K}(T)\geq \mathbf{0}\}})$ at maturity time $T$ (in the network formed by banks solvent through $t_{\ell-1}$). Therefore, just by using this static model, we can find that the dynamic clearing system (4) need not have a unique solution.

**Remark 3.6.** Consider a clearing solution for the tree model $(\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_{l\Delta t})_{l=0}^{T/\Delta t}, \mathbb{P}^n)$ of He [1990]. Then one can show that, for each bank $i = 1,\dots,n$, there is a predictable process $\boldsymbol{\theta}_i$ such that $\Delta P_i(t) = \boldsymbol{\theta}_i(t)^\top \Delta\tilde{\mathbf{x}}(t)$, for every time-step $[t, t+\Delta t]$, where $\tilde{\mathbf{x}}(t) := e^{-rt}\mathbf{x}(t)$ is the vector of discounted external asset values. This highlights how the impact of contagion adjusts 'continuously' to the movements of the external assets. It also highlights a stark contrast to earlier models based, either explicitly or implicitly, on historical price accounting whereby the value of interbank debt only updates at an actualized default due solely to the realized losses (c.f. Feinstein and Søjmark [2021] and references therein). In the notation of martingale transforms, we have $P_i(t) = (\theta \bullet \tilde{\mathbf{x}})(t)$, also known as a predictable representation of $P_i$. Using the martingale property of $P_i$ and exploiting the particular tree structure with $n+1$ branches for every node and a non-degenerate covariance matrix for the $n$ external assets, this predictable representation can be deduced from a system of $n$ linear equations in $n$ unknowns. Analogously, a predictable representation can be given for the discounted net worths $e^{-rt}\mathbf{K}(t)$, noting that these discounted processes are martingales by Assumption 2.3. For a general tree, such representations are much more complicated; we refer the interested reader to, e.g., [Protter, 2005, Section IV.3] and [Ararat and Feinstein, 2021, Section 3].

### 3.3 An explicit forward-backward representation

Within (4), we defined the single maturity clearing problem as a fixed point problem. However, the formulation of the clearing system $\Psi^T$ is seemingly complex to compute due to the need to find the probability of solvency $\Psi^T_{\mathbf{P}}$. Within this section, we will focus on a backwards recursion that can be used to simplify this computation. The basic concept is formulated within (5) below such that the clearing problem is rewritten as $\bar{\Psi}^T : \mathbb{D}^T \to \mathbb{D}^T$. Within Proposition 3.7, we demonstrate that the fixed points of $\Psi^T$ and $\bar{\Psi}^T$ coincide. Let $(\mathbf{K}, \mathbf{P}, \boldsymbol{\tau}) \in \mathbb{D}^T$ then:

$$\bar{\Psi}^T(\mathbf{K}, \mathbf{P}, \boldsymbol{\tau}) := \left(\Psi^T_{\mathbf{K}}(t_l, \mathbf{P}(t_l)) \ , \ \bar{\Psi}^T_{\mathbf{P}}(t_l, \mathbf{P}(t_{[l+1]\wedge \ell}), \boldsymbol{\tau}) \ , \ \Psi^T_{\boldsymbol{\tau}}(\mathbf{K})\right)_{l=0}^{\ell} \tag{5}$$

$$\bar{\Psi}^T_{\mathbf{P},i}(t_l, \tilde{\mathbf{P}}, \boldsymbol{\tau}, \omega_{t_l}) := \begin{cases} \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}})\tilde{P}_i(\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} & \text{if } l < \ell \\ \mathbb{1}_{\{\tau_i(\omega_T) > T\}} & \text{if } l = \ell \end{cases} \quad \forall \omega_{t_l} \in \Omega_{t_l}, \ i = 1, ..., n. \tag{6}$$

**Proposition 3.7.** $(\mathbf{K}, \mathbf{P}, \boldsymbol{\tau}) \in \mathbb{D}^T$ *is a clearing solution of* (4) *if and only if it is a fixed point of* (5)*.*

*Proof.* Let $(\mathbf{K}, \mathbf{P}, \boldsymbol{\tau}) \in \mathbb{D}^T$ be a fixed point of (4). This is a fixed point of (5) if and only if $P_i(t_l, \omega_{t_l}) = \bar{\Psi}^T_{\mathbf{P},i}(t_l, \mathbf{P}(t_{[l+1]\wedge \ell}), \boldsymbol{\tau}, \omega_{t_l})$.

- At $l = \ell$: $P_i(T, \omega_T) = \mathbb{P}(\tau_i > T | \mathcal{F}_T)(\omega_T) = \mathbb{1}_{\{\tau_i(\omega_T) > T\}}$ for every $\omega_T \in \Omega_T$ by construction of $\mathcal{F}_T = \mathcal{F}$.
- At $l < \ell$: Fix $\omega_{t_l} \in \Omega_{t_l}$,

$$\begin{aligned} P_i(t_l, \omega_{t_l}) = \mathbb{P}(\tau_i > T | \mathcal{F}_{t_l})(\omega_{t_l}) = \frac{\mathbb{P}(\tau_i > T, \omega_{t_l})}{\mathbb{P}(\omega_{t_l})} &= \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\tau_i > T, \omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} \\ = \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}})\mathbb{P}(\tau_i > T | \mathcal{F}_{t_{l+1}})(\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} &= \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}})P(t_{l+1}, \omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})}. \end{aligned}$$

Let $(\mathbf{K},\mathbf{P},\boldsymbol{\tau}) \in \mathbb{D}^T$ be a fixed point of (5). This is a fixed point of (4) if and only if $P_i(t_l) = \Psi^T_{\mathbf{P},i}(t_l,\boldsymbol{\tau})$ almost surely very every time $l = 0,1,...,\ell$.

- At $l = \ell$: $P_i(T,\omega_T) = \mathbb{1}_{\{\tau_i(\omega_T)>T\}} = \mathbb{P}(\tau_i > T|\mathcal{F}_T)(\omega_T)$ for every $\omega_T \in \Omega_T$ by construction of $\mathcal{F}_T = \mathcal{F}$.
- At $l < \ell$: Assume $P_i(t_{l+1}) = \Psi^T_{\mathbf{P},i}(t_{l+1},\boldsymbol{\tau})$ at time $t_{l+1}$ almost surely. Fix $\omega_{t_l} \in \Omega_{t_l}$,

$$\begin{aligned}P_i(t_l,\omega_{t_l}) &= \sum_{\omega_{t_{l+1}}\in\mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}})P(t_{l+1},\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} = \sum_{\omega_{t_{l+1}}\in\mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}})\mathbb{P}(\tau_i > T|\mathcal{F}_{t_{l+1}})(\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})}\\ &= \sum_{\omega_{t_{l+1}}\in\mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\tau_i > T,\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} = \frac{\mathbb{P}(\tau_i > T,\omega_{t_l})}{\mathbb{P}(\omega_{t_l})} = \mathbb{P}(\tau_i > T|\mathcal{F}_{t_l})(\omega_{t_l}).\end{aligned}$$

□

In fact, (5) can be viewed as a fixed point in $\mathbf{P} \in \prod_{l=0}^{\ell}[0,1]^{|\Omega_{t_l}|\times n}$ only. This can be done by explicitly defining the dependence of the net worths and default time on the probability of solvency. Specifically, this joint clearing problem in $(\mathbf{K},\mathbf{P},\boldsymbol{\tau})$ can be viewed as a forward-backward equation in $\mathbf{P}$ alone, i.e.,

$$\mathbf{P} = \left[\bar{\Psi}^T_{\mathbf{P}}\left(t_l,\mathbf{P}(t_{[l+1]\wedge\ell}),\Psi^T_{\boldsymbol{\tau}}\left(\Psi^T_{\mathbf{K}}(t_k,\mathbf{P}(t_k))_{k=0}^{\ell}\right)\right)\right]_{l=0}^{\ell}. \tag{7}$$

We refer to this as a forward-backward equation since $\mathbf{P} \mapsto \Psi^T_{\boldsymbol{\tau}}(\Psi^T_{\mathbf{K}}(\cdot,\mathbf{P}))$ is calculated forward-in-time whereas $\bar{\Psi}^T_{\mathbf{P}}$ is computed recursively backward-in-time.

**Corollary 3.8.** $(\mathbf{K},\mathbf{P},\boldsymbol{\tau}) \in \mathbb{D}^T$ *is a clearing solution of* (4) *if and only if* $\mathbf{P}$ *is a fixed point of* (7) *with* $\mathbf{K} = \Psi^T_{\mathbf{K}}(t_l,\mathbf{P}(t_l))_{l=0}^{\ell}$ *and* $\boldsymbol{\tau} = \Psi^T_{\boldsymbol{\tau}}(\Psi^T_{\mathbf{K}}(t_l,\mathbf{P}(t_l))_{l=0}^{\ell})$.

*Proof.* This is an immediate consequence of Proposition 3.7. □

## 3.4 Dynamic programming principle representation

With (5), we provided a recursive formulation for the clearing solutions. Within this section we will find an additional equivalent clearing problem that directly makes use of the dynamic programming principle for the *maximal* clearing solution. This formulation is used in Lemma 3.12 to prove that the maximal clearing solution $(\mathbf{K}^\uparrow,\mathbf{P}^\uparrow,\boldsymbol{\tau}^\uparrow) \in \mathbb{D}^T$ of (4) (and as is proven to exist in Theorem 3.4) is Markovian. Additionally, it is this dynamic programming formulation that allows us to study the multiple maturity setting in Section 4.

**Remark 3.9.** Throughout this section, we focus entirely on the maximal clearing solution. The minimal clearing solution can likewise be considered with only small alterations to the proofs.

To formalize the clearing problem as the dynamic programming principle, we need to introduce the operator FIX which we use to denote the *maximal* fixed point. With this fixed point operator, we can define the following equivalent clearing problem in $(\mathbf{K},\mathbf{P})$ reliant on the propagation of information forward-in-time through the auxiliary variables $\boldsymbol{\iota} \in \{0,1\}^n$. Specifically, consider $\hat{\Psi}^T : \mathbb{I} \to \mathcal{L}_T \times \mathcal{L}_T$ where $\mathbb{I} := \{(t_l, \boldsymbol{\iota}) \mid l \in \{0,1,...,\ell\},\ \boldsymbol{\iota} \in \{0,1\}^{|\Omega_{t_{[l-1]^+}}| \times n}\}$ (with $\hat{\Psi}^T(t,\boldsymbol{\iota}) \in \hat{\mathbb{D}}^T(t) := \{(\mathbf{K}(t),\mathbf{P}(t)) \mid (\mathbf{K},\mathbf{P},T\mathbf{1}) \in \mathbb{D}^T\}$ for any time $t$) constructed as:

$$\hat{\Psi}^T(t_l,\boldsymbol{\iota}) := \begin{pmatrix} \hat{\Psi}^T_{\mathbf{K}}(t_l,\boldsymbol{\iota}) \\ \hat{\Psi}^T_{\mathbf{P}}(t_l,\boldsymbol{\iota}) \end{pmatrix} \tag{8}$$
$$= \begin{cases} \mathrm{FIX}_{(\tilde{\mathbf{K}},\tilde{\mathbf{P}})\in\hat{\mathbb{D}}^T(t_l)} \begin{pmatrix} \Psi^T_{\mathbf{K}}(t_l,\tilde{\mathbf{P}}) \\ \left[\sum_{\omega_{t_{l+1}}\in\mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}})\hat{\Psi}^T_{\mathbf{P}}(t_{l+1},\mathrm{diag}(\boldsymbol{\iota}(\omega_{t_l})\mathbb{1}_{\{\tilde{\mathbf{K}}(\omega_{t_l})\geq\mathbf{0}\}})}{\mathbb{P}(\omega_{t_l})}\right]_{\omega_{t_l}\in\Omega_{t_l}} \end{pmatrix} & \text{if } l<\ell \\ \mathrm{FIX}_{(\tilde{\mathbf{K}},\tilde{\mathbf{P}})\in\hat{\mathbb{D}}^T(T)} \begin{pmatrix} \Psi^T_{\mathbf{K}}(T,\tilde{\mathbf{P}}) \\ \mathrm{diag}(\boldsymbol{\iota})\mathbb{1}_{\{\tilde{\mathbf{K}}\geq\mathbf{0}\}} \end{pmatrix} & \text{if } l=\ell \end{cases}$$

where $\boldsymbol{\iota}$ denotes the prior information on whether a bank has defaulted ($\iota_i = 0$) or not ($\iota_i = 1$) up to time $t_{l-1}$ when being used as an input for time $t_l$.

Before continuing, we wish to remark that $\hat{\Psi}^T(t,\boldsymbol{\iota})$ is well-defined insofar as the maximal fixed point exists for every time $t$ and set of solvent banks $\boldsymbol{\iota}$.

**Proposition 3.10.** *$\hat{\Psi}^T(t,\boldsymbol{\iota})$ is well-defined for every $(t,\boldsymbol{\iota}) \in \mathbb{I}$, i.e., the maximal fixed point exists (and is unique) for any combination of inputs.*

*Proof.* As with Theorem 3.4, this result follows from a trivial application of Tarski's fixed point theorem. □

**Proposition 3.11.** *Let $(\mathbf{K},\mathbf{P})$ be the realized solution from $\hat{\Psi}^T(0,\mathbf{1})$, i.e., $(\mathbf{K}(0,\omega_0),\mathbf{P}(0,\omega_0)) = \hat{\Psi}^T(0,\mathbf{1},\omega_0)$ and $(\mathbf{K}(t_l,\omega_{t_l}),\mathbf{P}(t_l,\omega_{t_l})) = \hat{\Psi}^T(t_l,\mathbb{1}_{\{\inf_{k<l}\mathbf{K}(t_k,\omega_{t_k})<\mathbf{0}\}},\omega_{t_l})$ for every $\omega_{t_l} \in \Omega_{t_l}$ and $l = 1,2,...,\ell$. Then $(\mathbf{K},\mathbf{P},\Psi^T_{\boldsymbol{\tau}}(\mathbf{K}))$ is the* maximal *fixed point to* (4).

*Proof.* Define $(\mathbf{K},\mathbf{P})$ to be the realized solution from $\hat{\Psi}^T(0,\mathbf{1})$. Let $\boldsymbol{\tau} := \Psi^T_{\boldsymbol{\tau}}(\mathbf{K})$ be the associated default times and $\boldsymbol{\iota}(t_l,\omega_{t_l}) := \prod_{k=0}^{l-1}\mathbb{1}_{\{\mathbf{K}(t_k,\omega_{t_l})\geq\mathbf{0}\}}$ be the realized (auxiliary) solvency process at time $t$ and in state $\omega_{t_l} \in \Omega_{t_l}$ (and $\boldsymbol{\iota}(0,\omega_0) = \mathbf{1}$). (We wish to note that $\boldsymbol{\iota}(t_l) \in \mathcal{L}^n_{t_{[l-1]^+}}$ and as such could be indexed by the preceding states $\omega_{t_{[l-1]^+}} \in \Omega_{t_{[l-1]^+}}$ instead; we leave the use of $\omega_{t_l}$ as we find it is clearer notationally.) First, we will show that $(\mathbf{K},\mathbf{P},\boldsymbol{\tau})$ is a clearing solution of (4) via the representation (5), i.e., $(\mathbf{K},\mathbf{P},\boldsymbol{\tau}) = \bar{\Psi}^T(\mathbf{K},\mathbf{P},\boldsymbol{\tau})$. Second, we will show that this solution must be the maximal clearing solution as proven to exist in Theorem 3.4.

(i) By construction of $\hat{\Psi}^T$, $(\mathbf{K},\mathbf{P},\boldsymbol{\tau}) = \bar{\Psi}^T(\mathbf{K},\mathbf{P},\boldsymbol{\tau})$ if and only if $\mathbf{P}(t_l) = \bar{\Psi}^T_{\mathbf{P}}(t_l,\mathbf{P}(t_{[l+1]\wedge\ell}),\boldsymbol{\tau})$ for every time $l \in \{0,1,...,\ell\}$. At maturity, $\mathbf{P}(T) = \bar{\Psi}^T_{\mathbf{P}}(T,\mathbf{P}(T),\boldsymbol{\tau})$ trivially by construction of $\boldsymbol{\tau}$. Consider now $l < \ell$ and assume $\mathbf{P}(t_{l+1}) = \hat{\Psi}^T_{\mathbf{P}}(t_{l+1},\iota(t_{l+1})) = \bar{\Psi}^T_{\mathbf{P}}(t_l,\mathbf{P}(t_{l+1}),\boldsymbol{\tau})$. By construction, $\mathbf{P}(t_l,\omega_{t_l}) = \bar{\Psi}^T_{\mathbf{P}}(t_{l+1},\mathbf{P}(t_{l+1}),\boldsymbol{\tau},\omega_{t_l})$ and the result is proven.

(ii) Now assume there exists some clearing solution $(\mathbf{K}^\dagger,\mathbf{P}^\dagger,\boldsymbol{\tau}^\dagger) \gneq (\mathbf{K},\mathbf{P},\boldsymbol{\tau})$. Then we can rewrite the form of $(\mathbf{K}^\dagger,\mathbf{P}^\dagger)$ as:

$$(\mathbf{K}^\dagger,\mathbf{P}^\dagger) = (\Psi^T_{\mathbf{K}}(t_l,\mathbf{P}^\dagger(t_l)),\bar{\Psi}^T_{\mathbf{P}}(t_l,\mathbf{P}^\dagger(t_{[l+1]\wedge\ell}),\Psi^T_{\tau}(\mathbf{K}^\dagger)))_{l=0}^{\ell}$$

through the use of the clearing formulation $\bar{\Psi}^T$ and explicitly applying $\boldsymbol{\tau}^\dagger = \Psi^T_{\boldsymbol{\tau}}(\mathbf{K}^\dagger)$. Following the logic of the prior section of this proof, it must follow that

$$\mathbf{P}^\dagger(t_l,\omega_{t_l}) = \begin{cases} \operatorname{diag}(\mathbb{1}_{\{\inf_{k<l}\mathbf{K}(t_k,\omega_{t_l})\geq\mathbf{0}\}})\sum_{\omega_{t_{l+1}}\in\mathcal{S}(\omega_{t_l})}\frac{\mathbb{P}(\omega_{t_{l+1}})\mathbf{P}(t_{l+1},\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} & \text{if } l<\ell \\ \mathbb{1}_{\{\inf_{k\in[0,\ell]}\mathbf{K}(t_k,\omega_T)\geq\mathbf{0}\}} & \text{if } l=\ell \end{cases}$$

for every time $t_l$ and state $\omega_{t_l} \in \Omega_{t_l}$. That is, $(\mathbf{K}^\dagger(t_l),\mathbf{P}^\dagger(t_l)) = \hat{\Psi}^T(t_l,\mathbb{1}_{\{\inf_{k<l}\mathbf{K}(t_k,\omega_{t_l})\geq\mathbf{0}\}})_{\omega_{t_l}\in\Omega_{t_l}}$ satisfies all of the fixed point problems within the construction of $\hat{\Psi}^T$ at all times $t_l$. We will complete this proof via backwards induction with, to simplify notation, $\boldsymbol{\iota}(t_l) = \mathbb{1}_{\{\inf_{k<l}\mathbf{K}^\dagger(t_k)\geq\mathbf{0}\}}$. Consider maturity $T$, it must follow that $(\mathbf{K}^\dagger(T),\mathbf{P}^\dagger(T)) \leq \hat{\Psi}^T(T,\boldsymbol{\iota}(T))$ by the definition of the fixed point operator FIX.. Consider some time $t_l < T$ and assume $(\mathbf{K}^\dagger(t_{l+1}),\mathbf{P}^\dagger(t_{l+1})) \leq \hat{\Psi}^T(t_{l+1},\boldsymbol{\iota}(t_{l+1}))$. By the backward recursion used within $\hat{\Psi}^T_{\mathbf{P}}(t_l,\boldsymbol{\iota}(t_l))$, it follows that

$$\begin{aligned}\mathbf{P}^\dagger(t_l,\boldsymbol{\iota}(t_l)) &= \operatorname{diag}(\boldsymbol{\iota}(t_l))\left[\sum_{\omega_{t_{l+1}}\in\mathcal{S}(\omega_{t_l})}\frac{\mathbb{P}(\omega_{t_{l+1}})\mathbf{P}(t_{l+1},\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})}\right]_{\omega_{t_l}\in\Omega_{t_l}} \\ &\leq \operatorname{diag}(\boldsymbol{\iota}(t_l))\left[\sum_{\omega_{t_{l+1}}\in\mathcal{S}(\omega_{t_l})}\frac{\mathbb{P}(\omega_{t_{l+1}})\hat{\Psi}^T_{\mathbf{P}}(t_{l+1},\boldsymbol{\iota}(t_{l+1},\omega_{t_{l+1}}),\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})}\right]_{\omega_{t_l}\in\Omega_{t_l}}.\end{aligned}$$

Further, by the monotonicity of $\Psi^T_{\mathbf{K}}$, it would follow that $(\mathbf{K}^\dagger(t_l),\mathbf{P}^\dagger(t_l)) \leq \hat{\Psi}^T(t_l,\boldsymbol{\iota}(t_l))$. This, together with the trivial monotonicity of $\hat{\Psi}^T$ w.r.t. the solvency indicator $\boldsymbol{\iota}$, forms a contradiction to the original assumption.

□

We conclude this discussion of the single maturity clearing problem by proving that the maximal clearing solution is Markovian on the extended state space $(\mathbf{x}, \mathbf{K}, \mathbf{P}, \boldsymbol{\iota})$. Without the inclusion of the auxiliary variable $\boldsymbol{\iota}$, the solution $(\mathbf{x}, \mathbf{K}, \mathbf{P})$ is clearly not Markovian.

**Lemma 3.12.** *Consider Markovian external assets $\mathbf{x}(t_l) = f(\mathbf{x}(t_{l-1}), \tilde{\epsilon}(t_l))$ for independent perturbations $\tilde{\epsilon}$. Let $(\mathbf{K}, \mathbf{P})$ be the realized solution from $\hat{\Psi}^T(0, \mathbf{1})$ with $\boldsymbol{\iota}(t_l) := \prod_{k=0}^{l-1} \mathbb{1}_{\{\mathbf{K}(t_k) \geq \mathbf{0}\}}$ denoting the realized solvency process. Then the joint process $(\mathbf{x}, \mathbf{K}, \mathbf{P}, \boldsymbol{\iota})$ is Markovian.*

*Proof.* Note that $\boldsymbol{\iota}(t_l) = \operatorname{diag}(\boldsymbol{\iota}(t_{l-1}))\mathbb{1}_{\{\mathbf{K}(t_{l-1}) \geq \mathbf{0}\}}$. Therefore Markovianity follows directly from (8) as $(\mathbf{K}(t_l), \mathbf{P}(t_l)) = \hat{\Psi}^T(t_l, \boldsymbol{\iota}(t_l); \mathbf{x}(t_l)) = \hat{\Psi}^T(t_l, \operatorname{diag}(\boldsymbol{\iota}(t_{l-1}))\mathbb{1}_{\{\mathbf{K}(t_{l-1}) \geq \mathbf{0}\}}; f(\mathbf{x}(t_{l-1}), \tilde{\epsilon}(t_l)))$.

□

### 3.5 Case studies

For the case studies within this section, recall the geometric random walk (2) constructed on the tree $(\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_{l\Delta t})_{l=0}^{T/\Delta t}, \mathbb{P}^n)$ within Section 2.2. That is, we consider the assets $x_k$ of each bank $k$ to be generated by

$$x_k(t + \Delta t, \omega^n_{t+\Delta t,(n+1)(i-1)+j}) = x_k(t, \omega^n_{t,\omega_{t,i}}) \exp\left((r - \frac{\sigma^2_{kk}}{2})\Delta t + \sigma_k^\top \tilde{\epsilon}_j \sqrt{\Delta t}\right)$$

for some initial point $x_k(0, \Omega^n) \in \mathbb{R}^n_{++}$, correlation structure $\sigma$, and such that $\tilde{\epsilon} = (\tilde{\epsilon}_1, ..., \tilde{\epsilon}_{n+1}) \in \mathbb{R}^{n \times (n+1)}$ is generated from an $(n + 1) \times (n + 1)$ orthogonal matrix as in He [1990]. Define the discounted assets as

$$\tilde{\mathbf{x}}(t, \omega^n_t) := \exp(-rt)\mathbf{x}(t, \omega^n_t)$$

for every time $t$ and state $\omega^n_t \in \Omega^n_t$. One can check that these discounted asset values $\tilde{\mathbf{x}}$ are martingales.

We will consider two primary case studies within this section to demonstrate the details of this single maturity model. First, we will present sample paths of the clearing solution over time to investigate the contagion mechanism in practice. Second, we will vary the correlation structure between banks in order to investigate the sensitivity of the model to this parameter. For simplicity, we will assume the recovery rate $\beta = 0$ throughout these case studies so that we generalize, e.g., the Gai–Kapadia system (Gai and Kapadia [2010]).

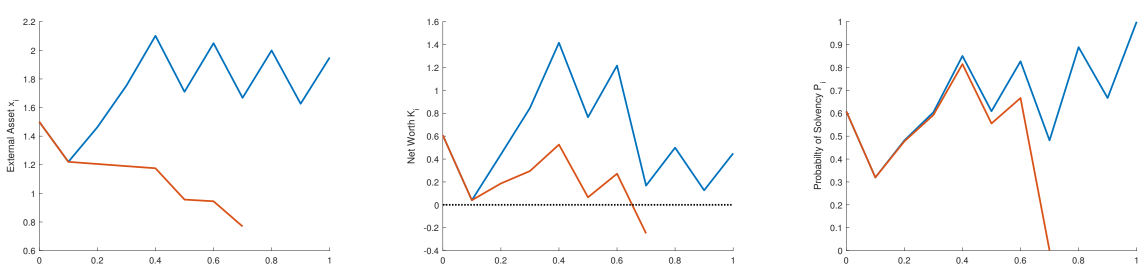
### 3.5.1 Sample paths

In order to demonstrate the network dynamics of this contagion model, we will consider an illustrative simple 2 bank (plus society node) system with zero risk-free rate $r = 0$. We will consider the terminal time $T = 1$ with time intervals $\Delta t = 0.1$. At the maturity $T$, the network of obligations is given by

$$\mathbf{L}_0 = \begin{pmatrix} 0.5 & 1 & 0 \\ 0.5 & 0 & 1 \end{pmatrix}$$

where the first column indicates the external obligations. The external assets have the same variance $\sigma^2 = 0.25$ with correlation of $\rho = 0.5$; both banks begin with $x_i(0) = 1.5$ in external assets for $i \in \{1, 2\}$.

With this simple setting, we are able to use the forward-backward model (5) to compute the clearing net worths $\mathbf{K}$ and probabilities of solvency $\mathbf{P}$. In Figure 2, a single sample path of the external assets $\mathbf{x}(\omega)$, net worths $\mathbf{K}(\omega)$, and probabilities of solvency $\mathbf{P}(\omega)$ are displayed. Along this sample path, bank 2 (displayed in red) defaults on its obligations at time $t = 0.7$ though bank 1 (displayed in blue) remains solvent until maturity. This is clearly seen in both Figure 2b of the net worths and Figure 2c of the probability of solvency. Notably, even at time $t = 0.7$, bank 2's external assets are approximately $x_2(0.7) \approx 0.769$; this means that, without considering default contagion (i.e., marking all interbank assets in full) bank 2 would *not* be in default. This demonstrates the impact of using this endogenous network valuation adjustment since, along this path, bank 1 does not default, yet the possibility that it fails to repay its obligations due to uncertainty in the future forces bank 2 to mark down its interbank assets enough so that it is driven into insolvency. Thus, a type of default contagion from bank 1 to itself over time (in unrealized paths of the tree) leads to the interbank default contagion that is more typically studied in the Gai–Kapadia model.

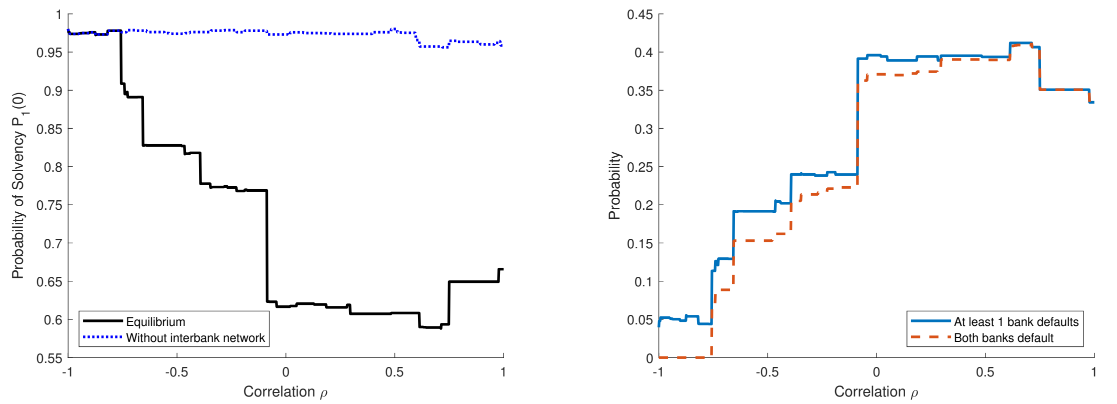



(a) External assets $\mathbf{x}$ (b) Net worths $\mathbf{K}$ (c) Probabilities of solvency $\mathbf{P}$

Figure 2: Section 3.5.1: A single sample path for the external assets $\mathbf{x}$, net worths $\mathbf{K}$, and probabilities of solvency $\mathbf{P}$ for both banks.

### 3.5.2 Dependence on correlation

Consider again the simple $n = 2$ bank network of Section 3.5.1. Rather than study a single sample path of this system, we will instead vary the correlation $\rho \in (-1, 1)$ to characterize the nontrivial behavior of this clearing model. As we will only investigate the behavior of the system at the initial time $t = 0$, and because the system is symmetric, throughout this case study we will without loss of generality only discuss bank 1.




(a) The probability of solvency $P_1(0)$ at time $t = 0$ for bank 1.

(b) The probability of one or both banks being in default as measured at time $t = 0$.

Figure 3: Section 3.5.2: The impact of the correlation $\rho \in (-1, 1)$ between the external asset values on health of the financial system as measured by probabilities of solvency/default.

The probability of solvency $P_1(0)$ at time $t = 0$ as a function of the correlation $\rho$ between the banks' external assets is provided in Figure 3a; we wish to note that, due to the construction of this network, $K_1(0) = P_1(0)$. Notably, the response of the probability of solvency to $\rho$ follows a staircase structure. This structure is due to the discrete nature of the tree model considered within this work. Specifically, because of the discrete time points and asset values, for a branch of the tree to move sufficiently to cause a bank to move from solvency into default (or vice versa) requires a sufficient change in the system parameters (e.g., the correlation); once such a sufficient change occurs to the tree, due to the possibility of contagion across time and between banks, there can be knock-on effects that generate large jumps in the health of the financial system. These probabilities of solvency can be compared to the probability of solvency without consideration for the interbank network (i.e., setting $L_{12} = L_{21} = 0$). Without an interbank network there is no avenue for contagion within this system and, as such, bank 1 is solvent in more scenarios under this setting. In fact, the only influence that the asset correlations $\rho$ have in this no-interbank network setting is due to the construction of the tree following He [1990].

In Figure 3b, we investigate the probability of defaults. For this we take the view of a regulator who would be concerned with the number of defaults but not which institution is failing. The blue solid line displays the probability that at least one of the banks will default (as measured at time $t = 0$). As with the probability that bank 1 is solvent $P_1(0)$, this has a step structure that is non-monotonic in the correlation $\rho$. The red dashed line displays the probability that both banks will default (as measured at time $t = 0$). Here we see that, though there is roughly a 5% chance that there is a default in the system when $\rho \approx -1$, there is no possibility of a joint default when the banks' external assets are highly negatively correlated. As the correlation increases, so does the likelihood of contagion in which both banks default (up until the banks default simultaneously, without the need for contagion, if $\rho = 1$).

We wish to conclude this case study by comparing the non-monotonicity exhibited here with the network valuation adjustment considered in Banerjee and Feinstein [2022]. In that work, which focuses on the Eisenberg–Noe clearing model (Eisenberg and Noe [2001], Rogers and Veraart [2013]) without early defaults, the comonotonic scenario (i.e., $\rho = 1$) is found to have the largest default contagion. Herein, with early defaults, we find that this no longer holds. There does exist general downward trend in the probability of solvency is found with the highest probability of solvency occurring near $\rho \approx -1$. When the correlations are highly negative, for one bank to be in distress (through the downward drift of its external assets) directly means the other bank has a large surplus; in this way the interbank assets act as a diversifying investment to reduce the risk of default. However, in contrast to the downward trend evidenced in the probability of solvency, a significant upward jump upward is evidenced at $\rho \approx 0.75$. This non-monotonicity makes clear the non-triviality of the constructed systemic model.

# 4 Multiple maturity setting

In contrast to the prior section in which all obligations are due at the same maturity $T$, we now wish to consider the possibility of obligations at every time $t_l$. At each maturity time $t_l$ a network of obligations $\mathbf{L}^l$ exists; the balance sheet otherwise is constructed as in the single maturity setting as presented in Section 3.1. As in Kusnetsov and Veraart [2019], Banerjee et al. [2022] if a bank defaults on an obligation then it also defaults on all subsequent obligations as well. Herein, as opposed to the structure of the single maturity setting in Section 3, we need to distinguish between solvency and liquidity as a bank may have positive capital but be unable to satisfy its short-term obligations.

We review the balance sheet of the banks in the system within Section 4.1 with emphasis on the additional considerations when multiple maturities are included. The constructed mathematical model for this multiple maturity setting is provided within Section 4.2. Rather than providing detailed equivalent forward-backward and dynamic programming formulations, we only comment on how such models can be presented in this setting. This model is then used in Section 4.4 to consider the implications of default contagion on the term structures for bank liabilities.

## 4.1 Balance sheet construction

Following the balance sheet constructed within Section 3.1, but with minor modification, banks hold *three* types of assets at time $t_l$: external (risky) assets $x_i(t_l) \in \mathcal{L}_{t_l}$, external (risk-free) assets, and interbank assets $\sum_{j=1}^n L_{ji}^k$ at time $t_k$ with $k > l$ where $L_{ji}^k \geq 0$ is the total obliged from bank $j$ to $i$ at time $t_k$ ($L_{ii}^k = 0$ so as to avoid self-dealing). Notably, as the bank may have received interbank payments at time $t_k$ with $k \leq l$ as well, the bank may have assets held in the risk-free asset. Specifically, at time $t_l$, the bank can split its cash holdings between its external risky and risk-free assets so that the (simple) return from time $t_l$ to $t_{l+1}$ is:

$$R_i(t_{l+1}, \alpha_i(t_l)) := \left[e^{r(t_{l+1}-t_l)}\alpha_i(t_l) + \frac{x_i(t_{l+1})}{x_i(t_l)}(1 - \alpha_i(t_l))\right] - 1$$

where $\alpha_i(t_l) \in \mathcal{L}_{t_l}$ provides the fraction of the cash account invested at time $t_l$ in the risk-free asset and, accordingly, $1 - \alpha_i(t_l)$ provide the fraction of the cash account invested at time $t_l$ in the risky asset. We will assume throughout this section that $\alpha_i(t_l, \omega_{t_l}) \in [0, 1]$ so that bank $i$ is long in both assets. Likewise, the bank has liabilities $\sum_{j=1}^n L_{ij}^k + L_{i0}^k$ at time $t_k$ where $L_{i0}^k \geq 0$ denotes the external obligations of bank $i$ at time $t_k$. To simplify the mathematical expressions below we will define $L_{0i}^l = 0$ for all times $t_l$; furthermore to avoid the need to consider defaults due to the initial setup of the system, we will assume $L_{ij}^0 = 0$ for all pairs of banks $i, j$ so that no obligations are due at time $t_0 = 0$.

As in Kusnetsov and Veraart [2019], when a bank defaults on its obligations at time $t_l$, it also does so for all of its obligations at time $t_k > t_l$ as well. However, in contrast to that work but similarly to Banerjee et al. [2022], the recovery on defaulted obligations occurs immediately after the clearing date (i.e., at time $t_l^+$) to account for any delays associated with the bankruptcy procedure. That is, a 0 recovery rate is assumed at the clearing time, but a $\beta \in [0, 1]$ recovery of liabilities occurs immediately after.

As mentioned above, thus far, we have ignored the considerations for the cash account

$V_i(t_l)$. The evolution of the cash account is due to the reinvestment of the cash account from the prior time step (including any recovery of defaulting assets at time $t_{l-1}$) and the (realized) net payments (following the Gai–Kapadia setting (Gai and Kapadia [2010])). This provides a recursive formulation for the cash account:

$$V_i(t_l) = (1+R_i(t_l, \alpha_i(t_{l-1}))) \left( V_i(t_{l-1}) + \beta \sum_{k=l}^{\ell} e^{-r(t_k - t_l)} \sum_{j=1}^{n} L_{ji}^k \mathbb{1}_{\{j \text{ defaulted at } t_{l-1}\}} \right)$$
$$+ \sum_{j=0}^{n} \left[ L_{ji}^l \mathbb{1}_{\{j \text{ is solvent at } t_l\}} - L_{ij}^l \right]$$

for $l = 1, ..., \ell$ and initial condition $V_i(t_0) := x_i(t_0)$. As in the single maturity setting, for consideration of the capital account, the interbank assets will not be valued at their face value, but rather based on the probability of default (i.e., to distinguish the book value and the realized value). That is, the obligation at time $t_k$ from bank $j$ to $i$ is valued as $p_{ji}(t_l, t_k) := e^{-r(t_k - t_l)} L_{ji}^k (\beta + (1-\beta)\mathbb{E}[P_j(t_k, t_k) \mid \mathcal{F}_{t_l}]) \in \mathcal{L}_{t_l}$ that takes value (almost surely) in the interval $e^{-r(t_k - t_l)} L_{ji}^k \times [\beta, 1]$ where $P_j(t_k, t_k, \omega_{t_k})$ is the realized indicator of solvency for bank $j$ at time $t_k$. Defining $P_j(t, t_k) := \mathbb{E}[P_j(t_k, t_k) \mid \mathcal{F}_t] \in \mathcal{L}_t$ as the (conditional) probability of solvency for obligations with maturity at time $t_k$ as measured at time $t$, we can write the discounted payments as $p_{ji}(t_l, t_k) = e^{-r(t_k - t_l)} L_{ji}^k (\beta + (1-\beta) P_j(t_l, t_k))$. Thus, the realized balance sheet for bank $i$ has possible write-downs in the value of assets and, therefore, the realized net worth at time $t_l$ is given by

$$K_i(t_l) = V_i(t_l) + \beta \sum_{j=1}^{n} L_{ji}^l \mathbb{1}_{\{j \text{ defaults at } t_l\}}$$
$$+ \sum_{k=l+1}^{\ell} e^{-r(t_k - t_l)} \sum_{j=0}^{n} [L_{ji}^k (\beta + (1-\beta) P_j(t_l, t_k)) \mathbb{1}_{\{j \text{ did not default before } t_l\}} - L_{ij}^k].$$

**Remark 4.1.** Recall from Remark 3.2 that throughout this work we assume that the interbank network is fixed. As such even though future interbank assets have a value determined by $\mathbf{P}$, these assets are treated as nonmarketable and cannot be used to increase short-term liquidity as encoded in the cash account. Notably, ignoring the effects of changing the network structure, if these interbank assets are treated as both liquid and marketable then the cash account $\mathbf{V}$ would be identical to $\mathbf{K}$.

As previously mentioned, as opposed to the single maturity setting, herein a default can be due to either insolvency (if the bank net worths drop below 0 so that, e.g., either shareholders are able to increase their risk-neutral value by declaring bankruptcy or some covenants have forced the early default) or illiquidity (if the bank cannot meet its obligations with its current cash account). That is, bank $i$ will default at the stopping time $\tau_i$ given by

$$\tau_i := \inf\{t \geq 0 \mid \min\{K_i(t)\ ,\ V_i(t)\} < 0\}$$

so that default occurs at the first time that either the realized capital or cash account become negative. This default condition was studied in Banerjee et al. [2022] in a different dynamic network context.

**Remark 4.2.** As in Remark 3.1(i), if only the modeling of illiquidity is desired then the default time can be reformulated such that it only accounts for the cash account, i.e., $\tau_i = \inf\{t \geq 0 \mid V_i(t) < 0\}$.

**Remark 4.3.** Consider the following three meaningful levels of the rebalancing parameter $\alpha_i$:

- If $\alpha_i^0(t_l) := 0$ then bank $i$ will reinvest its entire cash account into the external asset $x_i$ at time $t_l$.
- If $\alpha_i^1(t_l) := 1$ then bank $i$ will move all of its investments into the risk-free bond at time $t_l$; this includes any prior investment in the external asset $x_i$. Though this is feasible in our setting, we generally consider this to be an extreme scenario.
- If

$$\begin{aligned}\alpha_i^L(t_l) &:= \left[\frac{\sum_{k=0}^{l} e^{r(t_l-t_k)} \sum_{j=0}^{n} [L_{ji}^k \mathbb{1}_{\{\tau_j > t_k\}} + \beta \sum_{h=k}^{\ell} e^{-r(t_h-t_k)} L_{ji}^h \mathbb{1}_{\{\tau_j = t_k\}} - L_{ij}^k]}{V_i(t_l) + \beta \sum_{k=l}^{\ell} e^{-r(t_k-t_l)} \sum_{j=1}^{n} L_{ji}^k \mathbb{1}_{\{\tau_j = t_l\}}}\right]^+ \\ &= \left[1 - \frac{x_i(t_l)}{V_i(t_l) + \beta \sum_{k=l}^{\ell} e^{-r(t_k-t_l)} \sum_{j=1}^{n} L_{ji}^k \mathbb{1}_{\{\tau_j = t_l\}}}\right]^+\end{aligned}$$

  then bank $i$ will use its (risky) external asset position to cover any realized net liabilities, but will never increase its external position above its original level (i.e., if bank $i$ has net interbank assets, the external asset position will be made whole and all additional assets are invested in cash). Notably, if bank $i$ is solvent, this rebalancing parameter falls between the prior two cases, i.e., $\alpha_i^L(t_l) \in [0, 1]$.

If $\alpha_i(t_l) < 0$ then, implicitly, bank $i$ is shorting its own external assets; by a no-short selling constraint, we assume this cannot occur. If $\alpha_i(t_l) > 1$ then, similarly, bank $i$ is borrowing at the risk-free rate solely to purchase additional units of the risky asset; as this would produce new obligations, study of such a scenario is beyond the scope of this work.

## 4.2 Mathematical model

As highlighted within the balance sheet modeling, we can immediately define an equilibrium model for default contagion that is jointly on the net worths ($\mathbf{K} = (K_1, ..., K_n)^\top$), cash accounts ($\mathbf{V} = (V_1, ..., V_n)^\top$), survival probabilities ($\mathbf{P} = (P_1, ..., P_n)^\top$), and default times ($\boldsymbol{\tau} = (\tau_1, ..., \tau_n)^\top$). As in the single maturity setting, by convention and without loss of generality we will assume $\tau_i(\omega) = T + 1$ if bank $i$ does not default on $\omega \in \Omega$. In contrast to the single maturity setting, we consider the simpler domain $\mathbb{D} := \prod_{l=0}^{\ell}(\mathcal{L}_{t_l}^n \times \mathcal{L}_{t_l}^n \times [\mathbf{0}, \mathbf{1}]^{|\Omega_{t_l}| \times (\ell - l + 1)} \times \{t_0, t_1, ..., t_\ell, T + 1\}^{|\Omega| \times n})$. That is, we specify $\mathbf{K}, \mathbf{V}$ are adapted processes, $\mathbf{P}$ is a collection of adapted processes between 0 and 1, and $\boldsymbol{\tau}$ is a vector of stopping times for this model.

This clearing system $\Psi : \mathbb{D} \to \mathbb{D}$ is mathematically constructed in (9) with an explicit dependence on the rebalancing parameter $\boldsymbol{\alpha}$:

$$
\begin{aligned}
(\mathbf{K}, \mathbf{V}, \mathbf{P}, \boldsymbol{\tau}) &= \Psi(\mathbf{K}, \mathbf{V}, \mathbf{P}, \boldsymbol{\tau}; \boldsymbol{\alpha}) \qquad (9)\\
&:= \left(\Psi_{\mathbf{K}}(t_l, \mathbf{V}(t_l), \mathbf{P}(t_l, \cdot), \boldsymbol{\tau})\ ,\ \Psi_{\mathbf{V}}(t_l, \mathbf{V}(t_{[l-1]\vee 0}), \boldsymbol{\tau}; \boldsymbol{\alpha})\ ,\ \Psi_{\mathbf{P}}(t_l, t_k, \boldsymbol{\tau})_{k=[l+1]\wedge \ell}^{\ell}\ ,\ \Psi_{\boldsymbol{\tau}}(\mathbf{K}, \mathbf{V})\right)_{l=0}^{\ell}
\end{aligned}
$$

where, for any $i = 1, ..., n$,

$$
\begin{cases}
\Psi_{\mathbf{K},i}(t_l, \tilde{\mathbf{V}}, \tilde{\mathbf{P}}, \boldsymbol{\tau}) = \tilde{V}_i + \beta \sum_{j=1}^n L_{ji}^l \mathbb{1}_{\{\tau_j = t_l\}} + \sum_{k=l+1}^{\ell} e^{-r(t_k - t_l)} \sum_{j=0}^n \left[L_{ji}^k(\beta + (1-\beta)\tilde{P}_j(t_k))\mathbb{1}_{\{\tau_j \geq t_l\}} - L_{ij}^k\right] \\
\Psi_{\mathbf{V},i}(t_l, \tilde{\mathbf{V}}, \boldsymbol{\tau}; \boldsymbol{\alpha}) = \begin{cases} [1 + R_i(t_l, \alpha_i(t_{l-1}))]\left(\tilde{V}_i + \beta \sum_{k=l-1}^{\ell} e^{-r(t_k - t_{l-1})} \sum_{j=1}^n L_{ji}^k \mathbb{1}_{\{\tau_j = t_{l-1}\}}\right) & \\ \quad + \sum_{j=0}^n \left[L_{ji}^l \mathbb{1}_{\{\tau_j > t_l\}} - L_{ij}^l\right] & \text{if } l > 0 \\ x_i(0) & \text{if } l = 0 \end{cases} \\
\Psi_{\mathbf{P},i}(t_l, t_k, \boldsymbol{\tau}) = \mathbb{P}(\tau_i > t_k \mid \mathcal{F}_{t_l}) \\
\Psi_{\boldsymbol{\tau},i}(\mathbf{K}, \mathbf{V}) = \inf\{t \geq 0 \mid \min\{K_i(t), V_i(t)\} < 0\}.
\end{cases}
$$

In comparison to the single maturity setting consider in Section 3 above, due to recovery rate $\beta$ and the potential dependence of the rebalancing strategy to the capital and cash accounts (see, for instance, $\boldsymbol{\alpha}^L$ in Remark 4.3), we can no longer guarantee monotonicity of the clearing problem. However, as will be demonstrated in Theorem 4.4 below, we will prove the existence of a clearing solution constructively using an extension of the dynamic programming principle formulation of the problem (as in Section 3.4 for the single maturity setting). This, additionally, allows us to immediately conclude that we can construct a Markovian clearing solution.

**Theorem 4.4.** *Fix the rebalancing strategies $\boldsymbol{\alpha}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}) \in [\mathbf{0}, \mathbf{1}]^{|\Omega_{t_l}|}$ so that they depend only on the current time $t_l$, capital $\hat{\mathbf{K}} \in \mathcal{L}_{t_l}^n$, and cash account $\hat{\mathbf{V}} \in \mathcal{L}_{t_l}^n$. There exists a (finite) clearing* *solution* $(\mathbf{K}^*, \mathbf{V}^*, \mathbf{P}^*, \boldsymbol{\tau}^*) = \Psi(\mathbf{K}^*, \mathbf{V}^*, \mathbf{P}^*, \boldsymbol{\tau}^*)$ *to* (9). *Furthermore, if we have Markovian external assets,* $\mathbf{x}(t_l) = f(\mathbf{x}(t_{l-1}), \tilde{\epsilon}(t_l))$ *for i.i.d. perturbations* $\tilde{\epsilon}$*, then there exists a clearing solution such that* $(\mathbf{x}(t_l), \mathbf{K}^*(t_l), \mathbf{V}^*(t_l), \mathbf{P}^*(t_l, t_k)_{k=l}^{\ell}, \boldsymbol{\iota}^*(t_l))_{l=0}^{\ell}$ *is Markovian where* $\boldsymbol{\iota}^*(t_l) := \mathbb{1}_{\{\boldsymbol{\tau}^* \geq t_l\}}$ *is the realized solvency process.*

*Proof.* We will prove the existence of a clearing solution $(\mathbf{K}^*, \mathbf{V}^*, \mathbf{P}^*, \boldsymbol{\tau}^*)$ to (9) constructively. Specifically, as in the dynamic programming principle formulation (8) for the single maturity setting, consider the mappings $\hat{\Psi}$ of the time, solvent institutions, and prior cash account into the current clearing solution. That is, we define $\hat{\Psi}$ by

$$\hat{\Psi}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) := \begin{pmatrix} \hat{\Psi}_{\mathbf{K}}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) \\ \hat{\Psi}_{\mathbf{V}}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) \\ \hat{\Psi}_{\mathbf{P}}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) \end{pmatrix}$$

$$= \operatorname*{FIX}_{(\tilde{\mathbf{K}}, \tilde{\mathbf{V}}, \tilde{\mathbf{P}}) \in \hat{\mathbb{D}}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota})} \begin{pmatrix} \tilde{\mathbf{V}} + \beta (\mathbf{L}^l)^\top \operatorname{diag}(\boldsymbol{\iota}) \mathbb{1}_{\{\tilde{\mathbf{K}} \wedge \tilde{\mathbf{V}} < \mathbf{0}\}} \\ \quad + \sum_{k=l+1}^{\ell} e^{-r(t_k - t_l)} \left[ (\mathbf{L}^k)^\top \operatorname{diag}(\boldsymbol{\iota}) (\beta \mathbf{1} + (1-\beta) \tilde{\mathbf{P}}(t_k)) - \mathbf{L}^k \mathbf{1} \right] \\ \begin{cases} (I + \operatorname{diag}(\mathbf{R}(t_l, \boldsymbol{\alpha}(t_{l-1}, \hat{\mathbf{K}}, \hat{\mathbf{V}})))) \hat{\mathbf{V}} + (\mathbf{L}^l)^\top \operatorname{diag}(\boldsymbol{\iota}) \mathbb{1}_{\{\tilde{\mathbf{K}} \wedge \tilde{\mathbf{V}} \geq \mathbf{0}\}} - \mathbf{L}^l \mathbf{1} & \text{if } l > 0 \\ \mathbf{x}(0) & \text{if } l = 0 \end{cases} \\ \left( \begin{cases} \left[ \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}}) \hat{\Psi}_{\mathbf{P}, t_k}(t_{l+1}, \mathbf{F}_l(\tilde{\mathbf{K}}, \tilde{\mathbf{V}}, \boldsymbol{\iota})(\omega_{t_l}))(\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} \right]_{\omega_{t_l} \in \Omega_{t_l}} & \text{if } l < k \\ \operatorname{diag}(\boldsymbol{\iota}) \mathbb{1}_{\{\tilde{\mathbf{K}} \wedge \tilde{\mathbf{V}} \geq \mathbf{0}\}} & \text{if } l = k \end{cases} \right)_{k=l}^{\ell} \end{pmatrix}$$

with

$$\mathbf{F}_l(\tilde{\mathbf{K}}, \tilde{\mathbf{V}}, \boldsymbol{\iota}) := \left( \tilde{\mathbf{K}}\,,\ \tilde{\mathbf{V}} + \beta \sum_{k=l}^{\ell} (\mathbf{L}^k)^\top \operatorname{diag}(\boldsymbol{\iota}) \mathbb{1}_{\{\tilde{\mathbf{K}} \wedge \tilde{\mathbf{V}} < \mathbf{0}\}}\,,\ \operatorname{diag}(\boldsymbol{\iota}) \mathbb{1}_{\{\tilde{\mathbf{K}} \wedge \tilde{\mathbf{V}} \geq \mathbf{0}\}} \right)$$

for any $l = 0, 1, ..., \ell$ and the fixed points taken on the lattice

$$\mathbb{D}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) := \mathbb{D}_{\mathbf{K}}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) \times \mathbb{D}_{\mathbf{V}}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) \times [\mathbf{0}, \mathbf{1}]^{|\Omega_{t_l}| \times (\ell - l + 1)}$$

$$\mathbb{D}_{\mathbf{K}}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) := \left\{ \mathbf{K} \in \mathcal{L}_{t_l}^n \;\middle|\; \mathbf{K} \in \mathbb{D}_{\mathbf{V}}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) + [\mathbf{0}, \beta (\mathbf{L}^l)^\top \boldsymbol{\iota}] + \sum_{k=l+1}^{\ell} e^{-r(t_k - t_l)} ([\beta (\mathbf{L}^k)^\top \boldsymbol{\iota}, (\mathbf{L}^k)^\top \boldsymbol{\iota}] - \mathbf{L}^k \mathbf{1}) \right\}$$

$$\mathbb{D}_{\mathbf{V}}(t_l, \hat{\mathbf{K}}, \hat{\mathbf{V}}, \boldsymbol{\iota}) := \begin{cases} \left\{ \mathbf{V} \in \mathcal{L}_{t_l}^n \;\middle|\; \mathbf{V} \in (I + \operatorname{diag}(\mathbf{R}(t_l, \boldsymbol{\alpha}(t_{l-1}, \hat{\mathbf{K}}, \hat{\mathbf{V}})))) \hat{\mathbf{V}} + [\mathbf{0}, (\mathbf{L}^l)^\top \boldsymbol{\iota}] - \mathbf{L}^l \mathbf{1} \right\} & \text{if } l > 0 \\ \{\mathbf{x}(0)\} & \text{if } l = 0. \end{cases}$$

As the problem used to define $\hat{\Psi}$ is a monotonic mapping on a lattice (proven by inspection for $\hat{\Psi}_{\mathbf{K}}, \hat{\Psi}_{\mathbf{V}}$ and by a simple induction argument for $\hat{\Psi}_{\mathbf{P}}$), we can apply Tarski's fixed point theorem to guarantee the existence of a greatest fixed point (i.e., the mapping FIX is well-defined in this case for every feasible combination of inputs). Let $(\mathbf{K}^*, \mathbf{V}^*, \mathbf{P}^*)$ be the realized solution from $\hat{\Psi}(0, \mathbf{0}, \mathbf{x}(0), \mathbf{1})$ and define $\boldsymbol{\tau}^* := \Psi_{\boldsymbol{\tau}}(\mathbf{K}^*, \mathbf{V}^*)$. (We wish to note that the initial value for $\hat{\mathbf{K}}$ taken here as $\mathbf{0}$ is an arbitrary choice as this term is never utilized.) As in the proof of Proposition 3.11, we will seek to demonstrate that $(\mathbf{K}^*, \mathbf{V}^*, \mathbf{P}^*, \boldsymbol{\tau}^*)$ is a fixed point to (9) by proving that $\mathbf{P}^*(t_l, t_k) = \Psi_{\mathbf{P}}(t_l, t_k, \boldsymbol{\tau}^*)$ for all times $t_l \leq t_k$. First, at maturity times, $\mathbf{P}^*(t_l, t_l) = \prod_{k=0}^{l} \mathbb{1}_{\{\min\{\mathbf{K}^*(t_l), \mathbf{V}^*(t_l)\} \geq 0\}} = \mathbb{1}_{\{\boldsymbol{\tau}^* > t_l\}} = \mathbb{P}(\boldsymbol{\tau}^* > t_l | \mathcal{F}_{t_l})$ by construction. Second, consider $l < k$ and assume $\mathbf{P}^*(t_{l+1}, t_k) = \Psi_{\mathbf{P}}(t_{l+1}, t_k, \boldsymbol{\tau}^*)$. By definition of $\hat{\Psi}_{\mathbf{P}, t_k}$, we can construct $\mathbf{P}^*(t_l, t_k)$ as follows:

$$
\begin{aligned}
P_i^*(t_l, t_k, \omega_{t_l}) &= \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}}) \hat{\Psi}_{\mathbf{P}, t_k, i}(t_{l+1}, \mathbf{K}^*(t_l), \mathbf{V}^*(t_l) + \beta \sum_{k=l}^{\ell} (\mathbf{L}^k)^\top \mathbb{1}_{\{\boldsymbol{\tau}^* = t_l\}}, \mathbb{1}_{\{\boldsymbol{\tau}^* > t_l\}})(\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} \\
&= \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}}) P_i^*(t_{l+1}, t_k, \omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} \\
&= \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}}) \Psi_{\mathbf{P}, i}(t_{l+1}, t_k, \boldsymbol{\tau}^*)(\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} \\
&= \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\omega_{t_{l+1}}) \mathbb{P}(\tau_i^* > t_k | \mathcal{F}_{t_{l+1}})(\omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} \\
&= \sum_{\omega_{t_{l+1}} \in \mathcal{S}(\omega_{t_l})} \frac{\mathbb{P}(\tau_i^* > t_k, \omega_{t_{l+1}})}{\mathbb{P}(\omega_{t_l})} \\
&= \mathbb{P}(\tau_i^* > t_k | \mathcal{F}_{t_l})(\omega_{t_l}) = \Psi_{\mathbf{P}, i}(t_l, t_k, \boldsymbol{\tau}^*)(\omega_{t_l}).
\end{aligned}
$$

Therefore, we have constructed a process that clears (9). To prove that the enlarged process is Markovian, as with the Lemma 3.12, we can simply observe that

$$
\begin{aligned}
(\mathbf{K}^*(t_l), \mathbf{V}^*(t_l), \mathbf{P}^*(t_l, t_k)_{k=l}^{\ell}) &= \hat{\Psi}(t_l, \boldsymbol{\iota}^*(t_l), \mathbf{V}^*(t_{l-1}); \mathbf{x}(t_l)) \\
&= \hat{\Psi}(t_l, \operatorname{diag}(\boldsymbol{\iota}^*(t_{l-1})) \mathbb{1}_{\{\mathbf{K}^*(t_{l-1}) \wedge \mathbf{V}^*(t_{l-1}) \geq \mathbf{0}\}}, \mathbf{V}^*(t_{l-1}); f(\mathbf{x}(t_{l-1}), \tilde{\epsilon}(t_l))).
\end{aligned}
$$

From this, the result is immediate. $\square$

## 4.3 Optimal rebalancing

Thus far, we have assumed that the rebalancing strategies $\boldsymbol{\alpha}$ are known and fixed solely based on the composition of the cash account. However, in practice, the banks will seek to optimize their own strategy $\alpha_i$ so as to maximize their utility subject to certain regulatory constraints. In particular, herein, we will consider the setting in which banks must satisfy a short-term capital adequacy constraint at each time point $t_l$ for $l = 0, 1, ..., \ell - 1$. This regulatory requirement, imposed by the Basel accords, is such that the ratio of the capital to the risk-weighted assets needs to exceed some minimal ratio. That is, assuming the risk-weights for the risk-free and all interbank assets are 0, $\alpha_i(t_l) \in [0,1]^{|\Omega_{t_l}|}$ is constrained such that

$$\frac{K_i(t_l)}{w_i(1-\alpha_i(t_l))(V_i(t_l) + \beta \sum_{k=l}^{\ell} e^{-r(t_k - t_l)} \sum_{j=1}^{n} L_{ji}^k \mathbb{1}_{\{\tau_j = t_{l-1}\}})} \geq \theta$$

for risk-weight $w_i > 0$ and minimal threshold $\theta > 0$. (As banks only rebalance when they are solvent, we can assume wlog that $K_i(t_l), V_i(t_l) \geq 0$.) That is, assuming bank $i$ is solvent at time $t_l$, bank $i$ is bounded on its investment in its external assets by

$$\alpha_i(t_l) \geq 1 - \frac{K_i(t_l)}{w_i\theta(V_i(t_l) + \beta \sum_{k=l}^{\ell} e^{-r(t_k - t_l)} \sum_{j=1}^{n} L_{ji}^k \mathbb{1}_{\{\tau_j = t_{l-1}\}})}.$$

We wish to note that $\alpha_i(t_l) \leq 1$ automatically so long as bank $i$ is solvent and this capital ratio constraint is satisfied.

Under this regulatory setting, bank $i$ aims to optimize:

$$\alpha_i^*(t_l, \omega_{t_l}) = \operatorname*{arg\,max}_{\alpha_i \in [0,1]} \left\{ U_i(\alpha_i; t_l, \omega_{t_l}) \mid \alpha_i \geq 1 - \frac{K_i(t_l, \omega_{t_l})}{w_i\theta(V_i(t_l, \omega_{t_l}) + \beta \sum_{k=l}^{\ell} e^{-r(t_k - t_l)} \sum_{j=1}^{n} L_{ji}^k \mathbb{1}_{\{\tau_j = t_{l-1}\}}(\omega_{t_l}))} \right\} \tag{10}$$

for some quasiconcave utility function $U_i : [0,1] \to \mathbb{R}$ (which may also depend on the time and current state of the external market) at every time $t_l$ and state $\omega_{t_l} \in \Omega_{t_l}$.

**Example 4.5.** Assume bank $i$'s utility $U_i$ is nondecreasing in its investment in its risky (external) asset, i.e., nonincreasing in $\alpha_i$. Then (10) can trivially be solved in closed form, i.e.,

$$\alpha_i^*(t_l, \omega_{t_l}) = \left[1 - \frac{K_i(t_l, \omega_{t_l})}{w_i\theta(V_i(t_l, \omega_{t_l}) + \beta \sum_{k=l}^{\ell} e^{-r(t_k - t_l)} \sum_{j=1}^{n} L_{ji}^k \mathbb{1}_{\{\tau_j = t_{l-1}\}}(\omega_{t_l}))}\right]^+$$

at every time $t_l$ and state $\omega_{t_l} \in \Omega_{t_l}$.

The monotonicity of the utility function is natural for two (overlapping) reasons. First, as portfolio managers are rewarded for high returns but have only limited costs on the downside, they are incentivized to take outsized risks; that is, a portfolio manager would attempt to invest as much as allowed in the risky asset. Second, as we assume the dynamics of the external assets are independent of the investments made, this can be viewed as the "ideal" portfolio for bank $i$; as such, the bank would seek to invest as much as possible into this risky position. In either case, the result for bank $i$ is a utility function that is nonincreasing in $\alpha_i$.

### 4.4 Case studies

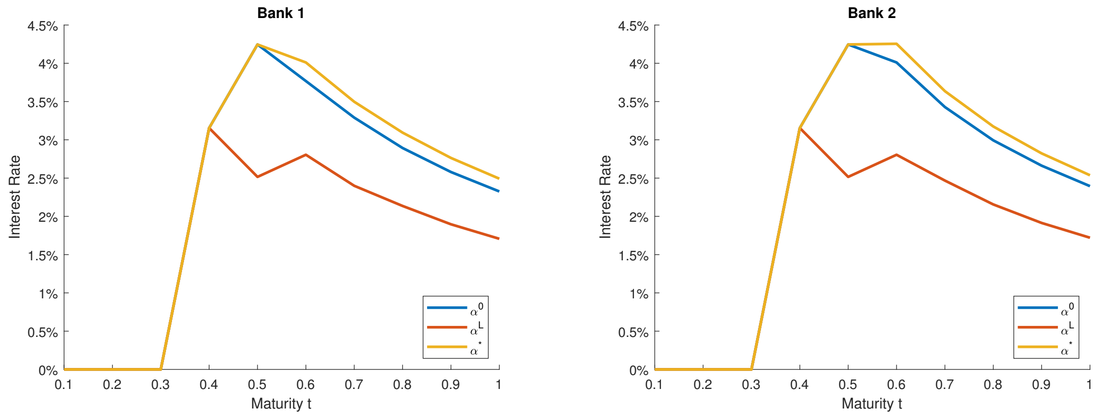
As in Section 3.5, for the purposes of these case studies, we will consider the tree model $(\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_{l\Delta t})_{l=0}^{T/\Delta t}, \mathbb{P}^n)$ as constructed in He [1990] and replicated in Section 2.2. Likewise, we will assume that the external assets follow the geometric random walk (2). Furthermore, we will assume the Gai–Kapadia setting (Gai and Kapadia [2010]) in which there is a zero recovery rate $\beta = 0$.

We will consider three primary case studies within this section to demonstrate the impacts of financial contagion on the term structure of bank debt. We define the term structure or yield curve of bank debt at time $t = 0$ through the interest rates $R_i^*(t_l) = P_i(0, t_l)^{-1/t_l} - 1$ so that $(1 + R_i^*(t_l))^{t_l} = 1/P_i(0, t_l)$. First, we will present a sample yield curve (as measured at time 0) under varying investment strategies $\boldsymbol{\alpha}$. Second, we will vary the leverage of the banks in the system in order to investigate the sensitivity of this systemic term structure to the initial balance sheet of the banks. Third, we will investigate a core-periphery network model to study a larger system and investigate the impacts of localized stresses on the term structure of larger system.

**Remark 4.6.** All clearing solutions computed herein are found via the Picard iteration of (9) beginning from the assumption that no banks will ever default. That is, these computations are accomplished without application of the constructive dynamic programming approach presented in the proof of Theorem 4.4. As this process converged for all examples considered in this work, we suspect that the monotonicity of (9) may be stronger than could be proven herein (which provides, e.g., a guarantee for the existence of a maximal clearing solution).

#### 4.4.1 Systemic term structure and investment strategies

First, we want to consider the yield curve (as measured at time $t = 0$) under the network valuation adjustment under varying investment strategies. Specifically, we will consider the same $n = 2$ network as in Section 3.5.1 but where all obligations $\mathbf{L}_0$ are split randomly (uniformly) over $(t_l)_{l=1}^{\ell}$. Due to the random split of obligations over time, the banks in this example are no longer symmetric institutions. We consider only a single split of the obligations as the purpose of this case study is to understand the possible shapes of the term structure under varying rebalancing strategies $\boldsymbol{\alpha}$. In particular, we will study three meaningful rebalancing structures: all investments are made in the external asset only ($\boldsymbol{\alpha}^0$ as provided in Remark 4.3), all surplus interbank payments are held in the risk-free asset ($\boldsymbol{\alpha}^L$ as defined in Remark 4.3), and the optimal investment strategy ($\boldsymbol{\alpha}^*$ as given in Example 4.5 with risk-weight $w_i = 2$ for $i \in \{1, 2\}$ and threshold $\theta = 0.08$ to match the Basel II Accords).




(a) Bank 1 (b) Bank 2

Figure 4: Section 4.4.1: The term structure for banks 1 and 2 for varying investment strategies $\alpha$.

Figure 4 displays the systemic term structure for both banks in this system. Notably, though these interest rates are similar for the two institutions, they are not identical due to the aforementioned random splitting of the obligations over the 10 time periods. We would also like to highlight that the yield curve for both banks has an inverted shape, i.e., the interest rate for longer dated maturities is lower than some short-term obligations. Inverted yield curves are typically seen as precursors of economic distress; here the probabilities of default are largely driven by contagion as, e.g., bank 1 will never default so long as bank 2 makes all of its payments in full (bank 2 will default in fewer than 0.7% of paths under all three investment strategies if bank 1 pays in full).

Comparing the different investment strategies makes clear that $\boldsymbol{\alpha}^L$ is the least risky strategy, i.e., the interest rates are dominated by those of $\boldsymbol{\alpha}^0, \boldsymbol{\alpha}^*$. This is as expected as, so long as the bank is healthy, it holds all surplus assets in the risk-free asset, otherwise it draws down its cash account to pay off obligations. Surprisingly, however, $\boldsymbol{\alpha}^*$ results in higher equilibrium interest rates than $\boldsymbol{\alpha}^0$. That is, imposing a regulatory constraint on investments through $\boldsymbol{\alpha}^*$ leads to greater probabilities of default than solely investing in the risky (external) asset $\boldsymbol{\alpha}^0$. Though, naively, it may seem counterintuitive that the most volatile investing strategy ($\boldsymbol{\alpha}^0$) is *not* the riskiest ex post, we conjecture this is due to the pro-cyclicality of the capital adequacy requirement (see also, e.g., Banerjee and Feinstein [2021]). Specifically, when ignoring the possibility of counterparty risks (i.e., assuming all interbank assets are fulfilled in full) bank

2 defaults under more scenarios when both banks follow $\boldsymbol{\alpha}^0$ than when they follow $\boldsymbol{\alpha}^*$. When accounting for the network effects, the pro-cyclicality of the capital adequacy regulation implies that banks are forced to move their investments into the risk-free asset when under stress. This means that once a bank is stressed, the cash account has lower volatility and the institution is less able to recover when the external asset value increases; because of default contagion, once a bank defaults it will drag its counterparties down as well potentially precipitating a cycle of contagion.

We wish to conclude this case study by commenting briefly on the dependence of $\boldsymbol{\alpha}^*$ on the risk-weights $w := w_1 = w_2$. If this risk-weight is set too low (below approximately 0.52 for this example), then the banks will not be constrained at all by the regulatory environment. Therefore, under such a setting, the resulting interest rates are identical to those under $\boldsymbol{\alpha}^0$. Conversely, if this risk-weight is set too high (above approximately 2.34 for this example), then the banks will not be able to invest in the risky (external) asset at all due to the regulatory constraints. Therefore, under such a setting, the resulting interest rates are 0 (identical to those under $\boldsymbol{\alpha}^1$ which are not displayed above) due to the construction of this system. Around these threshold interest rates, the term structures can be highly sensitive to the regulatory environment. Therefore naively, and heuristically, setting regulatory constraints can result in large unintended risks. We wish to highlight the recent works of Feinstein [2020], Banerjee and Feinstein [2021] which provide discussions on determining risk-weights to be consistent with systemic risk models.

### 4.4.2 Dependence on leverage

Having explored the impact of the investment strategy on the yield curve in Section 4.4.1, we now wish to explore how the (initial) leverage of the banking book can impact the shape of the term structure. Specifically, as will be demonstrated with the numerical experiments herein, we find a normal term structure when the leverage is low enough but becomes inverted for riskier scenarios.

As with Section 4.4.1, we consider a variation of the $n = 2$ network of Section 3.5.1 with $\ell = 10$ time steps where we will vary only the interbank assets and liabilities. The banking book leverage ratio, i.e., assets over equity assuming all debts are paid in full, at time $t = 0$ for bank 1 is given by:

$$\lambda_1 := \frac{x_1(0) + \sum_{l=0}^{\ell} L_{21}^l}{x_1(0) - \sum_{l=0}^{\ell} L_{10}^l} = 1.5 + \bar{L}_{21}$$

for $\bar{L}_{21} = \sum_{l=0}^{\ell} L_{21}^l \geq 0$ (where $x_1(0) = 1.5$ and $\sum_{l=0}^{\ell} L_{10}^{\ell} = 0.5$ by construction); similarly

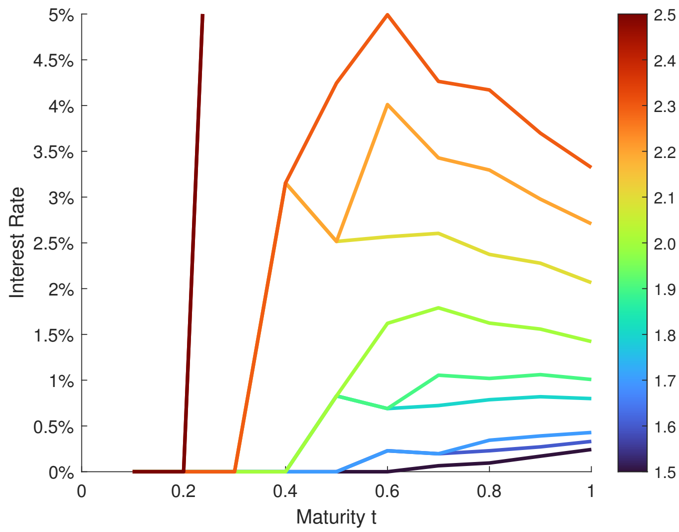



Figure 5: Section 4.4.2: The term structure for both banks 1 and 2 under varying leverage ratios $\lambda \in [1.5, 2.5]$.

$\lambda_2 := 1.5 + \bar{L}_{12}$ for $\bar{L}_{12} = \sum_{l=0}^{\ell} L_{12}^l \geq 0$. As in our prior case studies, we will assume the banking book for the two banks are symmetric so that $\bar{L} := \bar{L}_{12} = \bar{L}_{21}$ and, therefore also, $\lambda := \lambda_1 = \lambda_2$ throughout this example. In contrast to Section 4.4.1, here we assume all obligations are split deterministically so that:

$$L_{12}^l = L_{21}^l = \begin{cases} \bar{L}/3 & \text{if } l \in \{3,6,10\} \\ 0 & \text{else} \end{cases} \quad \text{and} \quad L_{10}^l = L_{20}^l = \begin{cases} 1/6 & \text{if } l \in \{3,6,10\} \\ 0 & \text{else.} \end{cases}$$

That is, only 3 times ($t \in \{0.3, 0.6, 1\}$) are maturities for debts and all liabilities are split equally over those dates. To complete the setup, we will assume that all banks follow the optimal rebalancing strategy $\boldsymbol{\alpha}^*$ (as proposed in Example 4.5) with $w := w_1 = w_2 = 2$ and $\theta = 0.08$.

First, we want to comment on the impact that increasing the leverage $\lambda$ of the two banks has on the health of the financial system. As seen in Figure 5, the system has higher (implied) interest rates $R_i^*(t_l)$ at every time $t_l$ under higher leverage. That is, the probability of a default (as measured at time 0) increases as the initial leverage $\lambda$ increases. This is as anticipated because larger leverages correspond with firms that are less robust to financial stresses, i.e., a smaller shock is required to cause a bank to default.

As obligations are only due at $t \in \{0.3, 0.6, 1\}$, we wish to consider the interest rates $R_i^*(t_l)$ for those dates specifically. As displayed in Table 1, when the leverage $\lambda$ is small ($\lambda < 2.0$),

| | Leverage $\lambda$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 | 2.0 | 2.1 | 2.2 | 2.3 | 2.4 | 2.5 |
| $t = 0.3$ | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 13.41% | 13.41% |
| $t = 0.6$ | 0.00% | 0.23% | 0.23% | 0.69% | 0.69% | 1.62% | 2.57% | 4.01% | 4.99% | 11.20% | 11.20% |
| $t = 1.0$ | 0.24% | 0.33% | 0.43% | 0.80% | 1.01% | 1.42% | 2.07% | 2.71% | 3.32% | 6.79% | 6.79% |

Table 1: Section 4.4.2: Yields for obligations as measured at time 0.

the interest rates charged are monotonically increasing over time, i.e., a normal yield curve. In particular, this occurs at $\bar{L} = 0$ (i.e., $\lambda = 1.5$) when no interbank network exists; such a scenario can be compared with, e.g., Black and Cox [1976] where a single firm is studied in isolation. As the leverage $\lambda$ grows via the increased size of the interbank network, the risk of defaults grows as well. This increased network size eventually leads to an inverted yield curve, i.e., in which the implied interest charged on obligations due at $t = 1$ is lower than on those due at $t = 0.6$. Until the leverage is sufficiently high ($\lambda \approx 2.5$), no defaults are realized at the first maturity $t = 0.3$ at all. The banks always make all payments due at $t = 0.3$ because of the tree structure considered herein; specifically, a bank fails to make payments on an early obligation only if it is already close to default at $t = 0$ as the tree model of He [1990] does not model extreme events.

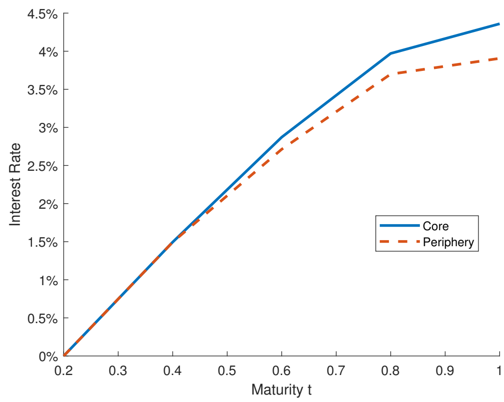
#### 4.4.3 A core-periphery network

In this final case study, we want to consider a larger financial network than used previously. Specifically, we will study the core-periphery structure for a financial network which has been found in several empirical studies (e.g., Craig and Von Peter [2014], Fricke and Lux [2015], Veld and Lelyveld [2014]). In this system we will assume that there are 2 core banks, 10 peripheral banks, and a societal node, i.e., $n = 12$ banks. Each core bank owes the other core bank \$3, all of the peripheral banks \$0.50, and society \$5. The peripheral banks owe both core banks \$0.50, nothing to the other peripheral banks, and \$1 to society. These obligations will be equally split at times $0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$ with $\ell = 5$. As before, we will assume the prevailing risk-free interest rate $r = 0$. As in Section 4.4.2, we will assume that all banks follow the optimal investment strategy $\boldsymbol{\alpha}^*$ presented in Example 4.5 with $w_i = 2$ for every bank $i$ and with regulatory threshold $\theta = 0.08$. Finally, the external assets are as follows. The core banks begin at time $t = 0$ with \$15 in external assets each. The peripheral banks begin with \$3 in external assets each. The correlation between any pair of bank assets is fixed at $\rho = 0.3$. We will study two scenarios for the external asset volatilities:

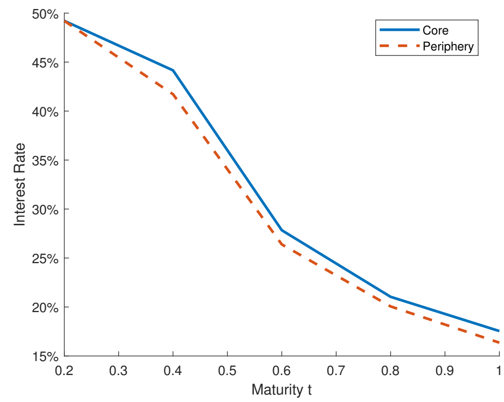
(i) a low volatility (unstressed) setting in which the volatility of the external assets for either core bank is $\sigma_C^2 = 0.75$ and the volatility for the external assets of any peripheral bank is

$\sigma_P^2 = 0.5$;

(ii) a high volatility (stressed) setting in which the volatility of the external assets for either core bank jumps upward to $\sigma_C^2 = 1$ while the volatility for the peripheral banks is unaffected ($\sigma_P^2 = 0.5$).




(a) The low volatility (unstressed) setting.




(b) The high volatility (stressed) setting.

Figure 6: Section 4.4.3: The term structure for the core and peripheral banks under the two stress scenarios (i.e., $\sigma_C^2 = 0.75$ and $\sigma_C^2 = 1$ respectively).

As depicted in Figure 6a, in the low volatility (unstressed) setting, all banks in the system have a normal yield curve with interest rates $\mathbf{R}^*(t_l)$ in the low single digits at each maturity date. In comparison, in the high volatility (stressed) setting displayed in Figure 6b, all banks have an inverted yield curve with interest rates in the double digits. Notably, the stress scenario only includes an increase in the core volatilities $\sigma_C^2$ without any direct change to the balance sheet of the peripheral banks. Therefore the change in the term structure for peripheral banks is being driven entirely by the contagion from stresses to the core institutions. Though a stylized system, this highlights the importance of regulating systemically important financial institutions (i.e., core banks) as their stress can readily spread throughout the entire financial system.

# 5 Conclusion

## 5.1 Policy implications

Within this work, we have constructed a dynamic model of the interbank network with forward-looking probabilities of default. In doing so, we found (numerically) that even small shocks that do not precipitate short term default, can still be relevant for stress testing purposes. For instance, as seen in Section 4.4.3 , stressing the volatility of core institutions can cause large impacts to the risk of future defaults through the feedback mechanisms. In this way, stress tests can be constructed so that the long term effects of the stress scenario can filter backwards in time to harm financial stability without having to trigger short term actualized defaults. This is distinct from how stress testing is typically carried out in which only the short term liquidity of firms is paramount (i.e., in line with the one period models of, e.g., Eisenberg and Noe [2001], Rogers and Veraart [2013], Gai and Kapadia [2010]).

Furthermore, recent works (see, e.g., Greenwood et al. [2022]) have explored the possibility of predicting *future* financial crises from the available data. Empirical works (such as Bluwstein et al. [2021], Babecký et al. [2014]) in this direction find that the shape of the yield curve can have strong predictive power. As we are able to construct a systemic term structure, our model is able to endogenize some of those effects due to the contagion through the probability of defaults. In particular, we construct a structural model for this type of contagion which provides some theoretical backing to the observations regarding the predictive power of yield curves. Such results point to the possibility of probabilistic stress tests in which the resulting systemic yield curves – and the resulting possibility of a financial crisis – are the primary outcomes.

## 5.2 Extensions

We wish to conclude by considering two important extensions of the framework presented herein. First, due to the exponential size of trees in the number of banks and the number of time steps, simulating large networks is computationally expensive. One important extension of this work is finding a Monte Carlo approach to approximate the clearing solutions for larger systems. Formulating such a numerical method requires further consideration due to the forward-backward construction of this model; specifically, we cannot simply simulate the process backwards in time with a least squares Monte Carlo approach. Furthermore, typical least squares Monte Carlo approaches perform poorly for this model because fixed points and equilibria are often sensitive to the parameters which leads to flawed regressions. A second natural extension would be to consider the continuous-time limit. In particular, one may ask if the discrete stochastic integral representations provided by Remark 3.6 propagate to the limit, similarly to the results of He [1990] for passing from discrete- to continuous-time within the classical Black–Scholes framework. Due to the dependence on the conditional probability of default, this may be viewed as a new type of McKean–Vlasov problem related to continuous-time McKean–Vlasov problems for hitting times that have recently been studied in the probability literature as per Remark 3.3. When restricting to the filtration generated by the Brownian motions, the particular structure of our model allows it to be recast as a system of forward-backward stochastic differential equations, but the analysis of this system does not seem to fit within the scope of any known results, thus calling for new mathematical developments.

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