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Existence of Equilibria in Infinite Horizon Finance Economies with Stochastic Taxation

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Abstract

This paper proves the existence of equilibria in the Infinite Horizon General Equilibrium with Incomplete Markets (GEI) model with insecure property rights. Insecure property rights come in the form of the stochastic taxes imposed on agents' endowments and assets' dividends. This paper finds that under reasonable assumptions, Financial Markets (FM) equilibria with Transversality Condition (TC) as well as equilibria with Implicit Debt Constraint (IDC) and Explicit Debt Constraint (EDC) exist in Infinite Horizon FM economies with stochastic taxes and with short-lived securities in zero net supply. Also, this paper finds that under reasonable assumptions, FM equilibria with TC as well as equilibria with IDC and EDC exist in Infinite Horizon FM economies with long-lived securities in zero net supply for a dense subset of the set of all stochastic dividend tax rates. Similarly, this paper finds that under reasonable assumptions, FM equilibria with TC as well as equilibria with IDC and EDC exist in Infinite Horizon FM economies with some long-lived securities in positive supply for a dense subset of the set of all stochastic endowment and dividend tax rates. Finally, we find that GEI implies that there is no such thing as the "optimal" buying or selling P/E ratio but instead that it is completely determined by constantly changing investors' expectations of company's after-tax profitability.

Keywords: Stochastic Taxation, GEI, Infinite Horizon, Complete Markets, CCAPM, Property Rights
JEL Classification: D50; D52; D53; E13; G10; H20.

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1. INTRODUCTION

Taxes are a part of individuals' and corporations' budget constraints. Therefore, taxes clearly affect equilibrium commodity and asset prices and allocations. Also, changes in various tax rates, especially income tax rates, are driven by the constantly changing political balance of power, and the direction of those changes seems to have been anything but predictable. Thus, it seems entirely appropriate to regard future taxation as stochastic.

But if taxation is stochastic, then it is clearly a risk factor affecting equilibrium asset prices through stochastic discount factors and after-tax dividends. Since this risk cannot be eliminated or substantially reduced by diversification, standard Finance Theory suggests that it ought to be an asset-pricing risk factor, which ought to affect asset prices and allocations.

Surprisingly, however, there has been very little research done to date on the effects of stochastic taxes on equilibrium asset prices and allocations. The research done so far relies on the CCAPM with identical agents and twice-differentiable utility functions and focuses primarily on resolving the so-called "Equity Premium Puzzle." See Magin (2014), Edelstein and Magin (2017) and (2013), DeLong and Magin (2009), Sialm (2009) and (2006).

While resolving the Equity Premium Puzzle is critically important for confirming the validity of the Lucas-Rubenstein CCAPM with identical agents, the role of insecure property rights (stochastic taxation) in Economic Theory is much broader.

For example, do Financial Markets (FM) equilibria exist in the Finite Horizon General Equilibrium of Incomplete Markets (GEI) model with stochastic taxation? Do sufficiently small changes in stochastic tax rates preserve the existence and completeness of FM equilibria? Magin (2015) finds that under reasonable assumptions, FM equilibria exist for all stochastic tax rates imposed on agents' endowments and dividends, except for a closed set of measure zero. Moreover, sufficiently small changes in stochastic taxation preserve the existence and completeness of FM equilibria.

Does an increase in current and future taxes reduce current prices of tradable assets? Magin (2017-4), (2017-2) and (2016-3) study comparative statics of FM equilibria in the Finite Horizon GEI model with respect to changes in stochastic tax rates imposed on agents' endowments and dividends. He shows that under reasonable assumptions, without assuming CRRA and identical agents, an increase in the current dividend tax rate unambiguously reduces current asset prices. The paper also finds that there exists a bound $\overline{B}$ such that for a coefficient of relative risk aversion less than $\overline{B}$, an increase in a future dividend tax rate reduces current price of tradable assets. At the same time, for a coefficient of relative risk aversion greater than $\overline{B}$, an increase in a future dividend tax rate boosts current prices of tradable assets. Finally, for a coefficient of relative risk aversion equal to $\overline{B}$, an increase in a future dividend tax rate leaves current consumption and current price of tradable assets unchanged. As a special case, under additional assumptions, $\overline{B}$ is equal to 1. Also, under reasonable assumptions, an increase in the current endowment tax rate reduces current asset prices, while an increase in a future endowment tax rate boosts current asset prices.

This paper proves the existence of equilibria in the Infinite Horizon GEI model with insecure property rights. Insecure property rights come in the form of the stochastic taxes imposed on agents' endowments and assets' dividends. This paper finds that under reasonable assumptions, Financial Markets (FM) equilibria with Transversality Condition (TC) as well as equilibria with Implicit Debt Constraint (IDC) and Explicit Debt Constraint (EDC) exist in Infinite Horizon FM economies with stochastic taxes and with short-lived securities in zero net supply. Also, this paper finds that under reasonable assumptions, FM equilibria with TC as well as equilibria with IDC and EDC exist in Infinite Horizon FM economies with long-lived securities in zero net supply for a dense subset of the set of all stochastic dividend tax rates. Similarly, this paper finds that under reasonable assumptions, FM equilibria with TC as well as equilibria with IDC and EDC exist in Infinite Horizon FM economies with some long-lived securities in positive supply for a dense subset of the set of all stochastic endowment and dividend tax rates. Finally, we find that GEI implies that there is no such thing as the "optimal" buying or selling P/E ratio but that instead it is completely determined by constantly changing investors' expectations of a company's after-tax profitability.

The paper is organized as follows. Section 2 defines Infinite Horizon FM economies with stochastic taxation of endowments and dividends. Section 3 proves existence of equilibria in Infinite Horizon FM economies with short-lived securities in zero net supply and with stochastic taxation of endowments and dividends. Section 4 proves existence of equilibria in Infinite Horizon FM economies with long-lived securities in zero net supply and with stochastic taxation of endowments and dividends. Section 5 proves existence of equilibria in Infinite Horizon FM economies with some securities in positive supply and with stochastic taxation of endowments and dividends. Section 6 demonstrates that GEI implies that there is no such thing as the "optimal" buying or selling P/E ratio but that instead it is completely determined by constantly changing investors' expectations of company's after-tax profitability. Section 7 concludes.

2. INFINITE HORIZON FM ECONOMIES WITH STOCHASTIC TAXATION OF DIVIDENDS AND ENDOWMENTS

First, we need to introduce several definitions to incorporate stochastic taxation imposed on agents' endowments and assets' dividends and used to finance public good G into the General Equilibrium Theory of Financial Markets.[1] Let ET be the event-tree, I be the set of infinitely living investors-consumers, L be the set of commodities traded on spot markets, K be the set of assets traded on financial markets, such that

$$\max\left[|ET|,\ |L|\right] = \infty,\ |I| < \infty,\ |K| < \infty.$$

We will start with definitions for commodities markets:
DEFINITION: *Let*

$$e_i(\tau_{e_i}) = \{e_i(\xi,\ l,\ \tau_{e_i})\}_{(\xi,\ l)\in ET\times L} \in \mathbb{R}_+^{|ET\times L|}$$

be the individual endowment of agent $i \in I$ and therefore

$$e_i(\xi,\ \tau_{e_i}) = \{e_i(\xi,\ l,\ \tau_{e_i})\}_{l\in L} \in \mathbb{R}_+^{|L|}$$

be the vector of the individual endowment of agent $i \in I$ at node $\xi \in ET$ and

$$e\left(\tau_e\right) = \{e_i(\tau_{e_i})\}_{i\in I} \in \mathbb{R}_+^{|ET\times L\times I|}$$

be the matrix of before-tax individual endowments, where

$$\tau_{e_i} = \{\tau_{e_i}(\xi,\ l)\}_{(\xi,\ l)\in ET\times L} \in [0,\ 1]^{|ET\times L|}$$

be the stochastic tax imposed on the individual endowment of agent $i \in I$ and therefore

$$\tau_{e_i}(\xi) = \{\tau_{e_i}(\xi,\ l)\}_{l\in L} \in [0,\ 1]^{|L|}$$

[1]For basic notions related to Infinite Horizon FM Economies without stochastic taxation see, for example, Magill and Quinzii (2008), (1996) and (1994).

be the vector of the stochastic tax imposed on the individual endowment of agent $i \in I$ at node $\xi \in ET$ and

$$\tau_e = \{\tau_{e_i}\}_{i \in I} = \left\{ \{\tau_{e_i}(\xi,\ l)\}_{(\xi,\ l) \in ET \times L} \right\}_{i \in I} \in [0,\ 1]^{|ET \times L \times I|}$$

be the matrix of taxes imposed on individual endowments.

DEFINITION: Let

$$c_i = \{c_i(\xi,\ l)\}_{(\xi,\ l) \in ET \times L} \in \mathbb{R}_+^{|ET \times L|}$$

be the consumption of agent $i \in I$ and therefore

$$c_i(\xi) = \{c_i(\xi,\ l)\}_{l \in L} \in \mathbb{R}_+^{|L|}$$

be the vector of consumption of agent $i \in I$ at node $\xi \in ET$.

DEFINITION: *Let*

$$p = \{p(\xi,\ l)\}_{(\xi,\ l) \in ET \times L} \in \mathbb{R}^{|ET \times L|}$$

be the matrix of spot prices, such that $p(\xi, 1) = 1\ \forall \xi \in ET$ and therefore

$$p(\xi) = \{p(\xi,\ l)\}_{l \in L} \in \mathbb{R}^{|L|}$$

be the vector of spot prices at node $\xi \in ET$.

We will now turn to financial markets:

DEFINITION: *Let*

$$d(\tau_d) = \{d(\xi,\ l,\ k,\ \tau_d)\}_{(\xi,\ l,\ k) \in ET \times L \times K} \in \mathbb{R}^{|ET \times L \times K|}$$ [2]

be the matrix of assets' dividends and therefore

$$d(\xi, \tau_d) = \{d\,(\xi,\ l,\ k,\ \tau_d)\}_{(l,\ k) \in L \times K} \in \mathbb{R}^{|L \times K|}$$

be the vector of assets' dividends at node $\xi \in ET$, where

$$\tau_d = \{\tau_d(\xi,\ l,\ k)\}_{(\xi,\ k) \in ET \times L \times K} \in [0,\ 1]^{|ET \times L \times K|}$$

be the matrix of taxes imposed on assets' dividends and therefore

$$\tau_d(\xi) = \{\tau_d(\xi,\ l,\ k)\}_{(l,\ k) \in L \times K} \in [0,\ 1]^{|L \times K|}$$

be the vector of taxes imposed on assets' dividends at node $\xi \in ET$.

[2] Following Magill and Quinzii (1996), dividends are paid in bundles of all $|L|$ goods. Also, consistent with the Dividend Clientele Hypothesis (DCH), it is reasonable to assume that assets' dividends d are decreasing functions $d(\tau_d)$ of dividend tax rates τ_d. See Kawano (2013), for example, for a review of the DCH. She estimated that a one percentage point decrease in the dividend tax rate relative to the long-term capital gains tax rate leads to a 0.04 percentage point increase in dividend yields. Several papers, including Chetty and Saez (2005), Brown, Liang and Weisbenner (2007) have documented an increase in dividend payments in response to the 2003 tax changes.

DEFINITION: *We define the space of asset dividends as*

$$D \subset \mathbb{R}^{|ET \times L \times K|}.$$

DEFINITION: *Let $\xi(k) \in ET$ be the node of issue for an asset $k \in K$. Define the set ζ of all nodes of issue of existing financial contracts as*

$$\zeta = \{\xi(k) \mid k \in K\}.$$

DEFINITION: *We define the set of all actively traded financial contracts at node $\xi \in ET$ as*

$$K(\xi) = \{k \in K \mid \xi \in ET\left(\xi(k)\right),\, \exists \xi' \in ET^{+}(\xi(k)) \ s.t.\ d\left(\xi',\ k,\ \tau_d\right) > 0\}.$$

DEFINITION: *Let ζ be the set of all nodes of issue of existing financial contracts and d be the matrix of dividends. Then we call the pair*

$$\mathcal{A}(\tau_d) = (\zeta,\, (1 - \tau_d)\, d(\tau_d))$$

the financial structure.

DEFINITION (Short-lived Security): *We call a security $k \in K$ short lived if*

$$d(\tau_d) = \{d\left(\xi,\ l,\ k,\ \tau_d\right)\}_{(\xi,\ l,\ k) \in ET \times L \times K} \in \mathbb{R}^{|ET \times L \times K|}$$

is s.t.

$$d\left(\xi,\ l,\ k,\ \tau_d\right) = 0\ \forall \xi \in ET \setminus \xi^{+}(k).$$

DEFINITION (Long-lived Security): *We call a security $k \in K$ long lived if*

$$d = \{d\left(\xi,\ l,\ k,\ \tau_d\right)\}_{(\xi,\ k) \in ET \times K} \in \mathbb{R}^{|ET \times K|}$$

is s.t.

$$\exists \xi \in ET \setminus \xi^{+}(k) \ \textit{with}\ d\left(\xi,\ l,\ k,\ \tau_d\right) > 0.$$

DEFINITION: *Let*

$$z_i = \{z_i(\xi,\ k)\}_{(\xi,\ k) \in ET \times K} \in \mathbb{R}^{|ET \times K|}$$

be the asset portfolio held by agent $i \in I$ and therefore

$$z_i(\xi) = \{z_i(\xi,\ k)\}_{k \in K} \in \mathbb{R}^{|K|}$$

be the asset portfolio held by agent $i \in I$ at node $\xi \in ET$, where the number of shares of asset $k \in K$ held by agent $i \in I$ at node $\xi \in ET$

$$z_i(\xi,\, k) \in \mathbb{R}$$

is s.t.

$$z_i(\xi,\, k) = 0\ \forall \xi \in ET \ s.t.\ k \in K \setminus K(\xi),\, \forall k \in K,$$

i.e., $z_i(\xi,\, k) = 0$ *if an asset* $k \in K$ *is not actively traded at node* $\xi \in ET$,

$$z_i(\xi) = \{z_i(\xi,\ k)\}_{k \in K} \in \mathbb{R}^{|K|}$$

be the vector of the number of shares of the $|K|$ *assets held by agent* $i \in I$ *at node* $\xi \in ET$,

$$z_i = \{z_i(\xi)\}_{\xi \in ET} \in \mathbb{R}^{|ET \times K|}$$

be the asset portfolio held by agent $i \in I$.

DEFINITION: *We define the portfolio space as*

$$\mathcal{Z} = \mathbb{R}^{|ET \times K|}.$$

DEFINITION: *Let*

$$q = \{q(\xi,\ k)\}_{(\xi,\, k) \in ET \times K} \in \mathbb{R}^{|ET \times K|}$$

be the matrix of asset prices and therefore

$$q(\xi) = \{q(\xi,\ k)\}_{k \in K} \in \mathbb{R}^{|K|}$$

be the vector of asset prices at node $\xi \in ET$.

DEFINITION: *Let*

$$q(\xi,\, k) \in \mathbb{R}$$

be the price of asset $k \in K$ *at node* $\xi \in ET$, *where*

$$q(\xi,\, k) = 0 \ \forall \xi \in ET \ s.t. \ k \in K \setminus K(\xi), \forall k \in K,$$

i.e., $q(\xi,\, k) = 0$ *if an asset* $k \in K$ *is not actively traded at node* $\xi \in ET$,

$$q(\xi) = \{q(\xi,\ k)\}_{k \in K} \in \mathbb{R}^{|K|}$$

be the vector of prices of the $|K|$ *assets at node* $\xi \in ET$,

$$q = \{q(\xi)\}_{\xi \in ET} \in \mathbb{R}^{|ET \times K|}$$

be the matrix of prices of the $|K|$ *assets.*

DEFINITION: *We define the space of asset prices as*

$$\mathcal{Q} = \mathbb{R}^{|ET \times K|}.$$

DEFINITION: *Banach lattice* $(X,\, \|\cdot\|)$ *is a normed space s.t.* $(X,\, \|\cdot\|)$ *is a Banach space s.t.* $\forall x, y \in X \ |x| \leq |y| \Rightarrow \|x\| \leq \|y\|$, *where* $|x| = x \vee (-x)$.

We are now ready to define an Infinite Horizon FM Economy with assets in zero net supply and with stochastic taxation imposed on agents' endowments and assets' dividends and used to finance government spending.

DEFINITION: *We denote by*

$$\mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e\,(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d))$$

an Infinite Horizon FM Economy with assets in zero net supply and with stochastic taxation

$$\tau = (\tau_e,\ \tau_d) \in [0,\ 1]^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|},$$

where
ET is the event tree,
X is the commodity space,
X and X' are Banach lattices, s.t. $\boldsymbol{\tau}$ is a consistent topology on X, i.e., $(X,\ \boldsymbol{\tau})' = X'$,
$X_i = X_+$ is the individual consumption set $\forall i \in I$,
$e = \{e_i\}_{i\in I}$ is the total endowment s.t. $e_i \in X_i\ \forall i \in I$ is the set of individual endowments,
$\succeq = \{\succeq_i\}_{i\in I}$ is the set of agents' preferences s.t. agent's preferences $\succeq_i$on $X_i = X_+$ are given by the utility function

$$U_i(c_i,\ G) = \sum_{(\xi,\ l)\in ET\times L} \Pr(\xi)\cdot b_i^{T(\xi)}\cdot[u_i(c_i(\xi,\ l)) + v_i(G(\xi,\ l))]\ \forall i \in I,$$

where the government spending $G = \{G(\xi,\ l)\}_{(\xi,\ l)\in ET\times L} \in X_+$ is given by

$$G(\xi,\ l) = \sum_{i\in I}\tau_{e_i}(\xi,\ l)\cdot e_i(\xi,\ l,\ \tau_{e_i}) + \sum_{k\in K}\overline{z}(k)\cdot\tau_d(\xi,\ l,\ k)\cdot d(\xi,\ l,\ k,\ \tau_d)\ \forall(\xi,\ l)\in ET\times\{1\}$$

and $\overline{z}(k)$ is the total number of outstanding shares of asset $k \in K$.

Next, $\forall(p,\ q) \in \mathbb{R}^{|ET\times K|}\times Q$ we will define the $ET\times ET$ payoff matrix $W(q,\ (1-\tau_d)\cdot p\cdot d(\tau_d))$, which will significantly simplify writing of agents' budget constraints and defining the notions of No Arbitrage Condition and Complete Markets.[3]

DEFINITION: *Let*

$$\mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e\,(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d))$$

be an Infinite Horizon FM Economy with stochastic taxation. Then

$$\forall(p,\ q) \in \mathbb{R}^{|ET\times K|}\times Q$$

we define the $ET\times ET$ payoff matrix

$$\begin{gathered} W(q,\ (1-\tau_d)\cdot p\cdot d(\tau_d))\ \forall(p,\ q)\in\mathbb{R}^{|ET\times K|}\times Q\ \textit{as} \\ W_{\xi,\ \xi^+}(q,\ (1-\tau_d)\cdot p\cdot d(\tau_d)) = q(\xi^+) + \left(1-\tau_d(\xi^+)\right)\cdot p\cdot d(\xi^+,\ \tau_d)), \\ W_{\xi,\ \xi}(q,\ (1-\tau_d)\cdot p\cdot d(\tau_d)) = -q(\xi), \\ W_{\xi,\ \xi'}(q,\ (1-\tau_d)\cdot p\cdot d(\tau_d)) = 0\ \forall\xi' \notin \xi^+,\ \xi' \neq \xi. \end{gathered}$$

[3] See Magill and Quinzii (1996) for the original definition of the payoff matrix without stochastic taxation.

MATRIX $W(q,\,(1-\tau_d)\cdot p\cdot d(\tau_d))$

$\lvert K\rvert$ Columns for ξ_0		$\lvert K\rvert$ Columns for ξ^-	$\lvert K\rvert$ Columns for ξ		
$-q(\xi_0,\,\tau)$	0	0	0	0	ξ_0
$q(\xi_0^+,\tau)+$ $+\left(1-\tau_d(\xi_0^+)\right)\cdot$ $p(\xi_0^+,\tau_d)\cdot d(\xi_0^+,\,\tau_d)$	$\dots$	$\dots$	0	0	ξ_0^+
0	$\dots$	$\dots$	0	0	
0	0	$q(\xi,\,\tau)+$ $+(1-\tau_d(\xi))\cdot$ $p(\xi,\,\tau_d)\cdot d(\xi,\,\tau_d)$	$-q(\xi,\tau)$	0	ξ
0	$\dots$	$\dots$	0	$\dots$	
0	0	0	$q(\xi^+,\tau)+$ $+(1-\tau_d(\xi^+))\cdot$ $p(\xi^+,\tau_d)\cdot d(\xi^+,\,\tau_d)$	$\dots$	ξ^+
0	0	0	0	$\dots$	

We are now ready to define the notions of No-Arbitrage Condition (NAC) and Complete Markets.

DEFINITION: *Let*

$$\mathcal{E}_\infty(ET,\quad(X,\ X'),\,(1-\tau_e)\cdot e\,(\tau_e)\,,\,\succeq,\,\mathcal{A}(\tau_d))$$

be an Infinite Horizon FM Economy with stochastic taxation. Then we say that No-Arbitrage Condition (NAC) holds and a pair

$$(p,\,q)\in\mathbb{R}_+^{|ET\times L|}\times Q$$

constitutes a NA system of prices if

$$\exists\pi=\{\pi\,(\xi)\}_{\xi\in ET}\in\mathbb{R}_{++}^{|ET|}\ s.t.\ \pi\cdot W(q,\,(1-\tau_d)\cdot p\cdot d(\tau_d))=0.$$

and we say that markets are complete if

$$\exists!\pi=\{\pi\,(\xi)\}_{\xi\in ET}\in\mathbb{R}_{++}^{|ET|}\ s.t.\ \pi\cdot W(q,\,(1-\tau_d)\cdot p\cdot d(\tau_d))=0.$$

We are now ready to introduce the notion of the Explicit Debt Constraint (EDC). The Explicit Debt Constraint imposes an explicit bound on how short the investor is allowed to go.

DEFINITION: *Define the Explicit Debt Constraint* (EDC) $M > 0$ *for agent* $i \in I$ *as*

$$q\left(\xi\right) z_i\left(\xi\right) = \sum_{k \in K} q\left(\xi,\ k\right) \cdot z_i\left(\xi,\ k\right) \geqslant -M \ \forall \xi \in ET.$$

Next we will define the budget set with the Explicit Debt Constraint (EDC) $M > 0$.

DEFINITION: *Define the budget set with the* EDC $M > 0$ *for agent* $i \in I$ *as*

$$B_{\infty}^{M}(p,\ q,\ (1-\tau_{e_i})\, e_i(\tau_{e_i}),\ \mathcal{A}(\tau_d)) =$$
$$= \left\{ c_i \in X_+ \left| \begin{array}{c} \exists z_i \in \mathcal{Z} \ s.t. \ q\left(\xi\right) z_i\left(\xi\right) \geqslant -M \ \forall \xi \in ET \ and \\ p \cdot c_i - p \cdot (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}) = W(q,\ \ (1-\tau_d) \cdot p \cdot d(\tau_d)) \cdot z_i \end{array} \right. \right\}.$$

We will use it to define the notion of an FM equilibrium *with the* EDC $M > 0$ for an Infinite Horizon FM economy with stochastic taxation.

DEFINITION: *An FM equilibrium with* EDC $M > 0$ *of the Infinite Horizon economy*

$$\mathcal{E}_{\infty}(ET,\ (X,\ X')\,,\ (1-\tau_e) \cdot e\left(\tau_e\right),\ \succeq,\ \mathcal{A}(\tau_d))$$

is a combination

$$(\{(\overline{c}(\tau)_i,\ \overline{z}_i(\tau))\}_{i \in I}\,,\ (\overline{p}(\tau),\ \overline{q}(\tau))) \in \left(X_+^{|I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}^{|ET \times L|} \times Q\ \right) \ s.t.$$
$$(\overline{c}_i(\tau),\ \overline{z}_i(\tau)) \in \arg\max\left\{U_i(c_i) \mid (c_i,\ z_i) \in B_{\infty}^{M}(\overline{p}(\tau),\ \overline{q}(\tau),\ (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}),\ \mathcal{A}(\tau_d))\right\} \ \forall i \in I,$$
$$\sum_{i \in I} \overline{c}_i(\tau) = \sum_{i \in I} (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}),$$
$$\sum_{i \in I} \overline{z}_i(\tau) = 0.$$

We are now ready to introduce the notion of the Implicit Debt Constraint (IDC).The Implicit Debt Constraint imposes an Implicit bound on how much short the investor is allowed to go.

DEFINITION: *Define the Implicit Debt Constraint* (IDC) *for agent* $i \in I$ *as*

$$qz_i = \left\{ q\left(\xi\right) z_i\left(\xi\right) = \sum_{k \in K} q\left(\xi,\ k\right) \cdot z_i\left(\xi,\ k\right) \mid \xi \in ET \right\} \in L_{\infty}(ET).$$

Similarly, we will introduce the notion of the budget set with the Implicit Debt Constraint (IDC).

DEFINITION: *Define the budget set with the* IDC *for agent* $i \in I$ *as*

$$\mathcal{B}_{\infty}^{DC}(p,\ q,\ (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}),\ \mathcal{A}(\tau_d)) =$$
$$= \left\{ c_i \in X_+ \left| \begin{array}{c} \exists z_i \in \mathcal{Z} \ s.t. \ qz_i \in L_{\infty}(ET) \ and \\ p \cdot c_i - p \cdot (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}) = W(q,\ \ (1-\tau_d) \cdot p \cdot d(\tau_d)) \cdot z_i \end{array} \right. \right\}.$$

We will use it to define the notion of an FM equilibrium *with the* IDC for an Infinite Horizon FM economy with stochastic taxation.

DEFINITION: *An equilibrium with Implicit Debt Constraint IDC of the Infinite Horizon FM economy*

$$\mathcal{E}_\infty(ET, \ \ (X, \ X') \, , (1-\tau_e) \cdot e\left(\tau_e\right), \succeq, \ \mathcal{A}(\tau_d))$$

is a combination

$$\begin{gathered} (\{(\overline{c}_i(\tau), \ \overline{z}_i(\tau))\}_{i\in I} \, , (\overline{p}(\tau), \ \overline{q}(\tau))) \in \left(X_+^{|I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}^{|ET\times L|} \times Q \ \right) \ s.t. \\ (\overline{c}_i(\tau), \, \overline{z}_i(\tau)) \in \arg\max \left\{U_i(c_i) \mid (c_i, \ z_i) \in B_\infty^{DC}(\overline{p}, \ \overline{q}, \ (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}), \ \mathcal{A}(\tau_d))\right\} \, \forall i \in I, \\ \sum_{i\in I} \overline{c}_i(\tau) = \sum_{i\in I} (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}), \\ \sum_{i\in I} \overline{z}_i(\tau) = 0. \end{gathered}$$

As we will see later, under reasonable assumptions, the existence of equilibria with EDC and IDC in the Infinite Horizon FM economy are equivalent. Next, we will introduce the notion of the Transversality Condition (TC).

DEFINITION (Transversality Condition): *Let*

$$\mathcal{E}_\infty(ET, \ \ (X, \ X') \, , (1-\tau_e) \cdot e\left(\tau_e\right), \succeq, \ \mathcal{A}(\tau_d))$$

be an FM economy and

$$(p, \, q) \in \mathbb{R}_+^{|ET\times L|} \times Q$$

be a No-arbitrage system of prices, i.e.,

$$\exists! \pi_i = \{\pi\left(\xi\right)\}_{\xi\in ET} \in \mathbb{R}_{++}^{|ET|} \ s.t. \ \ \pi_i \cdot W(q, \, (1-\tau_d) \cdot p \cdot d(\tau_d)) = 0.$$

Suppose also that

$$(c_i, \, z_i) \in B(p, \ q, \ (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}), \ \mathcal{A}(\tau_d)).$$

Then we define the Transversality Condition (TC) for agent $i \in I$ as

$$\lim_{T\longrightarrow\infty} \sum_{\xi'\in ET_{t+T}} \pi_i\left(\xi'\right) q\left(\xi'\right) z_i\left(\xi'\right) = 0.$$

Finally, we will introduce the notion of the budget set with TC.

DEFINITION: *Define the budget set with the TC for agent $i \in I$ as*

$$\mathcal{B}_\infty^{TC}(p, \ q, \ \pi_i, \, (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}), \ \mathcal{A}(\tau_d)) =$$

$$= \left\{ c_i \in X_+ \left| \begin{array}{c} \exists z_i \in \mathcal{Z} \ s.t. \ \forall \xi \in ET \\ \lim\limits_{T\longrightarrow\infty} \sum\limits_{\xi'\in ET_{t+T}(\xi)} \pi_i\left(\xi'\right) q\left(\xi'\right) z_i\left(\xi'\right) = 0, \\ p \cdot c_i - p \cdot (1-\tau_{e_i}) \cdot e_i(\tau_{e_i}) = W(q, \ \ (1-\tau_d) \cdot p \cdot d(\tau_d)) \cdot z_i \end{array} \right. \right\}.$$

DEFINITION: *An equilibrium with TC of the Infinite Horizon FM economy with assets in zero net supply*

$$\mathcal{E}_\infty(ET,\ (X,\ X')\,,\ (1-\tau_e)\cdot e\,(\tau_e)\,,\ \succeq,\ \mathcal{A}(\tau_d))$$

is a combination

$$\begin{gathered}
\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I}\,,\ \left(\overline{p}(\tau),\ \overline{q}(\tau),\ \ \{\overline{\pi}_i(\tau)\}_{i\in I}\right)\right)\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\ \right)\times {X'_+}^{|I|}\ s.t.\\
(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\in\arg\max\left\{U_i(c_i)\mid(c_i,\ z_i)\in B_\infty^{TC}(\overline{p}(\tau),\ \overline{q}(\tau),\ \pi_i,\ \ (1-\tau_{e_i})\cdot e_i(\tau_{e_i}),\ \mathcal{A}(\tau_d))\right\}\ \forall i\in I,\\
\overline{\pi}_i\,(\xi,\ \tau)>0\ \forall\xi\in ET\ and\ \overline{P}_i(\tau)\in(L_1(ET\times L))_+\ \forall i\in I,\\
\overline{P}_i(\tau)=\left\{\overline{P}_i\,(\xi,\ \tau))\mid\xi\in ET\right\}=\left\{\overline{\pi}_i\,(\xi,\ \tau)\,\overline{p}_i\,(\xi,\ \tau)\mid\xi\in ET\right\},\\
\overline{c}_i(\tau)\in\arg\max\left\{U_i(c_i)\mid c_i\in B_\infty(\overline{P},\ e_i)\right\}\ \forall i\in I,\\
B_\infty^{CM}(\overline{P}_i(\tau),\ (1-\tau_{e_i})\cdot e_i\,(\tau_{e_i}))=\left\{c_i\in X_+\mid\overline{P}_i(\tau)(c_i-(1-\tau_{e_i})e_i\,(\tau_{e_i}))\leqslant 0\right\},\\
\overline{\pi}_i(\tau)\cdot W(\overline{q}(\tau),\ (1-\tau_d)\cdot\overline{p}(\tau)\cdot d(\tau_d))=0\ \forall i\in I,\\
\sum_{i\in I}\overline{c}_i(\tau)=\sum_{i\in I}(1-\tau_{e_i})\cdot e_i(\tau_{e_i}),\\
\sum_{i\in I}\overline{z}_i(\tau)=0.
\end{gathered}$$

DEFINITION: *We denote by*

$$\mathcal{E}_\infty(ET,\ (X,\ X')\,,\ (1-\tau_e)\cdot e\,(\tau_e)\,,\ \succeq)$$

an Infinite Horizon CM (Arrow-Debreu) Economy with assets in zero net supply and with stochastic taxation of endowments

$$\tau_e\in[0,\ 1]^{|ET\times L\times I|},$$

where all notations from the above definition of FM economies apply.

DEFINITION: *A CM (Arrow-Debreu) equilibrium of the Infinite Horizon CM (Arrow-Debreu) economy*

$$\mathcal{E}_\infty(ET,\ (X,\ X')\,,\ (1-\tau_e)\cdot e\,(\tau_e)\,,\ \succeq)$$

is a pair

$$\begin{gathered}
(\{\overline{c}_i(\tau)\}_{i\in I}\,,\ \overline{P}(\tau))\in X_+^{|I|}\times\mathbb{R}_+^{|ET\times L|}s.t.\\
\overline{c}_i(\tau)\in\arg\max\left\{U_i(c_i)\mid c_i\in B_\infty(\overline{P},\ e_i)\right\}\ \forall i\in I,\\
B_\infty^{CM}(\overline{P}_i(\tau),\ (1-\tau_{e_i})\cdot e_i\,(\tau_{e_i}))=\left\{c_i\in X_+\mid\overline{P}_i(\tau)(c_i-(1-\tau_{e_i})e_i\,(\tau_{e_i}))\leqslant 0\right\},\\
\sum_{i\in I}\overline{c}_i(\tau)=\sum_{i\in I}(1-\tau_{e_i})\cdot e_i(\tau_{e_i}).
\end{gathered}$$

As we will see later, under reasonable assumptions, the existence of equilibria with EDC, IDC and TC in the Infinite Horizon FM economy are equivalent.

For the rest of the paper, unless otherwise specified, we impose the following assumptions on

$$\mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d))$$

A1: *ET be s.t.*

$$b(\xi) = \mid \xi^+ \mid < \infty\ \forall \xi \in ET.$$

A2:

$$X = L_\infty(ET\times L) = \left\{ x = \{x(\xi,\ l)\}_{(\xi,\ l)\in ET\times L} \in \mathbb{R}^{|ET\times L|} \mid \parallel x \parallel_\infty = \sup_{(\xi,\ l)\in ET\times L} \mid x(\xi,\ l) \mid < \infty \right\},$$

$$X' = L_1(ET\times L) = \left\{ x = \{x(\xi,\ l)\}_{(\xi,\ l)\in ET\times L} \in \mathbb{R}^{|ET\times L|} \mid \parallel x \parallel_1 = \sum_{(\xi,\ l)\in ET\times L} \mid x(\xi,\ l) \mid < \infty \right\}.$$

A3: *The number of agents*

$$|I| < \infty.$$

A4: *Individual endowments*

$$(1-\tau_{e_i})\cdot e_i \in X_i = X_+\ \forall i\in I$$

be s.t.

$$\exists m,\ m' \in \mathbb{R}_{++},\ m' > m > 0$$

and

$$(1-\tau_{e_i})\cdot e_i \in [m,\ \infty[^{|ET\times L|}\ \forall i \in I.$$

A5: *Total endowment*

$$\sum_{i\in I}(1-\tau_{e_i})\cdot e_i \in [0,\ m'[^{|ET\times L|}.$$

A6: *Agents' preferences $\succeq_i$ on*

$$X_i = X_+$$

are given by the utility function

$$U_i(c_i,\ G) = \sum_{(\xi,\ l)\in ET\times L} \Pr(\xi)\cdot b_i^{T(\xi)}\cdot[u_i(c_i(\xi,\ l)) + v_i(G(\xi,\ l))]\ \forall i\in I$$

where $u_i(\cdot)$ is continuous, concave and monotone utility function s.t. $u_i(0) = 0\ \forall i \in I$, i.e., $\forall G$, $U_i(\cdot,\ G)$ is Mackey, i.e., $\boldsymbol{\tau}(X,\ X')$-continuous, concave and monotone utility function.

A7: *Financial structure $\mathcal{A}(\tau_d)$ is s.t. it is composed solely of short-lived securities and*

$$|K(\xi)| < \infty.$$

A8: *Existence of the risk-free bond*

$$\forall \xi \in ET\ \exists k_\xi \in K(\xi)\ s.t.\ d(\xi',\ l,\ k_\xi,\ \tau_d) = \begin{cases} 1\ \forall(\xi', l) \in \xi^+ \times \{1\} \\ 0\ \forall(\xi', l) \in [ET\times L] \setminus [\xi^+\times\{1\}] \end{cases}.$$

We are now ready to prove the existence of FM equilibria in Infinite Horizon FM economies with short-lived securities in zero net supply.

3. EXISTENCE OF FM EQUILIBRIUM IN INFINITE HORIZON FM ECONOMIES WITH SHORT-LIVED SECURITIES IN ZERO NET SUPPLY

THEOREM 1 (Existence of FM Equilibrium with TC and Short-lived Securities): *Let*

$$\mathcal{E}_\infty(ET,\,(X,\;X'),\,(1-\tau_e)\cdot e\,(\tau_e)\,,\;\;\succeq,\,\mathcal{A}(\tau_d))$$

be s.t. **A1-A8** *hold. Then for this economy*
a) *$\exists$ an FM equilibrium with TC*

$$\left(\{(\overline{c}_i\,(\tau)\,,\;\overline{z}_i\,(\tau))\}_{i\in I}\,,\left(\overline{p}\,(\tau)\,,\;\overline{q}\,(\tau)\,,\;\;\{\overline{\pi}_i(\tau)\}_{i\in I}\right)\right)\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right)\times X_+'^{\,|I|},$$

b) *$\exists$ an FM equilibrium with IDC*

$$\left(\{(\overline{c}_i\,(\tau)\,,\;\overline{z}_i\,(\tau))\}_{i\in I}\,,\,(\overline{p}\,(\tau)\,,\;\overline{q}\,(\tau))\right)\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right),$$

c) *$\exists$ an FM equilibrium with EDC $M\,(\tau)>0$*

$$\left(\{(\overline{c}_i\,(\tau)\,,\;\overline{z}_i\,(\tau))\}_{i\in I}\,,\,(\overline{p}\,(\tau)\,,\;\overline{q}\,(\tau))\right)\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right),$$

s.t.

$$EDC\;M\,(\tau)>0$$

is never binding.
d) *$\exists$ a CM equilibrium*

$$\left(\{(\overline{c}_i\,(\tau))\}_{i\in I}\,,\,\left(\overline{P}\,(\tau)\right)\right)\in X_+^{|I|}\times X_+'$$

for the CM economy

$$\mathcal{E}_\infty(ET,\,(X,\;X')\,,\,(1-\tau_e)\cdot e\,(\tau_e)\,,\;\;\succeq).$$

PROOF: a) Let us establish the existence of a TC equilibrium for this economy. Fix an arbitrary

$$\tau\in[0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times K|}\text{ s.t.}$$
$$\mathcal{E}_\infty(ET,\,(X,\;X')\,,\,(1-\tau_e)\cdot e\,(\tau_e)\,,\succeq',\,\mathcal{A}(\tau_d))$$

satisfies **Al-A8,** except that instead of **A6** we assume
A6': Agents' preferences $\succeq_i'$ on

$$X_i=X_+$$

are given by the utility function

$$U_i(c_i)=\sum_{(\xi,\,l)\in ET\times L}\Pr(\xi)\cdot b_i^{T(\xi)}\cdot u_i(c_i(\xi,\,l))\;\forall i\in I.$$

where u_i is continuous, concave and monotone s.t. $u_i(0)=0\;\forall i\in I$.
Then we can conclude by Theorem 5.1 on p. 868 of Magill and Quinzii (1994) that

$$\exists(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i\in I}))\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right)\times {X'_+}^{|I|}$$

an FM equilibrium with TC for

$$\mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq',\ \mathcal{A}(\tau_d)).$$

Clearly, using separability of agents' utility functions and the fact that government spending G is an exogenous variable, we can conclude

$$\begin{gathered}(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\in\\ \in\arg\max\left\{U_i(c_i)\mid(c_i,\ z_i)\in\mathcal{B}_\infty^{TC}(\overline{p}(\tau),\ \overline{q}(\tau),\ \overline{\pi}_i(\tau),\ \ (1-\tau)\cdot e_i(\tau),\ \mathcal{A}(\tau))\right\}=\\ \arg\max\left\{U_i(c_i,\ G)\mid(c_i,\ z_i)\in\mathcal{B}_\infty^{TC}(\overline{p}(\tau),\ \overline{q}(\tau),\ \overline{\pi}_i(\tau),\ \ (1-\tau)\cdot e_i(\tau),\ \mathcal{A}(\tau))\right\}\ \forall i\in I.\end{gathered}$$

So

$$\begin{gathered}(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\in\\ \in\arg\max\left\{U_i(c_i,\ G)\mid(c_i,\ z_i)\in\mathcal{B}_\infty^{TC}(\overline{p}(\tau),\ \overline{q}(\tau),\ \overline{\pi}_i(\tau),\ \ (1-\tau)\cdot e_i,\ \mathcal{A}(\tau))\right\}\ \forall i\in I.\end{gathered}$$

Therefore,

$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i\in I}))\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right)\times {X'_+}^{|I|}$$

is also an FM equilibrium with TC for

$$\mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d)).$$

b) Now, we can conclude by Theorem 5.2 on p. 868 of Magill and Quinzii (1994) that

$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},(\overline{p}(\tau),\ \overline{q}(\tau)))\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right)$$

is also an IDC equilibrium for this economy.
c) We can also conclude by Corollary 5.3 on p. 868 of Magill and Quinzii (1994) that

$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},(\overline{p}(\tau),\ \overline{q}(\tau)))\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right)$$

is an FM equilibrium with $EDC\ M(\tau)>0$ for this economy, s.t. $EDC\ M(\tau)>0$ is never binding. ■

This result is philosophically parallel to the finding of Magin (2015) that under reasonable assumptions in a Finite Horizon FM economy with a financial structure composed solely of short-lived securities, FM equilibria exist for all stochastic tax rates imposed on agents' endowments and assets' dividends.

We are now ready to prove the existence of FM equilibria in Infinite Horizon FM economies with long-lived securities in zero net supply.

4. EXISTENCE OF FM EQUILIBRIUM IN INFINITE HORIZON FM ECONOMIES WITH LONG-LIVED SECURITIES IN ZERO NET SUPPLY

THEOREM 2 (Existence of FM Equilibrium with TC and Long-lived Securities): *Let*

$$\mathcal{E}_{\infty}(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d))$$

be s.t. **A1-A8** *hold, except that instead of* **A7** *we assume*
A7': *Financial structure $\mathcal{A}(\tau_d)$ is s.t.*

$$d(\xi,\ l,\ k,\ \tau_d) \in L_{\infty}(ET \times L)\ \forall k \in K$$

and

$$|K(\xi)| < \infty.$$

A9: *Function*

$$f : \left([0,1]^{|ET\times L\times K|},\ \mathbb{T}\right) \longrightarrow (D,\ \|\cdot\|_{\infty}),$$

where $\mathbb{T}$ is the product topology on $[0,1]^{|ET\times K|}$ defined as

$$f(\tau_d) = (1-\tau_d)\cdot d(\tau_d)$$

is a homeomorphism. Fix

$$\tau_e = \overline{\tau}_e \in [0,1[^{|ET\times L\times I|}.$$

Then

$$\exists DT \subset [0,1]^{|ET\times L\times K|}\ s.t.$$
$$\overline{(DT)}_{\mathbb{T}} = [0,1]^{|ET\times L\times K|}$$

and

$$\forall \tau_d \in DT$$

for the economy

$$\mathcal{E}_{\infty}(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d))$$

a) $\exists$ *an FM equilibrium with TC*

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\left(\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i\in I}\right)\right) \in \left(X_+^{|I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}^{|ET\times L|} \times Q\right) \times {X'_+}^{|I|},$$

b) $\exists$ *an FM equilibrium with IDC*

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))\right) \in \left(X_+^{|I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}^{|ET\times L|} \times Q\right),$$

c) $\exists$ *an FM equilibrium with EDC $M(\tau) > 0$, s.t. EDC $M(\tau) > 0$ is never binding*

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))\right) \in \left(X_+^{|I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}^{|ET\times L|} \times Q\right).$$

d) $\exists$ *a CM equilibrium*

$$\left(\{(\overline{c}_i(\tau))\}_{i\in I},\ \left(\overline{P}(\tau)\right)\right) \in X_+^{|I|} \times X'_+$$

for the CM economy

$$\mathcal{E}_{\infty}(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq).$$

PROOF: a) Consider a function

$$f:\left([0,1]^{|ET\times L\times K|},\ \mathbb{T}\right)\longrightarrow (D,\ \|\cdot\|_\infty)\subset L_\infty(ET\times K\times L)$$

defined as

$$f(\tau_d)=(1-\tau_d)\cdot d\left(\tau_d\right).$$

By Theorem 5.1 on p. 151 of Magill and Quinzii (1996)

$$\exists \boldsymbol{D}\subset D$$

s.t.

$$\overline{(\boldsymbol{D})}_{\|\cdot\|_\infty}=D$$

and

$$\exists(\{(\overline{c}_i(d),\ \overline{z}_i(d))\}_{i\in I},\ \ (\overline{p}(d),\ \overline{q}(d),\ \ \{\overline{\pi}_i(d)\}_{i\in I}))\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\ \right)\times {X'_+}^{|I|}$$

an FM equilibrium with TC $\forall d\in\boldsymbol{D}$.
Set

$$DT=f^{-1}\left(\boldsymbol{D}\right)$$

Therefore, since

$$f:[0,1]^{|ET\times L\times K|}\longrightarrow D$$

is a bijection and thus surjective (onto) we can conclude that

$$f(DT)=ff^{-1}\left(\boldsymbol{D}\right)=\boldsymbol{D}\subset D.$$

Therefore,

$$f(DT)=\boldsymbol{D}\subset D.$$

Hence,

$$\overline{(f(DT))}_{\|\cdot\|_\infty}=\overline{(\boldsymbol{D})}_{\|\cdot\|_\infty}=D.$$

Since

$$f:\left([0,1]^{|ET\times L\times K|},\ \mathbb{T}\right)\longrightarrow(D,\ \|\cdot\|_\infty)$$

is a homeomorphism and therefore closed mapping we have that

$$\overline{(f(DT))}_{\|\cdot\|_\infty}=f(\overline{(DT)}_{\mathbb{T}}).$$

Thus,

$$f(\overline{(DT)}_{\mathbb{T}})=\overline{(\boldsymbol{D})}_{\|\cdot\|_\infty}=D.$$

But

$$f\left([0,1]^{|ET\times L\times K|}\right)=D.$$

Therefore, since

$$f:[0,1]^{|ET\times L\times K|}\longrightarrow D$$

is a bijection and thus injective (one-to-one), we can conclude that

$$\overline{(DT)}_{\mathbb{T}} = f^{-1}f(\overline{(DT)}_{\mathbb{T}}) = f^{-1}\left(\overline{(\boldsymbol{D})}_{\|\cdot\|_\infty}\right) = f^{-1}(D)$$

and

$$[0,1]^{|ET\times L\times K|} = f^{-1}f\left([0,1]^{|ET\times L\times K|}\right) = f^{-1}(D).$$

Thus,

$$\overline{(DT)}_{\mathbb{T}} = [0,1]^{|ET\times L\times K|}.$$

Therefore,

$$\forall \tau_d \in DT$$
$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i\in I})) \in \left(X_+^{|I|}\times \mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right)\times X'^{|I|}_+$$

is also an FM equilibrium with TC for the economy

$$\mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d)).$$

b) Now, we can conclude by Proposition 5.3 on p. 153 of Magill and Quinzii (1996) that

$$\forall \tau_d \in DT$$
$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))) \in \left(X_+^{|I|}\times \mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right)$$

is also an IDC equilibrium for this economy.
c) We can conclude by Corollary 5.4 on p. 154 of Magill and Quinzii (1996) that

$$\forall \tau_d \in DT$$
$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))) \in \left(X_+^{|I|}\times \mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right)$$

is an FM equilibrium with $EDC\ M(\tau) > 0$ for this economy, s.t. $EDC\ M(\tau) > 0$ is never binding.

d) Obvious. ■

The result is very intuitive. It simply states that the dividend tax rate for which an FM equilibrium with TC exists is never too far away.

This result is philosophically parallel to the finding of Magin (2015) that under reasonable assumptions in a Finite Horizon FM economy with a financial structure with long-lived securities, FM equilibria exist for all stochastic tax rates imposed on assets' dividends except for a closed set of measure zero.

We are now ready to prove the existence of FM equilibria in Infinite Horizon FM economies with some securities in positive supply.

5. EXISTENCE OF FM EQUILIBRIUM IN INFINITE HORIZON FM ECONOMIES WITH SOME SECURITIES IN POSITIVE SUPPLY

DEFINITION: *We denote by*

$$\mathcal{E}_\infty(ET,\,(X,\;X')\,,\,(1-\tau_e)\cdot e\,(\tau_e)\,,\,\succeq,\,\mathcal{A}(\tau_d),\,\delta)$$

an Infinite Horizon FM Economy with assets in positive supply and with stochastic taxation

$$\tau=(\tau_e,\,\tau_d)\in[0,1[^{|ET\times L\times I|}\times[0,\;1]^{|ET\times L\times K|},$$

where all notations are the same as in the Infinite Horizon FM Economy with assets in zero net supply except that

$$\delta=\{\delta_k\}_{k\in K}\in\mathbb{R}_+^{|K|}$$

be the total supply of shares s.t.

$$\delta_k=\begin{cases}\delta_k\geqslant 0\;\forall k\in K_0\\ \delta_k=0\;\forall k\notin K_0\end{cases},\;K_0\subset K(\xi_0),$$

where $\xi_0\in ET$ be the initial node ξ_0 of ET,

$$\delta_k=\sum_{i\in I}\delta_{ik},$$
$$\delta_{ik}=\overline{z}_i(\xi_0,\,k,\,\tau)\in\mathbb{R}_+$$

be the number of shares of asset $k\in K$ held by agent $i\in I$ at *the initial node ξ_0 of ET.*

$$G(\xi,\,l)=\sum_{i\in I}\tau_{e_i}(\xi,\,l)\cdot e_i(\xi,\,l,\,\tau_{e_i})+\sum_{k\in K}\delta_k\cdot\tau_d(\xi,\,l,\,k)\cdot d(\xi,\,l,\,k,\,\tau_d)\;\forall(\xi,\,l)\in ET\times L.$$

DEFINITION: *An equilibrium with TC of the Infinite Horizon FM economy with some assets in positive supply δ*

$$\mathcal{E}_\infty(ET,\,(X,\;X')\,,\,(1-\tau_e)\cdot e\,(\tau_e)\,,\,\succeq,\,\mathcal{A}(\tau_d),\,\delta)$$

is a combination

$$\left(\{(\overline{c}_i\,(\tau)\,,\;\overline{z}_i\,(\tau))\}_{i\in I}\,,\;\left(\overline{p}\,(\tau)\,,\;\overline{q}\,(\tau)\,,\;\;\{\overline{\pi}_i\,(\tau)\}_{i\in I}\right)\right)\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\right)\times {X'_+}^{|I|}$$

where all notations are the same as in the Infinite Horizon FM Economy with assets in zero-net supply except that

$$\sum_{i\in I}\overline{z}_i\,(\tau)=\delta,$$

where

$$\sum_{i\in I}\overline{z}_i(\xi,\,k,\,\tau)=\delta_k\;\forall(\xi,\,k)\in ET\times K,$$
$$\delta_k=0\;\forall k\notin K_0.$$

THEOREM 3 (Existence of FM Equilibrium with TC and Long-lived Securities): *Let*

$$\mathcal{E}_\infty(ET,\ \ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d),\ \delta)$$

be s.t. **A1-A8** *hold, except that instead of* **A7** *we assume*
A7": *Financial structure $\mathcal{A}(\tau_d)$ is s.t.*

$$\delta_{ik}=\begin{cases}\delta_{ik}\geqslant 0\ \forall k\in K\ s.t.\ \delta_k=\sum\limits_{k\in K}\delta_{ik}>0\\ \delta_{ik}=0\ \forall k\in K\ s.t.\ \delta_k=\sum\limits_{k\in K}\delta_{ik}=0\end{cases}\quad \forall i\in I,$$

$$d(\xi,\ l,\ k,\ \tau_d)=\begin{cases}d(\xi,\ l,\ k,\ \tau_d)\in L_\infty(ET\times L)_+\ \forall k\in K\ s.t.\ \delta_k=\sum\limits_{k\in K}\delta_{ik}>0\\ d(\xi,\ l,\ k,\ \tau_d)\in L_\infty(ET\times L)\ \forall k\in K\ s.t.\ \delta_k=\sum\limits_{k\in K}\delta_{ik}=0\end{cases}$$

and

$$|K(\xi)|<\infty.$$

A9': *Function*

$$f:\left([0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times L\times K|},\ \mathbb{T}\right)\longrightarrow(ED,\ \ \|\cdot\|)\subset L_\infty(ET\times L\times I)\times L_\infty(ET\times K\times L)$$

where $\mathbb{T}$ is the product topology on

$$[0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times K|}$$

defined as

$$f(\tau_e,\tau_d)=((1-\tau_e)\cdot e(\tau),\ (1-\tau_d)\cdot d(\tau_d))$$

is a homeomorphism. Then

$$\begin{gathered}\exists EDT\subset[0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times L\times K|}\ s.t.\\ \overline{(EDT)}_{\mathbb{T}}=[0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times L\times K|}\end{gathered}$$

and for the economy

$$\mathcal{E}_\infty(ET,\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d),\ \delta)\ \forall\tau_d\in EDT$$

a) $\exists$ *an FM equilibrium with TC*

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ \left(\overline{p}(\tau),\ \overline{q}(\tau),\ \ \{\overline{\pi}_i(\tau)\}_{i\in I}\right)\right)\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\ \right)\times X_+^{\prime\ |I|},$$

b) $\exists$ *an FM equilibrium with IDC*

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))\right)\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right),$$

c) $\exists$ *an FM equilibrium with EDC $M(\tau)>0$, s.t. EDC $M(\tau)>0$ is never binding*

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))\right)\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right).$$

d) $\exists$ *a CM equilibrium*

$$\left(\{(\overline{c}_i(\tau))\}_{i\in I},\ \left(\overline{P}(\tau)\right)\right)\in X_+^{|I|}\times X_+'$$

for the CM economy

$$\mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \ \succeq).$$

PROOF: a) Consider a function

$$f:\left([0,1]^{|ET\times L\times I|}\times[0,1]^{|ET\times K|},\ \mathbb{T}\right)\longrightarrow(ED,\ \|\cdot\|)\subset L_\infty(ET\times L\times I)\times L_\infty(ET\times K\times L),$$

defined as

$$f(\tau_e,\tau_d)=\left((1-\tau_e)\cdot e\left(\tau\right),\ (1-\tau_d)\cdot d\left(\tau_d\right)\right).$$

By Theorem 5.5 on p. 156 of Magill and Quinzii (1996)

$$\exists\boldsymbol{ED}\subset ED$$

s.t.

$$\overline{(\boldsymbol{ED})}_{\|\cdot\|}=ED$$

and

$$\exists(\{(\overline{c}_i\left(e,\ d\right),\ \overline{z}_i\left(e,\ d\right))\}_{i\in I},\ \left(\overline{p}\left(e,\ d\right),\ \overline{q}\left(e,\ d\right),\ \{\overline{\pi}_i\left(e,\ d\right)\}_{i\in I}\right))\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}^{|ET\times L|}\times Q\right)\times X_+'^{|I|}$$

an FM equilibrium with TC $\forall\left(e,\ d\right)\in\boldsymbol{ED}$.
Set

$$EDT=f^{-1}(\boldsymbol{ED})$$

Therefore, since

$$f:[0,1]^{|ET\times L\times I|}\times[0,1]^{|ET\times K|}\longrightarrow ED$$

is a bijection and thus surjective (onto), we can conclude that

$$f\left(EDT\right)=ff^{-1}(\boldsymbol{ED})=\boldsymbol{ED}\subset ED.$$

Therefore,

$$f(EDT)=\boldsymbol{ED}\subset ED.$$

Hence,

$$\overline{f(EDT)}_{\|\cdot\|}=\overline{(\boldsymbol{ED})}_{\|\cdot\|}=ED$$

Since

$$f:\left([0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times L\times K|},\ \mathbb{T}\right)\longrightarrow(ED,\ \|\cdot\|)$$

is a homeomorphism and therefore closed mapping we have that

$$\overline{f(EDT)}_{\|\cdot\|}=f(\overline{(EDT)}_{\mathbb{T}}).$$

Thus,

$$f(\overline{(EDT)}_{\mathbb{T}})=\overline{(\boldsymbol{ED})}_{\|\cdot\|}=ED.$$

But

$$f\left([0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times K|}\right)=ED.$$

Therefore, since

$$f:[0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times K|}\longrightarrow ED$$

is a bijection and thus injective (one-to-one) we can conclude that

$$f^{-1}f(\overline{(EDT)}_{\mathbb{T}})=f^{-1}\left(\overline{(\boldsymbol{ED})}_{\|\cdot\|}\right)=f^{-1}\left(ED\right).$$

and

$$f^{-1}f\left([0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times L\times K|}\right)=f^{-1}\left(ED\right).$$

Thus,

$$\overline{(EDT)}_{\mathbb{T}}=[0,1[^{|ET\times L\times I|}\times[0,1]^{|ET\times L\times K|}.$$

Proofs of **b), c)** and **d)** are similar to Theorem 2. ■

The result is very intuitive. It simply states that the endowment and dividend tax rates for which an FM equilibrium with TC exists are never too far away.

This result is philosophically parallel to the finding of Magin (2015) that in a Finite Horizon FM economy FM equilibria exist for all stochastic tax rates imposed on agents' endowments and dividends except for a closed set of measure zero.

We are now ready to characterize the existence of FM equilibria in Infinite Horizon FM economies with some securities in positive supply.

6. EQUILIBRIUM ASSET PRICES IN INFINITE HORIZON FM ECONOMIES WITH SOME SECURITIES IN POSITIVE SUPPLY

DEFINITION: *Let*

$$\mathcal{E}_{\infty}(ET,\,(X,\;X')\,,\,(1-\tau_e)\cdot e\,(\tau_e)\,,\,\succeq,\,\mathcal{A}(\tau_d),\,\delta)$$

be an Infinite Horizon FM economy with some securities in positive supply δ s.t.

$$(\{(\overline{c}_i(\tau),\;\overline{z}_i(\tau))\}_{i\in I}\,,\,(\overline{p}(\tau),\;\overline{q}(\tau),\;\;\{\overline{\pi}_i(\tau)\}_{i\in I}))\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\;\right)\times X_+'^{\,|I|}$$

be an FM equilibrium with TC for this economy.
Then we say that the security $k\in K$ is priced at its fundamental value if

$$\overline{q}\,(\xi,\;k,\;\tau)=\sum_{\xi'\in ET^+(\xi)}\frac{\overline{\pi}_i(\xi',\,\tau)}{\overline{\pi}_i(\xi,\,\tau)}\cdot\overline{p}\,(\xi',\;\tau)\cdot(1-\tau_d\,(\xi))\cdot d\,(\xi',\;k,\;\tau_d)\;\forall\,(i,\;\xi,\;k)\in I\times ET\times K.$$

THEOREM 4 (Security in Positive Supply Priced at its Fundamental Value): *Let*

$$\mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d),\ \delta)$$

be s.t. **A6** and **A7"** *hold and*

$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i\in I}))\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\right)\times X_+'^{\,|I|}$$

be an FM equilibrium with TC for this economy. Then the price of every security in positive supply ($\delta_k>0$) is equal to its fundamental value, i.e.,

$$\overline{q}(\xi,\ k,\ \tau)=\sum_{\xi'\in ET^+(\xi)}\tfrac{\overline{\pi}_i(\xi',\ \tau)}{\overline{\pi}_i(\xi,\ \tau)}\cdot\overline{p}(\xi',\ \tau)\cdot(1-\tau_d(\xi))\cdot d(\xi',\ k,\ \tau_d)\ \forall(i,\ \xi,\ k)\in I\times ET\times K\ s.t.\ \delta_k>0.$$

PROOF: Just apply Proposition 6.2. on p. 158 of Magill and Quinzii (1996). ■

Let us impose additional assumptions of differentiability of agents' utility functions. In case of differentiability of agents' utility functions it is more convenient to use traditional stochastic processes notations allowing us to use expectation operators. Indeed, set

$$\begin{gathered}d_{kt+T}=\{d(\xi,\ k,\ l)\}_{(\xi,\ l)\in ET_{t+T}\times L}\ \forall k\in K,\\ c_{it+T}=\{c_i(\xi,\ l)\}_{(\xi,\ l)\in ET_{t+T}\times L}\ \forall i\in I,\\ p_{t+T}=\{p(\xi,\ l)\}_{(\xi,\ l)\in ET_{t+T}\times L},\\ q_{kt+T}=\{q(\xi,\ k)\}_{\xi,\in ET_{t+T}}\ \forall k\in K.\end{gathered}$$

THEOREM 5 (Equilibrium Asset Prices with Differentiability of Utility Function): *Let*

$$\begin{gathered}\mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d))\ or\\ \mathcal{E}_\infty(ET,\ (X,\ X'),\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d),\ \delta)\end{gathered}$$

be an Infinite Horizon FM economy s.t. agents' preferences $\succeq_i$ on X_+ are given by the utility function

$$U_i(c_i)=\sum_{(\xi,\ l)\in ET\times L}\Pr_i(\xi)\cdot b_i^{T(\xi)}\cdot[u_i(c_i(\xi,\ l))]=E\left[\sum_{l\in L}\sum_{T=0}^{\infty}b_i^T u_{il}(c_{ilt+T})\right],$$

where $u_i\in C^2$, $u_i'(\cdot)<0$, $u_i''(\cdot)>0\ \forall i\in I$ and

$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i\in I}))\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\right)\times X_+'^{\,|I|}$$

be a TC equilibrium for this economy. Then

$$\tfrac{\overline{\pi}_i(\xi',\ \tau)}{\overline{\pi}_i(\xi,\ \tau)}=b_i^{T(\xi')}\cdot\tfrac{u_i'(\overline{c}_i(\xi',\ 1,\ \tau))}{u_i'(\overline{c}_i(\xi,\ 1,\ \tau))}\cdot\Pr_i(\xi')\ \forall(i,\ \xi')\in I\times ET^+(\xi).$$

and

$$\overline{q}(\xi,\ \tau)=\sum_{\xi'\in\xi^+}b_i\cdot\tfrac{u_i'(c_i(\xi',\ 1,\ \tau))}{u_i'(c_i(\xi,\ 1,\ \tau))}\cdot\Pr_i(\xi')\cdot(\overline{q}(\xi',\ \tau)+\overline{p}(\xi',\ \tau)\cdot(1-\tau_d(\xi'))\cdot d(\xi',\ \tau_d))\ \forall(i,\ \xi)\in I\times ET.$$

Or using expectation operators

$$\overline{q}_{kt}=E\left[b_i\tfrac{u_i'(\overline{c}_{i1t+1})}{u_i'(\overline{c}_{i1t})}\left(\overline{q}_{kt+1}+\overline{p}_{t+1}\cdot(1-\tau_{dkt+1})\cdot d_{kt+1}\right)\right]\ \forall(i,\ k)\in I\times K.$$

PROOF: Since

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ \left(\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i\in I}\right)\right) \in \left(X_+^{|I|}\times \mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\right)\times {X'}_+^{|I|}$$

is a TC equilibrium for the economy

$$\mathcal{E}_\infty(ET,\ (X,\ X'),(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d)) \text{ or } \mathcal{E}_\infty(ET,\ (X,\ X'),(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d),\delta)$$

we have that

$$\overline{c}_i \in \arg\max B_\infty^{CM}(\overline{P}_i,\ (1-\tau_{e_i})\cdot e_i).$$

Let us set up Lagrangian

$$\pounds_i^{CM}(\underbrace{c_i,\ \lambda_i}_{Unknowns},\underbrace{P_i,\ \ (1-\tau_{e_i})\cdot e_i}_{Parameters}) = U_i(c_i)-\lambda_i\cdot[P_i\cdot c_i - P_i\cdot(1-\tau_{e_i})\cdot e_i],$$

where $\lambda_i\in\mathbb{R}$ is the Lagrangian multiplier. Taking first-order conditions

$$D\pounds_i^{CM}(\underbrace{c_i,\ \lambda_i}_{Unknowns},\underbrace{P_i,\ \ (1-\tau_{e_i})\cdot e_i}_{Parameters}) = 0,$$

we obtain

$$D\pounds_i^{CM}(\underbrace{c_i,\ \lambda_i}_{Unknowns},\underbrace{P_i,\ \ (1-\tau_{e_i\ t+1})\cdot e_i}_{Parameters}) = \begin{cases}\frac{\partial\pounds_i^{CM}}{\partial c_i} = DU_i(c_i)-\lambda_i\cdot P_i = 0\\ \frac{\partial\pounds_i^{CM}}{\partial \lambda_i} = P_i\cdot c_i - P_i\cdot(1-\tau_{e_i})\cdot e_i = 0\end{cases}.$$

Therefore, in equilibrium

$$\frac{\partial\pounds_i^{CM}}{\partial c_i}\Big|_{\substack{c_i=\overline{c}_i\\ P_i=\overline{P}_i\\ \lambda_i=\overline{\lambda}_i}} = b_i^{T(\xi)}\cdot u_i'(\overline{c}_i(\xi,\ l))\cdot \Pr_i(\xi)-\overline{\lambda}_i\cdot\overline{P}_i(\xi,\ l) = 0\ \forall(i,\ \xi,\ l)\in I\times ET\times L.$$

In particular,

$$\begin{cases}\frac{\partial\pounds_i^{CM}}{\partial c_i(\xi',\,1)}\Big|_{\substack{c_i=\overline{c}_i\\ P_i=\overline{P}_i\\ \lambda_i=\overline{\lambda}_i}} = b_i^{T(\xi')}\cdot u_i'(\overline{c}_i(\xi',\ 1))\cdot\Pr_i(\xi')-\overline{\lambda}_i\cdot\overline{\pi}_i(\xi') = 0\ \forall(i,\ \xi')\in I\times ET^+(\xi)\\ \frac{\partial\pounds_i^{CM}}{\partial c_i(\xi,\,1)}\Big|_{\substack{c_i=\overline{c}_i\\ P_i=\overline{P}_i\\ \lambda_i=\overline{\lambda}_i}} = u_i'(\overline{c}_i(\xi,\ 1))-\overline{\lambda}_i\cdot\overline{\pi}_i(\xi) = 0\ \forall i\in I, \text{ where } \xi \text{ is the initial node of } ET.\end{cases}\Longrightarrow$$

$$\begin{cases} b_i^{T(\xi')}\cdot u_i'(\overline{c}_i(\xi',\ 1))\cdot\Pr_i(\xi') = \overline{\lambda}_i\cdot\overline{\pi}_i(\xi')\ \forall(i,\ \xi')\in I\times ET^+(\xi)\\ u_i'(\overline{c}_i(\xi,\ 1)) = \overline{\lambda}_i\cdot\overline{\pi}_i(\xi)\ \forall i\in I, \text{ where } \xi \text{ is the initial node of } ET.\end{cases}.$$

Hence,

$$\frac{\overline{\pi}_i(\xi',\,\tau)}{\overline{\pi}_i(\xi,\,\tau)} = b_i^{T(\xi')}\cdot\frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))}\cdot\Pr_i(\xi')\ \forall(i,\ \xi')\in I\times ET^+(\xi).$$

But we also know that in equilibrium

$$\overline{q}(\xi,\ \tau) = \sum_{\xi'\in\xi^+}\frac{\overline{\pi}_i(\xi',\,\tau)}{\overline{\pi}_i(\xi,\,\tau)}\cdot(\overline{q}(\xi',\ \tau)+\overline{p}(\xi',\ \tau)\cdot(1-\tau_d(\xi'))\cdot d(\xi',\ \tau_d))\ \forall(i,\ \xi)\in I\times ET.$$

Thus,

$$\overline{q}(\xi,\ \tau) = \sum_{\xi'\in\xi^+} b_i\cdot\frac{u_i'(c_i(\xi',\,1,\,\tau))}{u_i'(c_i(\xi,\,1,\,\tau))}\cdot\Pr_i(\xi')\cdot(\overline{q}(\xi',\ \tau)+\overline{p}(\xi',\ \tau)\cdot(1-\tau_d(\xi'))\cdot d(\xi',\ \tau_d))\ \forall(i,\ \xi)\in I\times ET.$$

Using expectation operators we can rewrite it as

$$\overline{q}_{kt} = E\left[b_i\frac{u_i'(\overline{c}_{i1t+1})}{u_i'(\overline{c}_{i1t})}\left(\overline{q}_{kt+1}+\overline{p}_{t+1}\cdot(1-\tau_{dkt+1})\cdot d_{kt+1}\right)\right]\ \forall(i,\ k)\in I\times K.$$

Dividing both sides of this equation by $\overline{q}_{kt}$ we obtain the Euler Equation

$$E\left[\frac{b_i u_i'(\overline{c}_{i1t+1})}{u_i'(\overline{c}_{i1t})}\frac{\overline{q}_{kt+1}+\overline{p}_{t+1}\cdot(1-\tau_{dkt+1})\cdot d_{kt+1}}{\overline{q}_{kt}}\right] = E\left[\frac{b_i u_i'(\overline{c}_{i1t+1})}{u_i'(\overline{c}_{i1t})}R_{kt+1}\right] = 1.\ \forall\,(i,\ k)\in I\times K.$$

Clearly, if a security $k \in K$ is risk-free, then

$$E\left[\frac{b_i u_i'(\overline{c}_{i1t+1})}{u_i'(\overline{c}_{i1t})}\right] = \frac{1}{R_f}.$$

Obviously, if a security $k \in K$ is short lived, then

$$\left.\begin{array}{r}\overline{q}_{kt} = E\left[b_i \frac{u_i'(\overline{c}_{ti+1})}{u_i'(\overline{c}_{it})}\left(\overline{p}_{t+1}\cdot(1-\tau_{dkt+1})\cdot d_{kt+1}\right)\right], \\ E\left[\frac{b_i u_i'(\overline{c}_{i1t+1})}{u_i'(\overline{c}_{i1t})}R_{kt+1}\right] = E\left[\frac{b_i u_i'(\overline{c}_{it+1})}{u_i'(\overline{c}_{it})}\frac{(1-\tau_{dkt+1})\cdot\overline{p}_{t+1}\cdot d_{kt+1}}{\overline{q}_{kt}}\right] = 1.\end{array}\right\}\forall i\in I.\ \blacksquare$$

THEOREM 6 (Security in Positive Supply Priced at Its Fundamental Value with Differentiability of Utility Function): *Let*

$$\mathcal{E}_\infty(ET,\quad (X,\ X'),(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d),\delta)$$

be an Infinite Horizon FM economy s.t. agents' preferences $\succeq_i$ on X_+ are given by the utility function

$$U_i(c_i) = \sum_{(\xi,\ l)\in ET\times L}\Pr(\xi)\cdot b_i^{T(\xi)}\cdot[u_i(c_i(\xi,\ l))] = E\left[\sum_{l\in L}\sum_{T=0}^{\infty} b_i^T u_{il}(c_{ilt+T})\right],$$

where $u_i\in C^2$, $u_i'(\cdot)<0$, $u_i''(\cdot)>0\ \forall i\in I$ and

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ \left(\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i\in I}\right)\right)\in\left(X_+\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\right)\times X_+'^{\,|I|}$$

be a TC equilibrium for this economy s.t. all assets are priced at fundamental value. Then

$$\overline{q}(\xi,\ \tau) = \sum_{\xi'\in ET^+(\xi)} b_i^{T(\xi')}\cdot\frac{u_i'(c_i(\xi',1,\tau))}{u_i'(c_i(\xi,1,\tau))}\cdot\Pr(\xi')\cdot\overline{p}(\xi',\ \tau)\cdot(1-\tau_d(\xi'))\cdot d(\xi',\ \tau_d)\ \forall\,(i,\ \xi)\in I\times ET$$

or using expectation operators

$$\overline{q}_{kt} = E\left[\sum_{T=1}^{\infty} b_i^T\frac{u_i'(\overline{c}_{i1t+T})}{u_i'(\overline{c}_{i1t})}(1-\tau_{dkt+1})\cdot\overline{p}_{t+1}\cdot d_{kt+1}\right]\ \forall\,(i,\ k)\in I\times K.$$

PROOF: Since

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ \left(\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i\in I}\right)\right)\in\left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\right)\times X_+'^{\,|I|}$$

is a TC equilibrium for the economy

$$\mathcal{E}_\infty(ET,\ (X,\ X'),(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d),\delta),$$

s.t. all assets are priced at fundamental value, we can conclude

$$\overline{q}(\xi,\ \tau) = \sum_{\xi'\in ET^+(\xi)}\frac{\overline{\pi}_i(\xi',\tau)}{\overline{\pi}_i(\xi,\tau)}\cdot\overline{p}(\xi',\ \tau)\cdot(1-\tau_d(\xi'))\cdot d(\xi',\ \tau_d)\ \forall\,(i,\ \xi)\in I\times ET.$$

We know from the previous Theorem

$$\frac{\overline{\pi}_i(\xi',\tau)}{\overline{\pi}_i(\xi,\tau)} = b_i^{T(\xi')}\cdot\frac{u_i'(\overline{c}_i(\xi',1,\tau))}{u_i'(\overline{c}_i(\xi,1,\tau))}\cdot\Pr_i(\xi')\ \forall\,(i,\ \xi')\in I\times ET^+(\xi).$$

Therefore,

$$\overline{q}(\xi,\ \tau) = \sum_{\xi' \in ET^{+}(\xi)} b_i^{T(\xi')} \cdot \frac{u_i'(\overline{c}_i(\xi',\ 1,\ \tau))}{u_i'(\overline{c}_i(\xi,\ 1,\ \tau))} \cdot \Pr(\xi') \cdot \overline{p}(\xi',\ \tau) \cdot (1 - \tau_d(\xi')) \cdot d(\xi',\ \tau_d)\ \forall (i,\ \xi) \in I \times ET$$

or using expectation operators

$$\overline{q}_{kt} = E\left[\sum_{T=1}^{\infty} b_i^T \frac{u_i'(\overline{c}_{i1t+T})}{u_i'(\overline{c}_{i1t})} \cdot \overline{p}_{t+1} \cdot (1 - \tau_{dkt+1}) \cdot d_{kt+1}\right] \forall (i,\ k) \in I \times K.\ \blacksquare$$

THEOREM 7 (No "Optimal" P/E Ratio): *Let*

$$\mathcal{E}_{\infty}(ET,\ (X,\ X'),(1 - \tau_e) \cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d),\ \delta)$$

be an Infinite Horizon FM economy s.t. agents' preferences $\succeq_i$ *on*

$$X_i = X_+$$

are given by the utility function

$$U_i(c_i) = \sum_{(\xi,\ l) \in ET \times L} \Pr(\xi) \cdot b_i^{T(\xi)} \cdot [u_i(c_i(\xi,\ l))] = E\left[\sum_{l \in L}\sum_{T=0}^{\infty} b_i^T u_{il}(c_{ilt+T})\right],$$

where

$$u_i(c) = \frac{c^{1-\lambda_i}}{1-\lambda_i}\ \forall i \in I$$

and

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i \in I},\ (\overline{p}(\tau),\ \overline{q}(\tau),\ \{\overline{\pi}_i(\tau)\}_{i \in I})\right) \in \left(X_+^{|I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}_+^{|ET \times L|} \times Q\right) \times X'^{\,|I|}_{+}$$

be a TC equilibrium for this economy s.t. all assets are priced at fundamental value. Assume further

$$\frac{\overline{c}_{i1t+T}}{\overline{c}_{i1t}} = \frac{\sum_{k \in K}(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}}{\sum_{k \in K}(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} = \frac{(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}}\ \forall (i,\ k,\ T) \in I \times K \times \{0, ..., \infty\},$$

i.e., all dividends are growing at the same rate and individuals' consumption is growing at the same rate as total dividends[4] *and*

$$\ln(b_i \cdot (\frac{(1-\tau_{dkt+1}) \cdot \overline{p}_{t+1} \cdot d_{kt+1}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}})^{1-\lambda_i}) \sim N(\mu_d,\ \sigma_d)\ \forall (i,\ T) \in I \times \{0, ..., \infty\}$$

with

$$COV\left[\ln\left(\frac{(1-\tau_{dkt+T_1}) \cdot \overline{p}_{t+T_1} \cdot d_{kt+T_1}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}}\right),\ \ln\left(\frac{(1-\tau_{dkt+T_2}) \cdot \overline{p}_{t+T_2} \cdot d_{kt+T_2}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}}\right)\right] = 0\ \forall (\ T_1,\ T_2) \in \{1, ..., \infty\} \times \{1, ..., \infty\},\ T_1 \neq T_2$$

and

$$e^{\mu_d + \frac{1}{2} \cdot \sigma_d^2} < 1^5\ \forall (i,\ k,\ T) \in I \times K \times \{0, ..., \infty\}.$$

Then

$$\boxed{q_{kt}(\mu_d,\ \sigma_d^2,\ \overline{p}_t \cdot (1 - \tau_{dkt}) \cdot d_{kt}) = \left[\frac{e^{\mu_d + \frac{1}{2}\sigma_d^2}}{1 - e^{\mu_d + \frac{1}{2}\sigma_d^2}}\right] \cdot \overline{p}_t(\tau) \cdot (1 - \tau_{dkt}) \cdot d_{kt}\ \forall k \in K.}$$

[4] Since our modified CCAPM describes a production economy, the total dividend $\sum_{k=1}^{n} d_{kt+T}$ represents total output, i.e., GDP.

[5] See Magin (2014) for calculations.

PROOF: By assumption of the Theorem we know that

$$\exists(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I}\,,\ \left(\overline{p}(\tau),\ \overline{q}(\tau),\ \ \{\overline{\pi}_i(\tau)\}_{i\in I}\right)) \in \left(X_+^{|I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\ \right)\times L_1(ET\times I)_+$$

a TC equilibrium for this economy s.t. all assets are priced at fundamental value, i.e.,

$$\overline{q}\,(\xi,\ \tau) = \sum_{\xi'\in ET^+(\xi)} b_i^{T(\xi')}\cdot\tfrac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))}\cdot\Pr(\xi')\cdot\overline{p}\,(\xi',\ \tau)\cdot(1-\tau_d\,(\xi'))\cdot d\,(\xi',\ \tau_d)\ \forall\,(i,\ \xi)\in I\times ET$$

or using expectation operators

$$\overline{q}_{kt} = E\left[\sum_{T=1}^{\infty} b_i^T\frac{u_i'\left(\overline{c}_{i1t+T}\right)}{u_i'\left(\overline{c}_{i1t}\right)}\cdot\overline{p}_{t+T}\cdot(1-\tau_{dkt+T})\cdot d_{kt+T}\right]\ \forall\,(i,\ K)\in I\times K. \tag{1}$$

Therefore,

$$\overline{q}_{kt} = E\left[\sum_{T=1}^{\infty} b_i^T\cdot\left(\frac{\overline{c}_{i1t+T}}{\overline{c}_{i1t}}\right)^{-\lambda_i}\cdot\overline{p}_{t+T}\cdot(1-\tau_{dkt+T})\cdot d_{kt+T}\right]\forall\,(i,\ K)\in I\times K. \tag{1}$$

By assumption of the Theorem, we have that

$$\tfrac{\overline{c}_{i1t+T}}{\overline{c}_{i1t}} = \tfrac{\sum\limits_{k\in K}(1-\tau_{dkt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}{\sum\limits_{k\in K}(1-\tau_{dkt})\cdot\overline{p}_t\cdot d_{kt}} = \tfrac{(1-\tau_{dt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}{(1-\tau_{dt})\cdot\overline{p}_t\cdot d_{kt}}\ \forall\,(i,\ k,\ T)\in I\times K\times\{0,...,\infty\}.$$

Hence,

$$\tfrac{\overline{c}_{i1t+T}}{\overline{c}_{i1t}} = \tfrac{(1-\tau_{dt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}{(1-\tau_{dt})\cdot\overline{p}_t\cdot d_{kt}}\ \forall\,(i,\ k,\ T)\in I\times K\times\{0,...,\infty\}.$$

Substituting the expression for $\frac{\overline{c}_{i1t+T}}{\overline{c}_{i1t}}$ into the previous equation (1), we obtain

$$\overline{q}_{kt} = E\left[\sum_{T=1}^{\infty} b_i^T\cdot\left(\tfrac{(1-\tau_{dkt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}{(1-\tau_{dkt})\cdot\overline{p}_t\cdot d_{kt}}\right)^{-\lambda_i}\cdot(1-\tau_{dkt+T})\cdot\overline{p}_{t+T}d_{kt+T}\right]\ \forall\,(i,\ k)\in I\times K.$$

Therefore,

$$\overline{q}_{kt} = E\left[\sum_{T=1}^{\infty} b_i^T\cdot\left(\tfrac{(1-\tau_{dkt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}{(1-\tau_{dkt})\cdot\overline{p}_t\cdot d_{kt}}\right)^{1-\lambda_i}\right]\cdot(1-\tau_{dkt})\cdot\overline{p}_t\cdot d_{kt}\ \forall\,(i,\ k)\in I\times K.$$

Moreover,

$$\ln\left(b_i\cdot\left(\tfrac{(1-\tau_{dkt+T+1})\cdot\overline{p}_{t+T+1}\cdot d_{kt+T+1}}{(1-\tau_{dkt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}\right)^{1-\lambda_i}\right) = \ln(b_i\cdot(\tfrac{\overline{c}_{i1t+T+1}}{\overline{c}_{i1t+T}})^{1-\lambda_i})\sim N(\mu_d,\,\sigma_d)$$

$\forall\,(i,\ k,\ T)\in I\times K\times\{0,...,\infty\}$. Also,

$$b_i^{\overline{T}} \cdot \left(\frac{\left(1-\tau_{dkt+\overline{T}}\right) \cdot \overline{p}_{t+\overline{T}} \cdot d_{kt+\overline{T}}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} \right)^{1-\lambda_i} = \prod_{T=0}^{\overline{T}-1} \left[b_i \cdot \left(\frac{(1-\tau_{dt+T+1}) \cdot \overline{p}_{t+T+1} \cdot d_{kt+T+1}}{(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}} \right)^{1-\lambda_i} \right]$$

$\forall \left(k,\ i,\ \overline{T}\right) \in I \times K \times \{1, ..., \infty\}$. Taking logarithms of both sides, we obtain

$$\ln \left(b_i^{\overline{T}} \cdot \left(\frac{\left(1-\tau_{dkt+\overline{T}}\right) \cdot \overline{p}_{t+\overline{T}} \cdot d_{kt+\overline{T}}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} \right)^{1-\lambda_i} \right) = \ln \left(\prod_{T=0}^{\overline{T}-1} \left[b_i \cdot \left(\frac{(1-\tau_{dkt+T+1}) \cdot \overline{p}_{t+T+1} \cdot d_{kt+T+1}}{(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}} \right)^{1-\lambda_i} \right] \right) =$$

$$= \sum_{T=0}^{\overline{T}-1} \ln \left(b_i \cdot \left(\frac{(1-\tau_{dkt+T+1}) \cdot \overline{p}_{t+T+1} \cdot d_{kt+T+1}}{(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}} \right)^{1-\lambda_i} \right) \ \forall \left(i,\ k,\ \overline{T}\right) \in I \times K \times \{1, ..., \infty\}.$$

Hence,

$$\ln \left(b_i^{\overline{T}} \cdot \left(\frac{\left(1-\tau_{dkt+\overline{T}}\right) \cdot \overline{p}_{t+\overline{T}} \cdot d_{kt+\overline{T}}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} \right)^{1-\lambda_i} \right) = \sum_{T=0}^{\overline{T}-1} \ln \left(b_i \cdot \left(\frac{(1-\tau_{dkt+T+1}) \cdot \overline{p}_{t+T+1} \cdot d_{kt+T+1}}{(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}} \right)^{1-\lambda_i} \right)$$

$\forall \left(i,\ k,\ \overline{T}\right) \in I \times K \times \{1, ..., \infty\}$. Clearly,

$$E \left[\ln \left(b_i^{\overline{T}} \cdot \left(\frac{\left(1-\tau_{dkt+\overline{T}}\right) \cdot \overline{p}_{t+\overline{T}} \cdot d_{kt+\overline{T}}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} \right)^{1-\lambda_i} \right) \right] = \sum_{T=0}^{\overline{T}-1} E \left[\ln \left(b_i \cdot \left(\frac{(1-\tau_{dkt+T+1}) \cdot \overline{p}_{t+T+1} \cdot d_{kt+T+1}}{(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}} \right)^{1-\lambda_i} \right) \right] = \overline{T} \cdot \mu_d$$

$\forall \left(i,\ k,\ \overline{T}\right) \in I \times K \times \{1, ..., \infty\}$ and, since by assumption of the Theorem

$$COV \left[\ln \left(\frac{\left(1-\tau_{dkt+T_1}\right) \cdot \overline{p}_{t+T_1} \cdot d_{kt+T_1}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} \right),\ \ln \left(\frac{\left(1-\tau_{dkt+T_2}\right) \cdot \overline{p}_{t+T_2} \cdot d_{kt+T_2}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} \right) \right] = 0 \ \forall \left(\ T_1,\ T_2\right) \in \{1, ..., \infty\} \times \{1, ..., \infty\},$$
$$T_1 \neq T_2.$$

we have that

$$VAR \left[\ln \left(b_i^{\overline{T}} \cdot \left(\frac{\left(1-\tau_{dt+\overline{T}}\right) \cdot \overline{p}_{t+\overline{T}} \cdot d_{kt+\overline{T}}}{(1-\tau_{dt}) \cdot \overline{p}_t \cdot d_{kt}} \right)^{1-\lambda_i} \right) \right] = \sum_{T=0}^{\overline{T}-1} VAR \left[\ln \left(b_i \cdot \left(\frac{(1-\tau_{dt+T+1}) \cdot \overline{p}_{t+T+1} \cdot d_{kt+T+1}}{(1-\tau_{dt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}} \right)^{1-\lambda_i} \right) \right] = \overline{T} \cdot \sigma_d^2$$

$\forall \left(i,\ k,\ \overline{T}\right) \in I \times K \times \{1, ..., \infty\}$. Therefore,

$$\ln \left(b_i^T \cdot \left(\frac{(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} \right)^{1-\lambda_i} \right) \sim N(T \cdot \mu_d, \sqrt{T} \cdot \sigma_d)$$

$\forall (i,\ k,\ T) \in I \times K \times \{1, ..., \infty\}$. Fix an arbitrary $(i,\ k,\ T) \in I \times K \times \{0, ..., \infty\}$. Let

$$x = \ln \left(b_i^T \left(\frac{(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} \right)^{1-\lambda_i} \right).$$

Therefore,

$$E \left[b_i^T \cdot \left(\frac{(1-\tau_{dkt+T}) \cdot \overline{p}_{t+T} \cdot d_{kt+T}}{(1-\tau_{dkt}) \cdot \overline{p}_t \cdot d_{kt}} \right)^{1-\lambda_i} \right] = E\left[e^x\right] = e^{\mu_x + \frac{\sigma_x^2}{2}} = e^{T \cdot \mu_d + \frac{1}{2} T \cdot \sigma_d^2}$$

$\forall\,(i,\ k,\ T) \in I \times K \times \{1, ..., \infty\}$. Thus,

$$E\left[b_i^T \cdot \left(\frac{(1-\tau_{dkt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}{(1-\tau_{dkt})\cdot\overline{p}_t\cdot d_{kt}}\right)^{1-\lambda_i}\right] = e^{T\cdot\mu_d+\frac{1}{2}T\cdot\sigma_d^2}\ \ \forall\,(i,\ k,\ T) \in I \times K \times \{1, ..., \infty\}.$$

Hence, summing over $\forall T \in \{1, ..., \infty\}$, we obtain

$$E\left[\sum_{T=1}^{\infty} b_i^T \cdot \left(\frac{(1-\tau_{dkt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}{(1-\tau_{dkt})\cdot\overline{p}_t\cdot d_{kt}}\right)^{1-\lambda_i}\right] = \sum_{T=1}^{\infty} E\left[b_i^T \cdot \left(\frac{(1-\tau_{dt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}{(1-\tau_{dt})\cdot\overline{p}_t\cdot d_{kt}}\right)^{1-\lambda_i}\right] =$$
$$= \sum_{T=1}^{\infty} e^{T\cdot\mu_d+\frac{1}{2}T\cdot\sigma_d^2}\ \forall\,(i,\ k) \in I \times K.$$

Taking into consideration that by assumption

$$E\left[b_i \cdot \left(\frac{(1-\tau_{dkt+T})\cdot\overline{p}_{t+T+1}\cdot d_{kt+T+1}}{(1-\tau_{dkt})\cdot\overline{p}_t\cdot d_{kt}}\right)^{1-\lambda_i}\right] = e^{\mu_d+\frac{1}{2}\cdot\sigma_d^2} < 1$$

$\forall\,(i,\ k,\ T) \in I \times K \times \{0, ..., \infty\}$ and summing over $\forall T \in \{1, ..., \infty\}$, we obtain

$$\sum_{T=1}^{\infty} e^{T\cdot\mu_d+\frac{1}{2}T\cdot\sigma_d^2} = \frac{e^{\mu_d+\frac{1}{2}\sigma_d^2}}{1-e^{\mu_d+\frac{1}{2}\sigma_d^2}}.$$

Therefore,

$$q_{kt} = E\left[\sum_{T=1}^{\infty} b_i^T \cdot \left(\frac{(1-\tau_{dkt+T})\cdot\overline{p}_{t+T}\cdot d_{kt+T}}{(1-\tau_{dkt})\cdot\overline{p}_t\cdot d_{kt}}\right)^{1-\lambda_i}\right] \cdot \overline{p}_t \cdot (1-\tau_{dkt}) \cdot d_{kt} = \frac{e^{\mu_d+\frac{1}{2}\sigma_d^2}}{1-e^{\mu_d+\frac{1}{2}\sigma_d^2}} \cdot \overline{p}_t \cdot (1-\tau_{dkt}) \cdot d_{kt}.$$

So

$$\boxed{q_{kt}(\mu_d,\ \sigma_d^2,\ \overline{p}_t \cdot (1-\tau_{dkt}) \cdot d_{kt}) = \left[\frac{e^{\mu_d+\frac{1}{2}\sigma_d^2}}{1-e^{\mu_d+\frac{1}{2}\sigma_d^2}}\right] \cdot \overline{p}_t(\tau) \cdot (1-\tau_{dkt}) \cdot d_{kt}\ \forall k \in K.}\ \blacksquare$$

7. CONCLUSION

This paper proves the existence of equilibria in the Infinite Horizon GEI model with insecure property rights. Insecure property rights come in the form of the stochastic taxes imposed on agents' endowments and assets' dividends. This paper finds that under reasonable assumptions, Financial Markets (FM) equilibria with Transversality Condition (TC) as well as equilibria with Implicit Debt Constraint (IDC) and Explicit Debt Constraint (EDC) exist in Infinite Horizon FM economies with stochastic taxes and with short-lived securities in zero net supply. This paper finds that under reasonable assumptions, FM equilibria with TC as well as equilibria with IDC and EDC exist in Infinite Horizon FM economies with long-lived securities in zero net supply for a dense subset of the set of all stochastic dividend tax rates. The paper also finds that under reasonable assumptions, FM equilibria with TC as well as equilibria with IDC and EDC exist in Infinite Horizon FM economies with some long-lived securities in positive supply for a dense subset of the set of all stochastic endowment and dividend tax rates. Finally, we find that GEI implies that there is no such thing as the "optimal" buying or selling P/E ratio but that instead it is completely determined by constantly changing investors' expectations of company's after-tax profitability.

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