Is Index Concentration an Inevitable Consequence of Market-Capitalization Weighting?

Abstract

Market-cap-weighted equity indexes are ubiquitous. However, there are growing concerns that such indexes are increasingly concentrated in a few stocks. We ask: Does market-cap weighting inevitably lead to increased concentration over time? The question of inevitability arises from research that develops probabilistic causal mechanisms for the dominance by a few firms over time. We show that while the concentration currently observed in major equity market indexes is substantial, it is not at an all-time high. Monte Carlo simulations calibrated to market data provide insight into various approaches to mitigate concentration, albeit at the expense of higher turnover.

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December 12, 2022

This material is for academic use and is not intended as the basis of investment advice or recommendations to buy or sell any securities. The authors thank Xuan Che for expert research assistance, and Andrew Ang, Robert Anderson, Julia Belford, Samara Cohen, Jennifer Hsui, Jason Ribando, Nogie Udevbulu, an anonymous reviewer and seminar participants at the London Quant Group and Berkeley's Consortium for Data Analytics in Risk (CDAR) for helpful comments.

Many traditional equity indexes—originally intended as performance benchmarks to capture the entire market in a region, asset class, or sector—are weighted by market capitalization.[1] Cap-weighted schema have strong justification on theoretical and practical grounds. Theory comes from the capital asset pricing model (Sharpe 1964) where investors hold combinations of the market portfolio and riskless bonds. From a practical perspective, cap-weighted indexes are naturally self-rebalancing, reducing turnover and transaction costs relative to alternative weighting schema that may require more extensive rebalancing. Indeed, the very first index fund (for the Samsonite Luggage pension plan) was initially equal-weighted, but the high turnover led to a transition to cap-weighting, specifically the S&P 500.[2] Yet, despite their advantages, concerns that cap-weighted indexes are overly concentrated in the largest stocks are increasingly common.[3]

We ask: Will the use of market-cap weighting in indexes *inevitably* lead to increased dominance over time by the largest firms? The question of inevitability is suggested by research that shows, under a variety of possible and plausible causal mechanisms, that the distribution of firm size may be increasingly dominated over time by just a few firms.

We begin by defining an intuitive metric for index concentration at a point in time. We then examine the time series of concentration for major equity indexes over the 25 years from 1996–2021. Many cap-weighted indices are currently very concentrated. The MSCI Emerging Markets Index and the S&P 500 index have seen the largest increases in concentration, with current levels above those of the peak of the dot-com era in 1999, but no such trend is evident for the MSCI EAFE Index. Given the prominence of the S&P 500 index, we examine in detail the concentration and turnover in that index from 1975 onward. Interestingly, although concentration in the S&P 500 Index is the highest in a quarter century, it was even higher in the 1970s.

Simple descriptive statistics provide useful insight but may be limited in our understanding of future dynamics. We turn to power laws that are pervasive in the social and physical sciences to model concentrated phenomena. A random variable follows a power law distribution if the probability of a value exceeding x is proportional to x to the power of a negative constant. Power law distributions occur widely in physics, biology, earth and planetary sciences, demography, and the social sciences, including economics and finance. Of particular interest, power laws allow us to model the entire distribution of index weights and, under simple assumptions about the random growth of firm size, provide insights into the long run concentration of cap weighted indexes as shown in the appendix. When we take these concepts to the data, we find that power laws can describe index concentration well, especially in the tails of the distribution, although we are not yet close to the extreme concentration that might occur in a steady state. In other words, although the weights of the largest stocks in many indexes is high, these are well below what we would expect over the long-run with random growth in firm size.

We then turn to the question of what can be done to mitigate rising concentration. Index reconstitution offers one potential tool for index providers. We study the impact of the annual reconstitution in June of the Russell 2000 from 1996-2022. Examining historical changes in index breadth between end-May and end-June in the Russell 2000, we find strong evidence that reconstitution significantly reduces concentration.[4] Concentration decreases significantly in all 27 years for the month-pairs around the reconstitution. Running the same analysis for other pairs of sequential months shows much smaller, mostly positive changes in concentration.

Beyond reconstitution, we explore the impact of alternative index construction methodologies, including capping approaches, that index constructors might take to mitigate concentration. Monte Carlo simulations calibrated to market data provide insight into an approach to diminishing concentration, albeit at the expense of higher turnover.

Our paper contributes to the literature on index weighting and cap weighting.[5] Several empirical studies have discussed how size and sector bias may limit diversification and how dominance by stocks whose valuations have risen quickly may add to risk.[6] Questions of concentration have also gained more prominence recently because of the growth of custom indexes that seek exposure to a particular sub-industry, style factor, or theme.[7] The proliferation of exchange-traded funds (ETFs) has also played an important role in the growth of custom indexes. These new indexes typically use market-cap weighting within their geographies or industries but may use alternative weighting schemes to avoid being dominated by the largest firms in their segments.[8] Our paper provides a framework to examine these topics from a quantitative viewpoint.

We note in passing that our analysis is distinct from fundamental indexing, which weights securities on accounting or fundamental characteristics such as profits or book value, to avoid investing in stocks that are potentially overpriced and hence may dominate a traditional market-cap-weighted index. Hsu (2006), Arnott and Hsu (2008), and Arnott et al. (2008) provide a rationale for alternative indexing, Chow et al. (2011) provide a survey, and Perold (2007), and Graham (2012) offer counter-arguments. We do not make any statements about the return properties of alternative weighting schema (see Fernholz 1999), choosing to focus on the time-series of concentration.

The paper proceeds as follows: Section I provides metrics for index breadth and concentration and offers descriptive statistics for major market indexes over time; Section II provides an introduction to power laws and explores the economic rationale for why concentration may rise over time; Section III uses power laws to explain the economics of why index concentration occurs in the first place and how it may change over time; Section IV analyzes ways to mitigate these effects; and Section V concludes.

I. Index Breadth and Concentration

i. Measurement

Figure 1 shows the weight of each stock in the S&P 500 index as proxied by the iShares S&P 500 Index ETF (ticker: IVV) as of December 31, 2020, where stocks are ranked with 1 being the greatest weight. (Because of corporate actions, the index has more than 500 constituents; this variation is not uncommon.) We observe that the very largest stocks account for a significant fraction of the index relative to the tail, a characteristic of power law distributions.

Simple measures of concentration (such as the weight of the top five constituents) miss the distribution outside the extreme largest stocks. There are other metrics for concentration such as the Gini coefficient which is very commonly used in the study of income or wealth disparities. The Gini coefficient requires computing a Lorenz curve based on all the data points and ranges between 0 (total equality) and 1 (perfect inequality). Here, we focus on a closely related metric that also looks at the distribution as a whole. Specifically, we focus on breadth, defined as the inverse of the Herfindahl-Hirschman Index (HHI) over *n* stocks in the universe:

$$B_n = 1/\sum_{i=1}^{n}(S_i)^2 \qquad (1)$$

where S_i is the weight (in decimal) of stock $i = 1, \ldots, n$. HHI is widely used as a concentration metric in industrial organization.[9] Breadth is intuitive: For example, in an equally weighted index of 500 stocks, the weight of stock i is S_i= 0.002 so B = 500; a fund holding a single asset, e.g., a gold ETF, would have $B = 1$. Using equation (1), we find that the breadth of the S&P 500 is 67.5 names as of December 31, 2021. Breadth then gives rise to a natural measure of concentration that we use to compare across indexes at a point in time or to measure changes in a particular index over time.

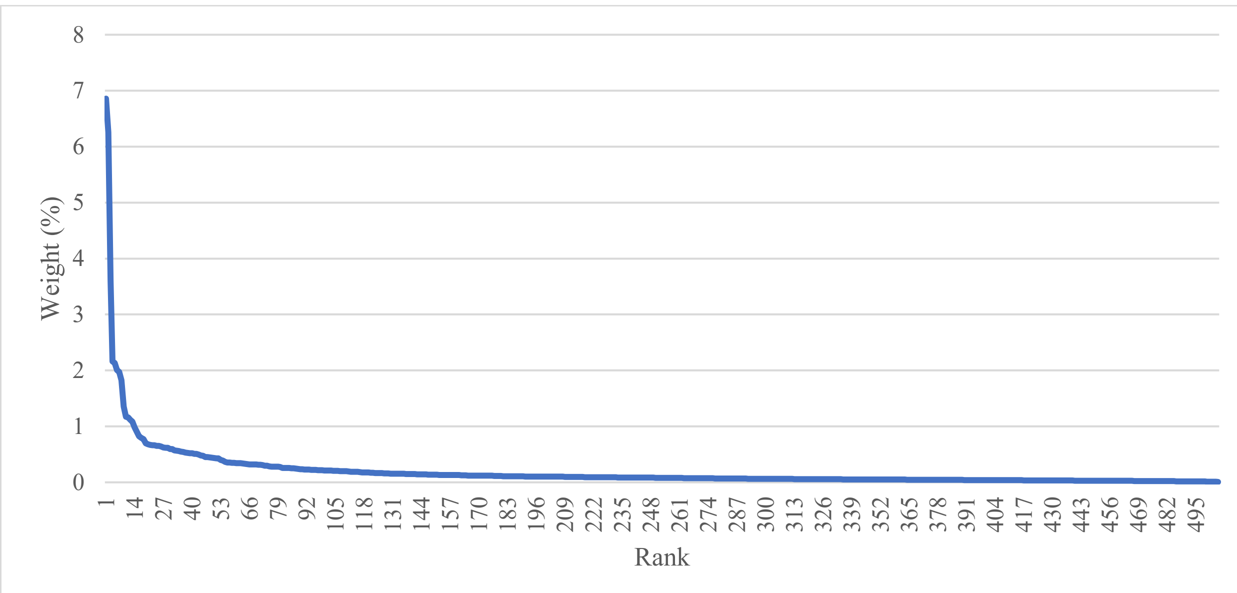

Figure 1 S&P 500 index weights as of December 31, 2020, by rank.

Source: BlackRock, based on index holdings data from MSCI on December 31, 2021.

Specifically, we scale B by the number of constituents, with $n \gg 1$ and define concentration as:

$$C_n = 1 - (B_n/n), \tag{2}$$

Concentration lies between 0 and 1. The case of $C_n = 0$ corresponds to an equal-weighting scheme, whereas $C_n \approx 1$ corresponds to dominance by one asset. One could also compute the *absorption ratio* (see Kritzman et al, 2011) which, based on a principal components analysis, equals the fraction of variability that is explained by the first few eigenvectors. In practice, all the metrics for concentration yield the same conclusions and we report results using equation (2) for simplicity.

Industry rotation may mitigate the impact of concentration because the composition of the topmost index ranks is not stable. In March 2020, Berkshire Hathaway was the fifth-largest stock until it was displaced by Alphabet, which itself was ultimately displaced by the entry of Tesla on December 21, 2020, as the largest weighting ever added to the index.[10] Indeed, none of the top five names in the S&P 500 index in December 2000 were in the top five in December 2020, as shown in Table 1.

Table 1 Rotation in top five names of S&P 500 index, 2000–2021.

December 2000	December 2010	September 2020	January 2021
General Electric	Exxon Mobil	Apple	Apple
Exxon Mobil	Apple	Microsoft	Microsoft
Pfizer	Microsoft	Amazon	Amazon
Cisco	General Electric	Facebook	Facebook
Citigroup	Chevron	Alphabet	Tesla

Sources: BlackRock; Bloomberg, as of December 31, 2021; and Debru (2020).

ii. Empirical evidence on concentration

To study the time series of concentration, we focus first on the past 25 years and four major equity indexes: S&P 500; Russell 2000; MSCI EAFE; and MSCI Emerging Markets. Figure 2 shows the history concentration (*C*) for the four indexes from 1996–2021. Since indexes are not directly investible, we use the respective iShares ETFs as proxies[11] and study their actual holdings as of December 31 of each year from 1996–2021. For one very large index, EAFE, there is no evidence that concentration increased from 1995–2021, and indeed concentration appears to have decreased, perhaps as a result of a broadening of the eligible universe.

Concentration was highest in emerging markets, first decreasing from 1996 to 2010 and then rising thereafter. But for the S&P 500, breadth declined from a maximum of 141.5 on December 31, 2016, to 67.5 on December 31, 2021, while concentration rose from 0.72 to 0.87 in that period, the highest on record in the past quarter century. The Russell 2000 is an interesting example because it is cap-weighted but focused on small cap stocks. It also has challenges with turnover, as the largest stocks in the index tend to leave upon annual rebalancing, which we will discuss later. We see rising concentration in the Russell 2000, but with a value of 0.54 as of December 31, 2021 (and breadth of 934 names), this index was less concentrated than the others.

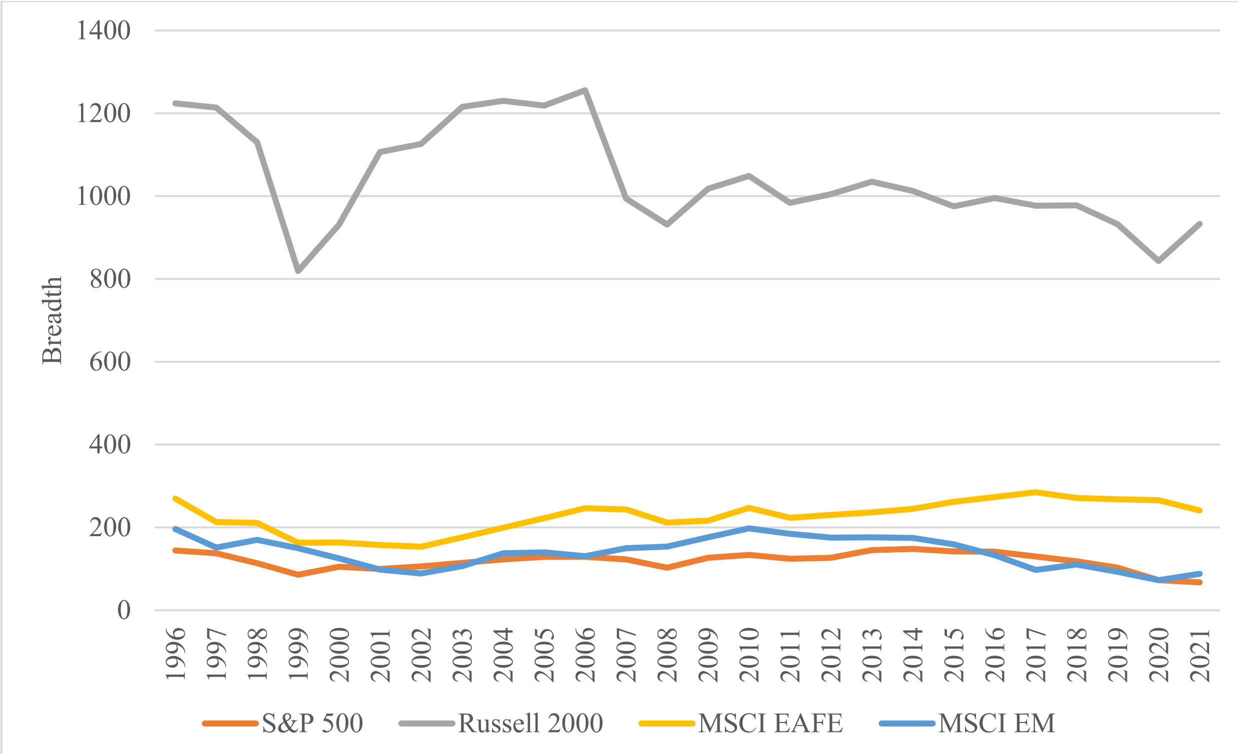

Figure 2 Time series in breadth from 1996–2021. Breadth is computed using index holdings as of December 31 of each year.

Source: Authors' estimates based on data from MSCI as of December 31, 2021.

Comparing concentration across indexes and time also suggests that there is no simple relation between concentration and the number of stocks in the index. We have seen periods when Canadian, Dutch, and Finnish markets were dominated by single stocks and very highly concentrated. While in general we expect a negative relation between the number of constituents and concentration, many other factors are at work. For example, an equally weighted S&P 500 index has a concentration that might be much less than a country fund with a few dozen stocks. We discuss this point later in the paper.

We turn now to a deeper analysis of concentration in the largest securities in an index. We use research on power laws to understand if there is systematic tendency for the upper tail to become more concentrated over time.

II. Economics of Index Concentration

i. Modeling extreme events

Power laws are ubiquitous in many fields. The distributions of the populations of cities, earthquake magnitudes, solar flares, frequency of use of words in language, firm sizes, income, and wealth have been modeled using power laws.[12] In financial markets, extreme return events are more likely and larger in magnitude than suggested by normality as witnessed by events such as the Quant Quake of 2007, Global Financial Crisis of 2009, Flash Crash of 2010, Covid Crisis of 2020, etc.[13] In this section, we discuss using power laws to (1) model the concentration in indexes, and (2) explain why concentration occurs and why it may change over time.

Mathematically, a power law is a relation of the type $Y = \zeta X^{-\alpha}$, where Y and X are variables of interest, $\alpha > 0$ is the power law exponent, and $\zeta > 0$ is a constant. Pareto's law (see Pareto [1896] and Mandelbrot [1960]) has the cumulative distribution function:

$$P(X > x) = \zeta x^{-\alpha}, \tag{3}$$

for $X > x_{min}$ where $x_{min} > 0$. This is based on Pareto's analysis of the proportion of individuals with an income above a certain level x. A lower α means a higher degree of inequality in the distribution, i.e., a greater probability of finding exceptionally large values.

A special but important case is Zipf's law where $\alpha = 1$. A linguist, Zipf (1949) found that the frequency of the i^{th} most common word in English text, denoted by S_i, is inversely proportional to its rank i out of n, so that we obtain:

$$S_i \sim 1/i. \tag{4}$$

Tests typically involve inverting Zipf's law in equation (4) together with (3) to obtain:

$$P(S > S_i) \approx Rank(S_i) = i \approx \zeta x^{-\alpha} \tag{5}$$

Equation (5) implies that:

$$\ln(i) = \ln(\zeta) - \alpha \cdot \ln(S_i) + \varepsilon \tag{6}$$

The power law coefficient in equation (6) can thus be estimated via a linear regression of log rank on log size.

ii.Economics of Power Laws

A considerable body of work has gone into explaining why power laws are prevalent in so many diverse areas. In this section, we sketch one candidate explanation for rising concentration in indexes—a phenomenon known as Gibrat's law. Following Gabaix (1999),[14] consider a stock i whose market capitalization (scaled by the total market cap of the index) at time t is denoted $S_{i,t}$. These weights evolve randomly $S_{i,t+1}= r_{i,t+1}\, S_{i,t}$ where $r_{i,t+1}= \exp(g_{i,t+1})$ and $g_{i,t+1}$ is a random growth rate (not necessarily normal) where we normalize the mean to zero. This is known as Gibrat's law. As time goes on, the distribution of firm weights will change.

In the appendix, we demonstrate that the resulting stationary distribution of cap weighted index weights is Zipf (although it need not be the only stationary distribution) when firm size follows Girbrat's law with a rule that prevents firms from becoming too small. Scale invariance drives this result. The Zipf distribution does not have a mean or any higher moments but it may be useful for understanding the extreme portion of the probability distribution above a certain size. Gibrat's Law is used here to motivate the dynamic forces giving rise to dominance of a few firms; indeed, there are other plausible causal mechanisms that might also give rise to power law distributions.[15] However, we do not make a statement that this outcome is inevitable: real world processes are more complex than the simple dynamics of these models. Further, actions by index providers, asset managers, and regulators also affect index concentration.

iii. Example: US city sizes

Before we delve into index concentration, we provide a motivating example based on Gabaix (1999). He examines the size of US cities and finds support for Zipf's law.[16] He ranks city size (1 = New York City, 2 = Los Angeles, ...) and regressed the log rank on the log of city population and estimates the power coefficient $\alpha \approx 1$. Figure 3 shows an update of Gabaix (1999) using US city population as of June 2020. Each dot is a city, and the line shown is the ordinary least squares (OLS) estimate of equation (5).

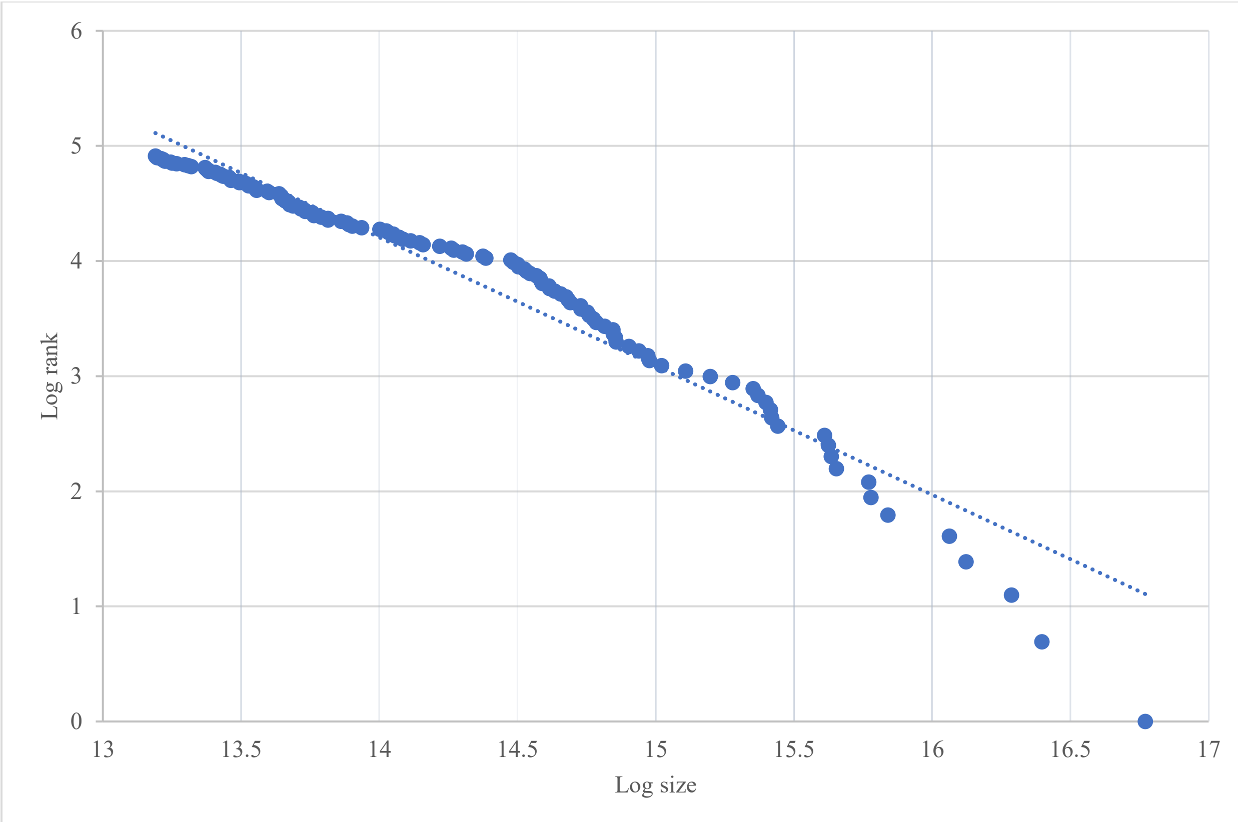

Figure 3 Log rank–log size regression for size of US cities: June 2020.

Source: Authors' estimates based on data from the Statistical Abstract of the United States (2020).

Consistent with Gabaix (1999), we focus on the top 135 metro regions and estimate a power law coefficient of α = 1.12 and R^2 = 0.95.[17] This is slightly higher in (absolute) magnitude than the estimate in Gabaix (1999). We also observe that Zipf does not fit well for the top ten cities shown on the right-hand side of the plot, although it does fit the lower tail well. When we turn to indexes, we will again see deviation from linearity in log rank/log size plots for the largest weights while the relatively substantial tails of the distributions are well fit with power laws.

While the OLS estimates are unbiased and consistent, the estimated standard error (0.022) needs modification because of autocorrelation induced by the ranking procedure. Gabaix and Ibragimov (2011) show that the correct standard error is:

$$SE = |\alpha|\sqrt{(2/n)} \tag{7}$$

where α is the OLS estimate of the slope coefficient in equation (6). This calculation yields an adjusted standard error of 0.136, so we cannot reject the null hypothesis that the distribution is Zipf. The regression is, however, very sensitive to the choice of truncation point. When we use 415 metro regions as our cutoff, we obtain $\alpha = 0.81$, and the null hypothesis that the power law coefficient is Zipf is rejected.[18] We conclude that Zipf may not capture the size distribution of the largest metro areas, although it does fit the lower tail well. We will see the same pattern expressed when we apply these tools to analyze index concentration.

III. Empirical Analysis Using Power Laws

i. Using power law distributions to model index concentration

Let us revisit the results on the four indexes featured in Figure 2 using the regression approach of equation (6) to investigate whether power laws capture the concentration in the largest names. Figure 4 shows a scatter plot of log rank on log size for four public equity indexes on December 31, 2021. There is a concave relationship between log rank and log size for all four indexes,[19] and each has a substantial, linear tail. This relationship is present throughout our data set. The concavity means that the distribution of firm sizes cannot, in its entirety, be well fit with a power law. Instead, we use formula (6) to fit power laws to the tails for the distributions, where the log rank/log size plots are approximately linear.[20] For each index on each date, we compute the mean m and standard deviation s of log size. Then, we fit a regression line to observations for which log size exceeds the cutoff $m + s$.

The table below shows the estimated power law coefficients, the standard error of the estimate computed by formula (7), the number of observations in each tail, and the goodness of fits measured by R-squared. The fits are excellent, with R-squared values greater than or equal to 0.97.

Index	Power law coefficient	Standard error	Number of observations	R^2
S&P 500	1.33	0.21	82	0.99
Russell 2000	3.08	0.22	377	0.97
MSCI EAFE	1.87	0.22	139	0.99
MSCI EM	1.40	0.14	207	0.99

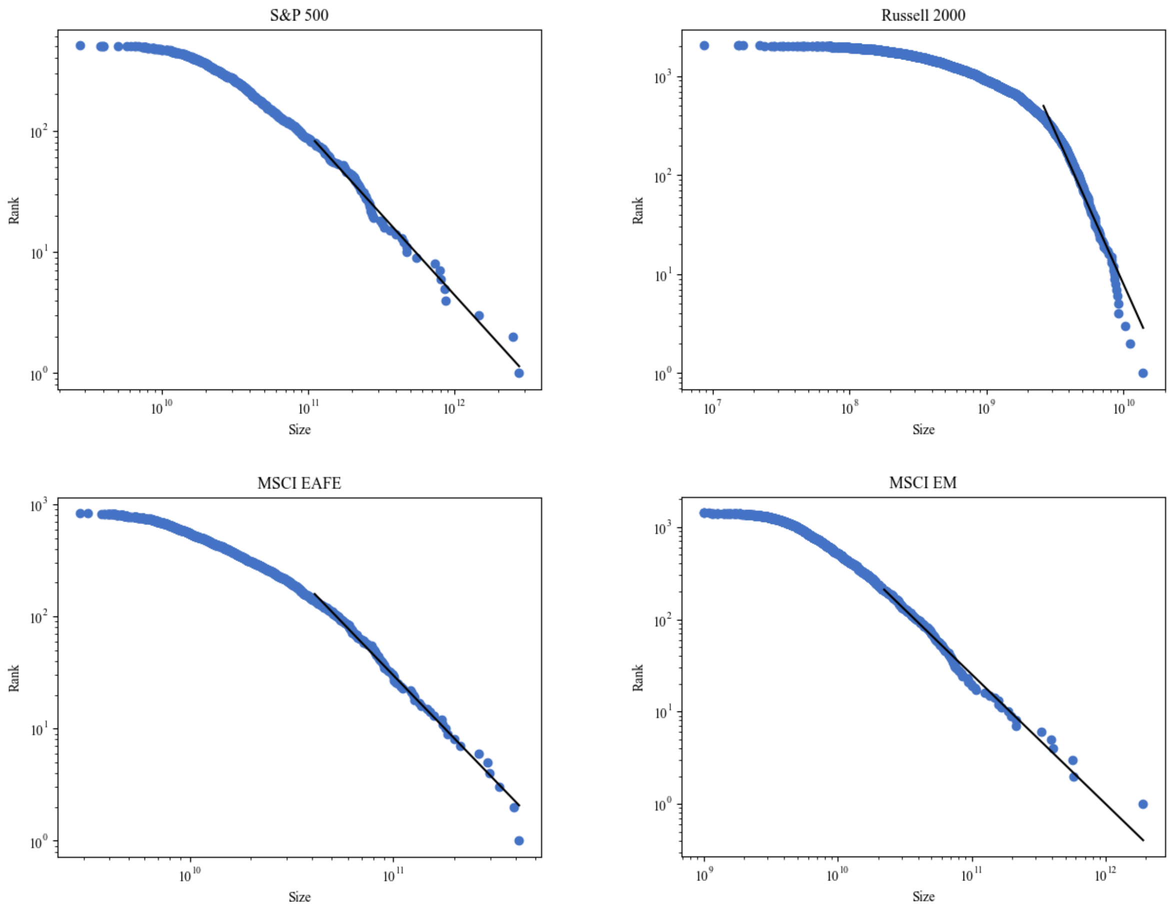

Figure 4 Rank-size scatter plots for four public equity indexes: December 2021.

Source: Authors' estimates based on data from BlackRock as of December 31, 2021.

We reject the Zipf distribution for three of the four indexes since the value of 1 is outside the interval alpha +/ SE. For the S&P 500, the value 1 is within the interval, and we fail to reject the Zipf distribution as a descriptor of the lower tail of the distribution, i.e., the stocks on the right of Figure 1.

Figure 5 plots the estimated power law coefficient α by year for the four indexes. A comparison of Figures 2 and 5 illustrates the inverse relationship between concentration and power law coefficients. Concentration was lowest and estimated power law coefficients were highest and least stable for the Russell 2000, the one index in the group that excludes the largest securities in its market.

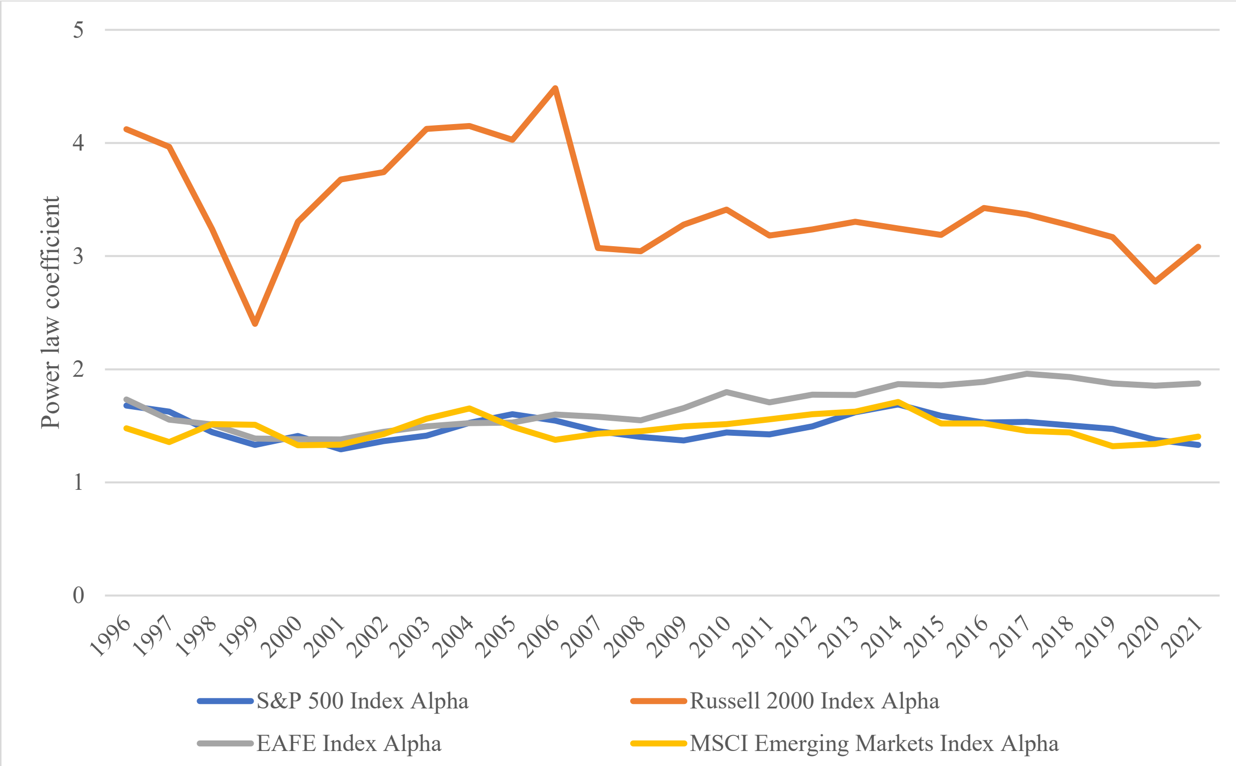

Figure 5 Estimated power law coefficients from 1996–2021 for major equity indexes.

Source: Authors' estimates based on data from BlackRock as of December 31 of each year.

Table 2 shows, for each index, averages over time of the estimated power law coefficient, the standard error computed by formula (7), the number of observations in the tail, and the goodness of fit measured by R-squared. The fits are good throughout the study period, with average R-squared values ranging from 0.94 for the Russell 2000 Index to 0.99 for the MSCI EM Index. Figure 5 implies that $\alpha = 1.5$ is a better fit to cap-weighted indices than the Zipfian power law ($\alpha = 1$). Indeed, the Zipf distribution is rejected throughout the study period for three of the four indexes, and it is borderline for the S&P 500 index.

Table 2 Average statistics for estimated power law coefficients from 1996–2021 for major equity indexes.

Index	Power law coefficient	Standard error	Number of observations	R^2
S&P 500	1.48	0.24	78.54	0.96
Russell 2000	3.43	0.26	349.15	0.94
MSCI EAFE	1.67	0.19	162.81	0.96
MSCI EM	1.48	0.18	143.69	0.99

Source: Authors' estimates based on data from 1996–2021 sourced from BlackRock as of December 31, 2021.

ii. Zipf and current index concentration

The discussion so far indicates that although the tails of market-cap distributions are well described by power laws, the coefficients are significantly different from 1. Another way of stating this fact is that while power laws (although not Zipf) work in describing the bulk of the distribution, they suggest higher levels of concentration for the largest firms than is observed in the data.

To illustrate, we compare empirical index weights directly to weights predicted by a Zipf distribution. The weight of constituent *i* under Zipf is given by:

$$S_i = 1/(i \sum_{j=1}^{n} 1/j), \tag{8}$$

where *n* is the number of constituents. Define $H_n^z = \sum_{j=1}^{n} 1/j \approx \ln(n) + \gamma$, where $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant, which we use to construct the Zipf weights. For $n > 1$, these weights are given by:

$$S_i = 1/iH_n^z \tag{9}$$

which is decreasing in *n*. For the largest five constituents from *n*, we obtain from equation (9): $S_1 = 1/H_n^z$, $S_2 = 1/2H_n^z$,…, $S_5 = 1/5H_n^z$. So, applying Zipf's Law to the S&P 500, the top five constituents with $n = 500$ would have weights, in percent, of 14.69, 7.34, 4.90, 3.67, and 2.94, respectively. Figure 6 compares the top 20 theoretical and actual weights (in percent) as of December 31, 2021, based on holdings of the iShares fund that seeks to track the S&P 500.

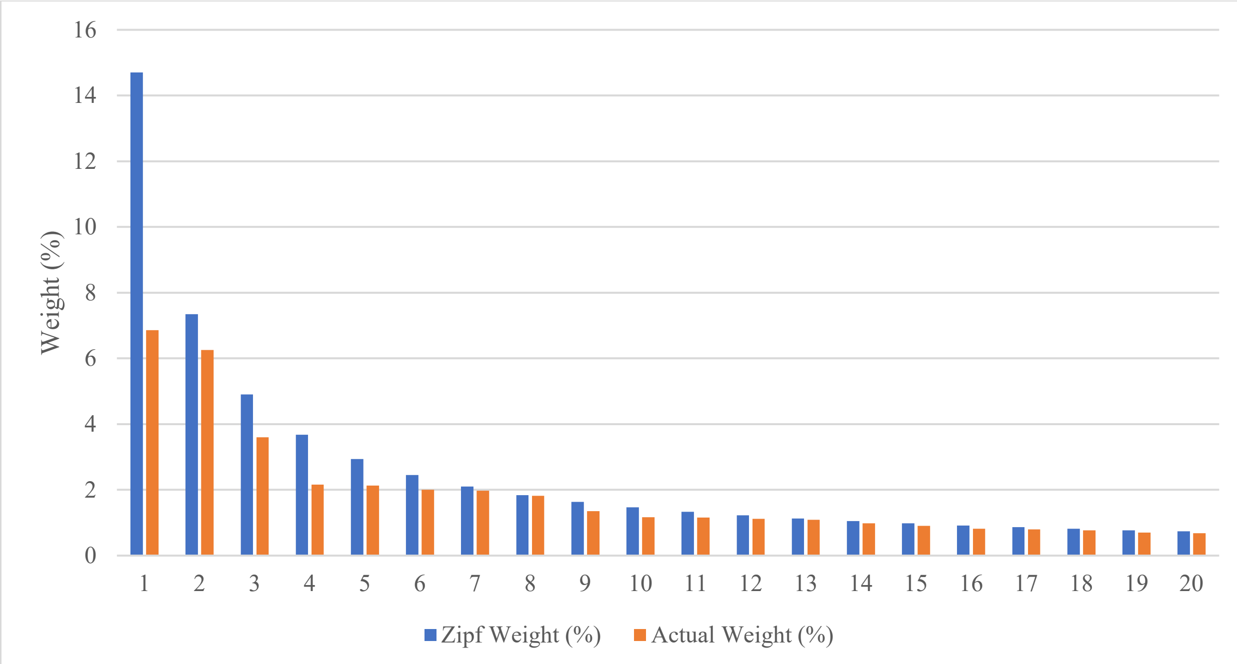

Figure 6 Top 20 Zipf and actual weights for the S&P 500 index.

Source: Authors' estimates as of December 31, 2021, based on index holdings data from MSCI.

Figure 6 shows that we are still far from Zipf in terms of concentration in the upper tail, that is the left-most 20 stocks of Figure 1 with $n = 500$. Indeed, the implied breadth of the Zipf distribution is $B = 28.2$, implying a high concentration $C = 0.95$, well above the actual index concentration. The actual top five holding weights (in percent, as of December 31, 2020) were 6.68, 5.30, 4.38, 2.07, and 1.69. As with the analysis of city size in Figure 3, we observe deviations from the Zipf weights in the largest stocks.

Will expanding the number of constituents in the index (n) mitigate concentration? Since the Zipfian weight $S_i = 1/iH_n^z$ from equation (9) is decreasing in n, this seems plausible. However, the answer is no. Figure 7 shows the Zipfian weights of the top security (lower line) and the top ten securities (upper line) as a function of n. We see that the weights of the top securities quickly decrease as n increases but approach a constant level asymptotically. So, even if the S&P 500 were expanded to 2,000 stocks, we still would not be able to lower the Zipf weight of the top stock to below 10% or the weight of the top ten to below 35%. This also illustrates why the link between the number of constituents and concentration is complex.

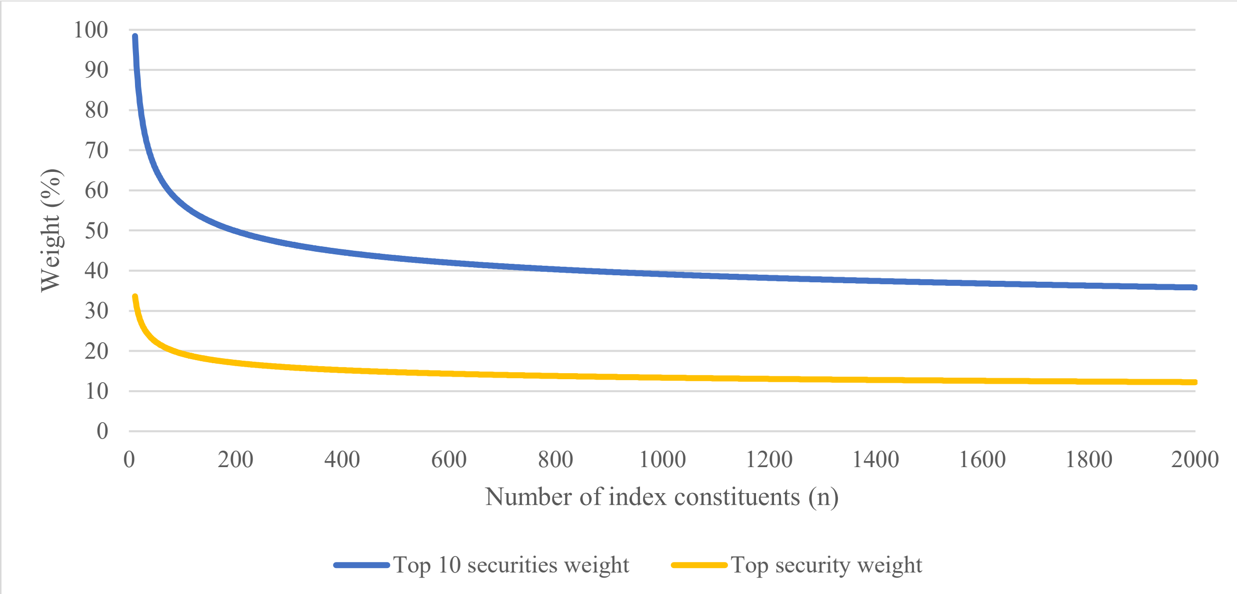

Figure 7 Zipfian weight of the top security (bottom line) and top ten securities (upper line) as a function of the number of constituents *n*.

Source: Authors' computations based on equation (9).

In summary, although Zipf is a stationary distribution for cap weights under a random growth process, we still have a long way to go before we get to levels of concentration that a Zipfian distribution would imply in the largest stocks. Rather, a power law with $\propto = 1.5$ may a be good descriptor of the tail of the distribution of index weights. We turn now to more a practical question: What can index providers do to mitigate rising concentration, and at what cost?

IV. Mitigating Concentration

i. Rebalancing and Reconstitutions

All indexes, including market capitalization weighted indexes, need to periodically reconstitute and rebalance due to changes in index inclusion criteria or corporate actions. Reconstitution refers to adding or deleting securities from an index based on the index criteria, while rebalancing refers to adjusting index weights as specified by the index provider to reflect corporate actions such as mergers or spin-offs, etc. Periodic reconstitution potentially offers index providers a tool to mitigate concentration by adding new constituents and migrating the largest to other indexes.

Finance theory does not offer much guidance regarding the impact of index reconstitution on concentration. Intuition suggests that reconstitution would have potentially large positive impacts on breadth for market cap weighted indexes that change at both the top and the bottom and also equal weighted indexes. We would expect the smallest effects for large cap indexes.

In this section, we study the impact on concentration of the annual reconstitution of the Russell 2000 small stock index from 1996-2022. A comparison of the effect of index reconstitution across indexes is beyond the scope of this paper but is an interesting topic for future research. The Russell 1000 and 2000 together constitute the Russell 3000 index, a broad-based index of US stocks. The Russell indexes are rebalanced on the last Friday in June the smallest in large cap indexes that use market cap weighting. The Russell 2000 is an ideal candidate for this analysis because the annual reconstitution is based on market capitalization rank and hence is free of subjective biases (see Madhavan, Ribando, and Udevbulu, 2022). Further, no other index provider rebalances that week.

We expect the reconstitution to increase breadth (i.e., lower concentration as shown in equations (1) and (2)) as larger firms migrate up to the Russell 1000 and smaller firms from outside the Russell 3000 enter at the bottom. We examine historical changes in breadth between end-May and end-June in the Russell 2000, and find strong evidence that reconstitution significantly increases breadth and, equivalently from equation (2), reduces concentration. The change in breadth is positive in all 27 reconstitutions, with a mean change of 192.2. Breadth increases dramatically post-reconstitution as shown in Figure 8. and this is especially evident at the end the dot com era in 1999-2000. The effect of increased breadth is to reduce concentration by 8% overall, a substantial figure.

As a check, we ran the same analysis for all other pairs of sequential months and found much smaller, mostly positive changes in concentration and negative changes in breadth. The changes in breadth for other pairs of months are nothing like those shown in Exhibit 8. This suggests that index reconstitutions are an avenue for index providers to mitigate concentration concerns. Index providers may choose to have larger or more frequent reconstitutions to maintain breadth.

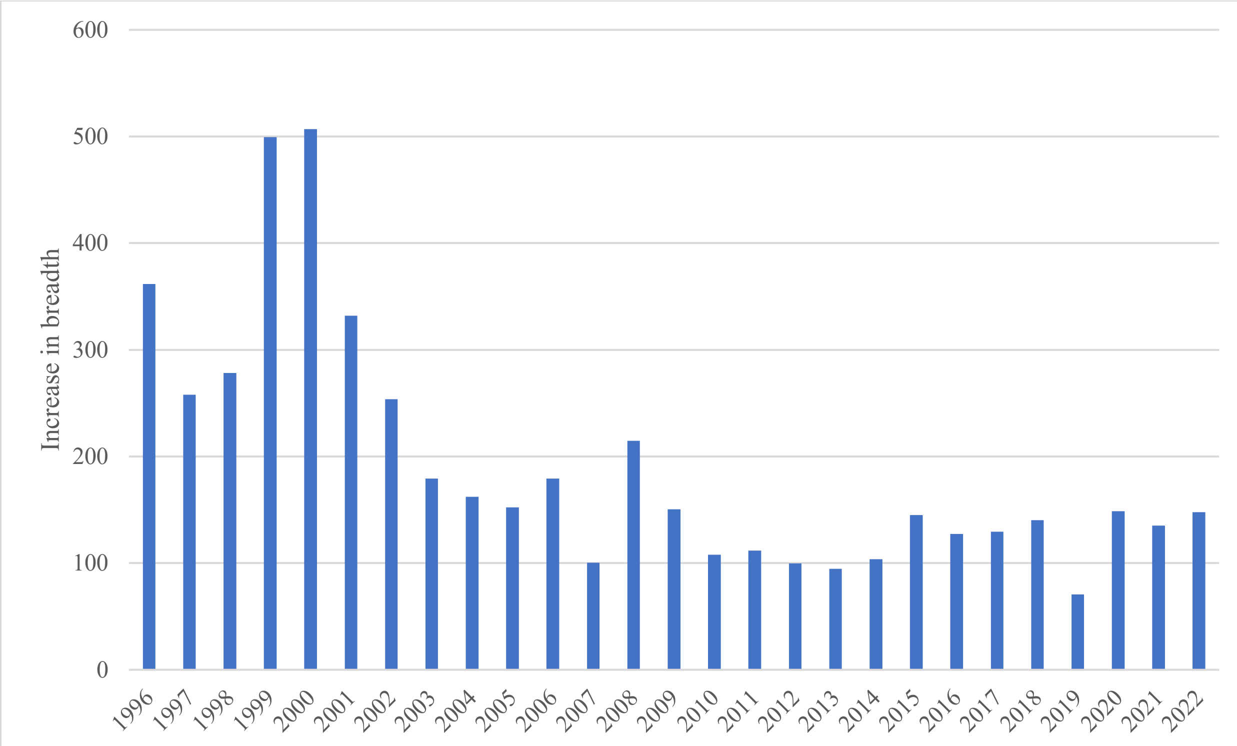

Figure 8 Changes in index breadth for the Russell 2000 index from 1996-2022 from end-May to end-June.

Source: Authors' computations based on equation (1).

ii. Directly Addressing Concentration

Beyond periodic reconstitutions, index providers have experimented with ways to diversify and have made changes to their methodologies to reflect new developments. We examine the impact of alternative weighting schemes on the concentration of cap-weighted indexes. A popular scheme is equal weighting (e.g., weight of $1/N$), which has the advantage of simplicity, albeit with higher turnover.[21] Our focus here is on approaches that preserve cap weighting to some extent. We first consider the MSCI 25/50 rule, which mandates that no single issuer should account for more than 25% of an index, and the aggregate weight of issuers each exceeding 5% of the index should not exceed 50% of the index's total assets. We next consider the UCITS 5/10/40 rule, which is similar in form: it limits ownership in a single issuer to 10% and caps total ownership of issuers with weight exceeding 5% to 40%.

An alternative is *diversity weighting* which derives from mathematical theory. Diversity weighting schema are of interest because of their properties. Define S_i to be the market cap weight of i. Fernholz (2005) defines a family of (diversity) index weighting schemes based on a parameter p (for $0 < p < 1$) where π_i, the weight of constituent i, is proportional to S_i^p:

$$\pi_i = S_i^p / \sum_{j=1}^{n} S_j^p \tag{10}$$

As p ranges from 0 to 1, this diversity weighting scheme in equation (10) interpolates between equal weighting and market-cap weighting.[22] From a practical perspective, diversity weighting offers a systematic way to interpolate between equal and cap weighting which may be of interest to sophisticated institutional clients in separately managed accounts.

iii. Can capping rules mitigate concentration?

We examine the ability of various rules to mitigate concentration in the S&P 500 index, expanding the date range from January 31, 1975, to December 31, 2021, with a monthly data frequency using data from MSCI. Over this period, neither the MSCI 25/50 nor UCITS 5/10/40 rules were binding on the S&P 500 index, so we focus our empirical analysis on diversity weighting.

In Figure 9, we show the time evolution of concentration in the S&P 500 index and its diversity-weighted counterpart. Trends in concentration, breadth, and power law exponent of the diversity-weighted and cap-weighted S&P 500 index are in sync, but the absolute levels are materially different. At the end of 2021, for example, the breadth of the S&P 500 index, or effective number of securities, was 67.5, while diversity weighting increased that value to 327.1. The corresponding power law coefficient increased from 0.89 to 1.78. We also include the equal-weighted index for comparisons in breadth, concentration, and turnover.

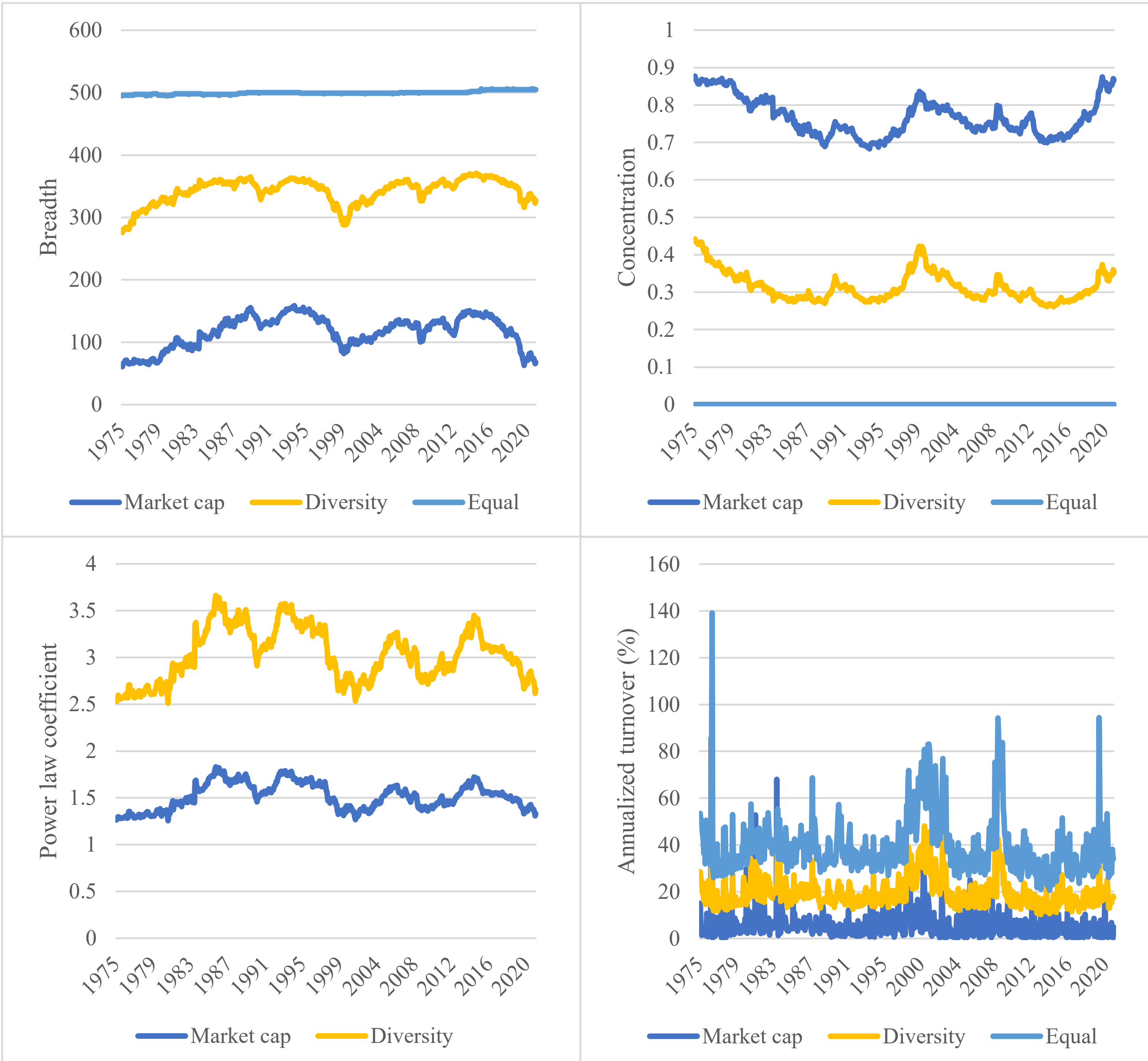

Figure 9 Concentration metrics and turnover for the S&P 500: January 1975-December 2021

Source: Authors' estimates using monthly market cap and index holdings data from MSCI, as of December 31, 2021

Diminished concentration in a cap-weighted index comes at the cost of excess turnover, which was generally increased by diversity and equal weighting, as shown in Figure 9. Exceptions include periods in which large-cap securities entered or exited the index. In these cases, the reduction in weights of these securities under diversity weighting trumped the excess turnover. One such example is the entrance of Tesla in December 2020. As of December 31, 2020, Tesla's weight in the S&P 500 index was 1.69%. Diversity-weighted, Tesla would comprise 0.72% of the index, enough reduction to bring the total month-to-month turnover down below that of the cap-weighted index. The enormous spike in turnover in the equal-weighted index in 1976 is explained by the large number of additions and deletions in that year. Figure 9 illustrates the positive association between breadth and power law. In our data sample, the correlation between the two measures is 0.45. Note the big turnover spikes for the equally weighted S&P 500 in the bottom right panel of Figure 9; these appear to correspond to a big drift upward in concentration just prior to reconstitution consistent with Figure 8.

Figure 10 shows the difference in annualized turnover between market-cap and diversity-weighted S&P 500 index. We see that diversity weighting leads to generally higher turnover and there are times when this is significantly larger.

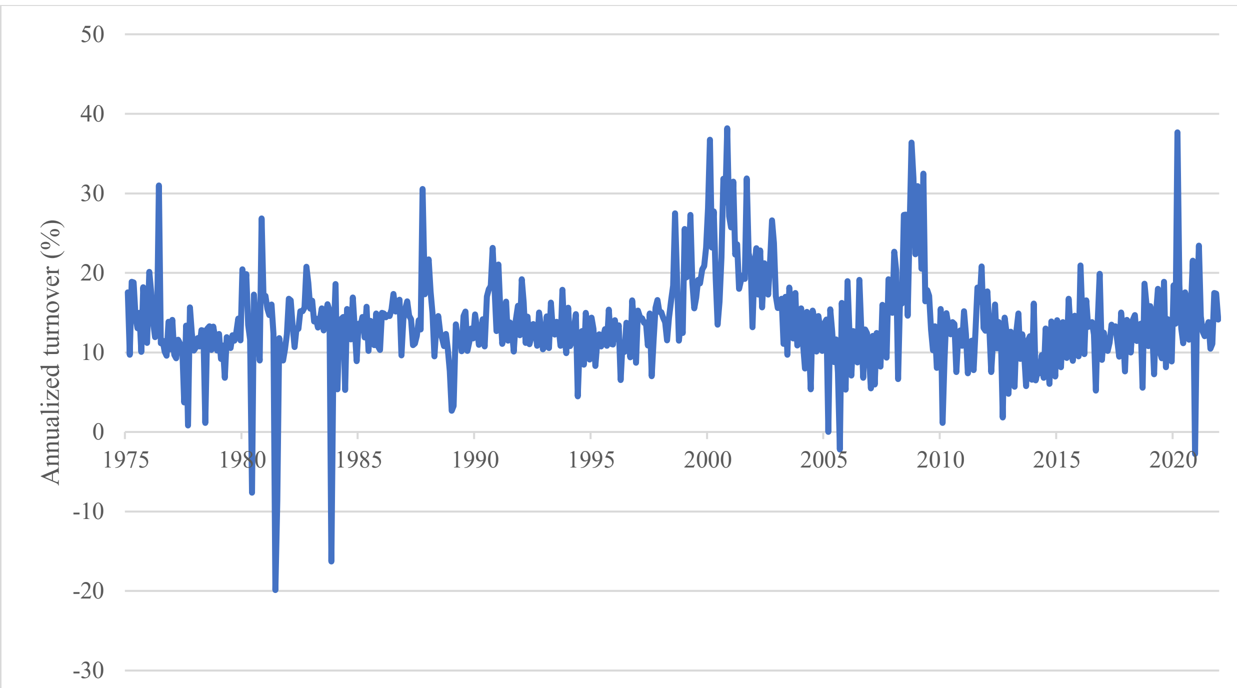

Figure 10 Difference in annualized turnover between market-cap and diversity-weighted S&P 500 index.

Source: Authors' estimates using monthly market cap and index holdings data from MSCI, as of 12/31/2021.

iv. Using simulation to evaluate concentration mitigation schemes

Simulation sheds light on concentration mitigation schemes. We follow Gabaix (1999), who models the evolution of normalized market cap, the fraction of the market accounted for by a firm, with a reflecting geometric Brownian motion, which we specify below.[23]

Suppose each firm in an index grows randomly so that $S_{i,t+1} = \gamma_{i,t+1} S_{i,t}$ where:

$$\gamma_{i,t+1} = \exp(g_{i,t+1} + e_{i,t+1}) \tag{10}$$

Here, $g_{i,t+1}$ is interpreted as the organic growth rate (which could be positive or negative) plus a random shock $e_{i,t+1}$ so that:

$$\ln(S_{i,T}) = \ln(S_{i,0}) + \sum g_{i,t} + \sum e_{i,t} \tag{11}$$

As T gets large, the distribution of size becomes a log normal distribution through the central limit theorem. In this distribution, unlike Zipf, the higher moments are all well-defined. Gabaix (1999) shows that with a minimum threshold for viable inclusion size (quite plausible in the index context), equation (11) leads to a Pareto distribution as specified in equation (3).

In the appendix, we specify our model in more detail including the key parameters. Here, we provide values for parameters, which are calibrated to market data. The simulation is run in monthly intervals with inputs set accordingly. We set volatility to 0.118, which is calibrated from monthly normalized growth of firms in the S&P 500 index between January 1975 and January 2022. This translates to an annualized volatility of approximately 41%, which is reasonable for individual stocks. Portfolio volatility is, of course, much lower. Our hypothetical index is taken to be the 500 largest firms in a universe of N=4,000 securities. The other key parameter is the smallest fraction of the market that can be realized by a firm, denoted by $S_{\min}$. We chose S_{min} so that NS_{min}= 0.35, which, as shown in the appendix, is equivalent to a power law exponent α of approximately 1.5, consistent with our reported results.

Continuing our focus from the empirical section on cap-weighting and diversity weighting with p = 0.5, we show breadth, concentration, power law exponent, and annualized turnover for our simulated indexes in Figure 11.[24] Since the simulated index is reconstituted monthly, we measure turnover across quarterly intervals to approximate the empirical index's reconstitution cycle. Simulation results are based on 1,000 experiments. Across our experiments, diversity weighting elevated the median breadth of the simulated cap-weighted index from 123 to 407. The corresponding median values of concentration were 0.75 and 0.19. Diversity weighting raised the power law exponent of the simulated cap-weighted index from 1.51 to 3.03.[25] The cost of diversity can be measured in turnover, whose annualized median value was 21% for the simulated cap-weighted index and 47% for its diversity-weighted counterpart.

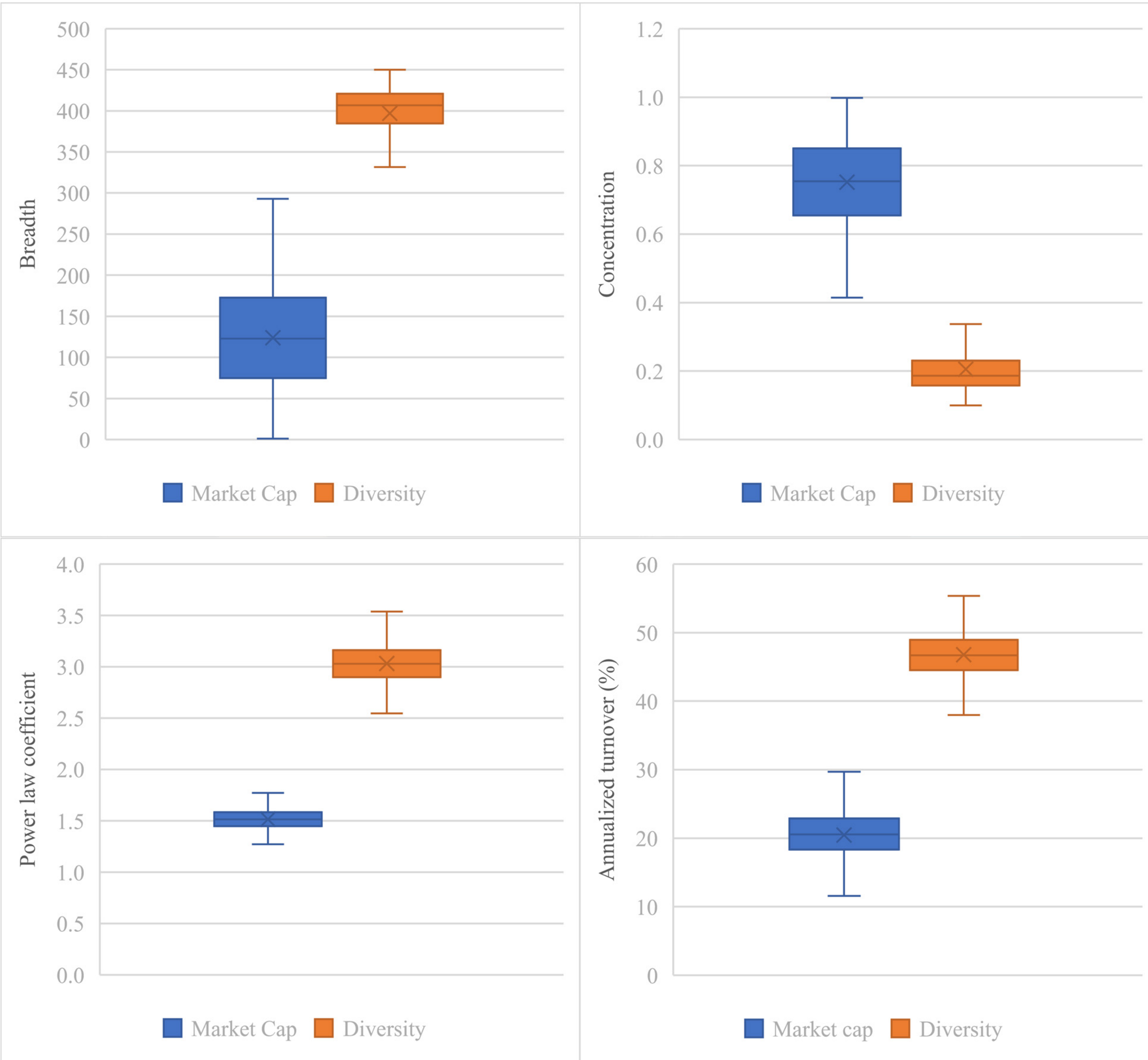

Figure 11 Concentration metrics and turnover for simulated indexes.

Source: Authors' computations.

So, with turnover over twice as high in the simulated diversity-weighted index, the corresponding transaction cost drag will similarly be higher. Assuming costs of 0.2% in individual stocks, this amounts to an annual drag of 0.094% vs. 0.04% under cap weighting. With, say, \$24 trillion following major indexes, this amounts to additional costs of $13 billion per annum, a huge amount to index investors.

In Figure 12, we look at the dependence of index concentration on the diversity weight exponent *p*. The figure shows this dependence for the S&P 500 index on December 31, 1993, when the market was relatively diversified, and on December 31, 2021, when the market was relatively concentrated.

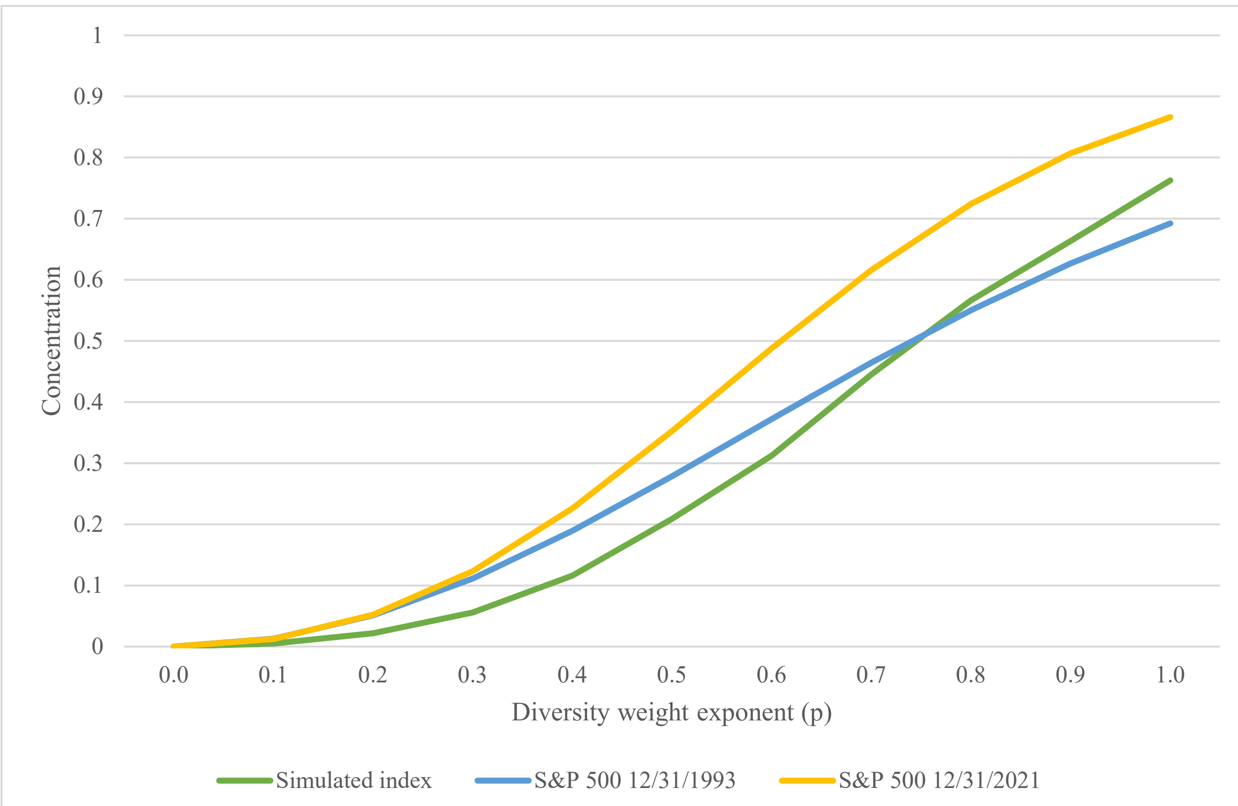

Figure 12 Dependence of concentration on diversity weight exponent for the S&P 500 index on two dates and a simulated index

Source: Authors' computation and market cap and index holdings data from MSCI.

As we vary the diversity weight exponent p, we obtain a *concentration curve*, which shows how the concentration of a diversity weighted index varies for a fixed set of market caps as p ranges between 0 and 1. We show the curve for the cap weights taken from the median path of our experiment along with two curves based on empirical cap weights of the S&P 500 index on two different dates. Moving from equal weights (p = 0) to cap weights ($p = 1$), concentration emerges rose more slowly for the median simulated index than the empirical indexes. For cap weights, concentration of the median simulated index, 0.76, lies in between the December 1993 value of 0.69 and the December 2021 value of 0.87.

Finally, we look at the time it takes to go from a starting distribution to a steady-state distribution of market caps in our simulation. We run three experiments distinguished by assumption on initial state. The simplest is that all firms begin at zero growth, which corresponds to an equally weighted initial distribution of normalized market caps. We also use trailing one-year growth rates for firms in the S&P 500 as of December 31,1993, when the market was relatively diversified, and as of December 31, 2021, when the market was relatively concentrated. The median time to steady state decreased from 550 months to 530 months to 496 months for the three cases, i.e., 40–45 years in our simulation. Of course, this assumes no mitigating efforts. We conclude that even under simplistic assumptions for the random growth of firms, the extreme dominance

V. Conclusion

Market-capitalization weighting is a traditional and popular approach to constructing indexes. Developed initially for performance measurement, market-cap-weighted indexes are now common benchmarks for equity index–tracking funds and ETFs. The relatively low turnover in cap-weighted indexes together with their association with the market portfolio are important rationales for their widespread adoption. However, there are increased concerns that some cap-weighted indexes may suffer from overly concentrated positions in a few stocks, limiting diversification and exposing investors to the risks that some of the largest holdings are overpriced.

We find that some, but not all, major indexes have seen increases in concentration in the past quarter century. To put this in context, we examine the historical concentration of major equity indexes. It turns out that concentration varied dramatically across indexes, and the current levels, albeit quite high, are not very unusual by historical standards. For example, the S&P 500 index was more concentrated in December 1975 than in December 2021.

The academic literature on concentrated phenomena, including distributions of words, sizes of cities, and other areas in which the largest players dwarf the smaller ones offers several plausible mechanisms that may help (1) explain concentration in firm size, and (2) provide insight into its evolution over time. These mechanisms suggest that the eventual distribution of cap weights in a market or index may be approximated by power laws in the tails. We generally reject the Zipf distribution for current indexes, meaning we are far from the possible extremes of concentration suggested by theory, particularly for the largest constituents.

We show that index reconstitution can be an effective tool to mitigate concentration, at least in the case of an index where firms can both exit above and enter below. We suspect this to be the case also for diversity weighted indexes including the special case of equal-weight indexes. However, this is a topic for future research. To assess both the efficacy of a basic theory that predicts concentration and the cost of concentration mitigation schemes, we develop a simulation based on a geometric Brownian motion with a reflecting barrier. This model fits empirical data remarkably well, suggesting that market concentration may, to some degree, be the inevitable consequence of dynamic forces.

The framework developed herein has several practical uses. For example, index providers may use our findings to use index reconstitutions proactively to manage concentration, perhaps by reconstituting more frequently or by changing the minimum criterion for index inclusion. In considering the choice of index benchmark, asset managers can use the approach developed here to assess the trade-off between lower concentration and higher turnover. We find that concentration mitigation schemes are effective, but they generate additional turnover and higher transaction costs, and so may reduce investors' returns relative to cap weighting. Our framework offers a potential to gauge the relative merits of these mitigation efforts. As usual, there is no free lunch, and we conclude that market-cap weighting is not likely to be displaced in the near term.

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Appendix

Under random growth for firm size, Zipf is a stationary distribution

Proof: This derivation follows Gabaix (1999).

Since the market capitalization weights add to 1, we have

$$\sum_{i=1}^{N} S_{i,t} = 1 \tag{A.1}$$

This implies that

$$\mathrm{E}[r] = \int_0^\infty r f(r) dr = 1 \tag{A.2}$$

where $f(r)$ is the distribution function of *r*, which in turn derives from the distribution function of the random growth rates. At time *t*, the distribution of weights is $G_t(S) = P(S_t > x)$.

If there exists a stationary distribution[26] $G(S)$ of index weights, then

$$G_{t+1}(S) = G_t(S) = G(S). \tag{A.3}$$

Combining equations (A.2) and (A.3):

$$G_{t+1}(S) = P(S_{t+1} > S) = P(r_{t+1}S_t > S) = G_t(\frac{S}{r}) = \int_0^\infty G(\frac{S}{r}) f(r) dr \tag{A.4}$$

We see that $G(S) = b/S$ where b is a constant is a solution (using equation (A.1) where $\int_0^\infty r f(r) dr = 1$. This establishes that Zipf is a stationary long-run distribution under random growth for firm size.

Simulation Formulation

We begin with the growth rate distribution, which is expressed in terms of an arithmetic Brownian motion that is 0 at time 0:

$$dX_t = m_t + \sigma B_t. \tag{A.4}$$

The evolution of normalized market caps over time is given by

$$S_t = S_{min}\, exp(X_t + L_t) \quad \text{(A.5)}$$

where $L_t = inf_0 \leq s \leq t \cdot min(0, X_s)$, and S_{min} is the smallest fraction of the market that can be realized by a firm. In this setting, S hits the barrier S_{min} precisely when the driving Brownian motion X hits a new low. As explained in Gabaix (1999), when the underlying Brownian motion's drift is negative and given by:

$$m = (-\sigma^2/2) * 1/(1 - S_{min}/S_{avg}), \quad \text{(A.5)}$$

the distribution of S achieves a steady-state power law distribution with exponent

$$\alpha = 1/(1 - S_{min}/S_{avg}) \quad \text{(A.6)}$$

Here, S_{avg} is the expected value of S_t, which is the same for each time t. In a market with N firms, we set S_{avg}= 1/N, which gives us the steady-state power law exponent:[27]

$$\alpha = 1/(1 - NS_{min}) \quad \text{(A.7)}$$

This completes the description of the simulation.

Endnotes

[1] See, e.g., Madhavan (2016). A prominent exception is the Dow Jones Industrial Average, which is price weighted.

[2] See, e.g., Wigglesworth (2022).

[3] Baxter (2022) discusses ETF concentration risk and notes "Even deliberately diffuse funds can still be caught by overexposure to once-fashionable companies." See also Morningstar (2020) at https://www.morningstar.com/articles/992504/the-sp-500-grows-ever-more-concentrated. Debru (2020) notes that as of November 2020, the five biggest stocks in the S&P 500 represented 21% of the index, almost double the long-term average of 12.5%. Academic concerns go back much further, as in Strongin et al. (2000), Bernstein (2003), Tabner (2007), and Malevergne et al. (2009).

[4] We thank the anonymous reviewer for this and many other excellent suggestions.

[5] Roll (1977) famously questions the common use of cap-weighted equity indexes as a proxy for the market portfolio. The point is carried further in Bohn et al. (2022), who document a large-

cap bias in cap-weighted subportfolios of market indexes. Using the methodology from their paper, they find the largest decile of securities has more than 90% of the active weight of the S&P 500 Index against the Russell 1000 Index. See also Jiang et al. (2020).

[6] See, e.g., Amenc et al. (2011) for a review. See also Haugen and Baker (1991), Grinold (1992), Amenc et al. (2012), Tabner (2007), and Malevergne et al. (2009), among others. Levy (2016) compares the market with a large number of randomly constructed and passively held portfolios and finds that 69% of these random portfolios yield higher Sharpe ratios than the market.

[7] Newer, factor, and thematic indexes are not just performance benchmarks but are designed also to be investible. Such indexes may be non–market cap weighted. Robertson (2019) finds a recent proliferation of indexes. She documents "substantial heterogeneity" across indexes and concludes that the overwhelming majority of the indexes in her sample "are used as a primary benchmark by only a single fund."

[8] In the factor space, concentration has grown larger, and investors need to be conscious of unintended factor allocations arising from following common indexes. Madhavan et al. (2018) show there is large time-series variation in the exposures of common indexes, implying that the implicit factor loadings of indexes are time varying. Their analysis breaks down the style factor exposure of each benchmark index using stock-by-stock style scores.

[9] The breadth of a portfolio or an index can be thought of as its effective number of securities. Consider three portfolios consisting of two stocks each, with weight distributions of 0.5–0.5, 0.8–0.2 and 0.99–0.01. These portfolios have breadth of, respectively, 2.00, 1.47, and 1.02.

[10] See, e.g., Santilli (2020).

[11] We use the actual iShares exchange-traded funds (ETFs) that seek to track these indexes.

[12] Power laws are possibly so prevalent across seemingly diverse applications because of their aggregation properties. The sum of two (independent) power law distributions yields another power law distribution. The product of two power laws or their max (or min) also yields a power law distribution.

[13] For applications in economics/finance, see Axtell (2001), Bouchard et al. (2009), Gabaix (1999, 2009), Saichev et al. (2010), and Fernholz and Fernholz (2020), among many others. Shankar et al. (2015) use power laws to analyze changes in the distribution of trading volumes of S&P 500 stocks and non-S&P 500 stocks over time.

[14] Gibrat (1931) proposes proportional random growth as the source of power law distributions. Other explanations include the transfer of that power law through matching and optimization and network effects.

[15] A Zipf distribution may also arise through optimization. In designing a language, for example, very long words would be rare while very common words would be short. This may also arise through chance, as for example, with monkeys typing on keyboards with a space marking the end of a word.

[16] Zipf's law has been used in index construction in the crypto space. The T3 Crypto-X Power Index, initiated in January 2014, seeks to represent a value of investment in a portfolio of top ten cryptocurrencies by market capitalization with weighting determined by Zipf's law chosen as "a middle ground between investing according to market capitalization, which heavily prioritizes Bitcoin and a few other major cryptocurrencies, and equal distribution which puts too high of a risk on less established assets." Source: https://t3index.com/indices/crypto-xp/.

[17] We take α as the negative of the estimated slope coefficient by convention.

[18] In the original 1999 study, data for only 135 metro regions was available. Today, the US data covers more than 400 cities.

[19] In the language of Fernholz and Fernholz (2020), the plots in Figure 5 are "quasi-Zipfian" and can be fit to rank-based Atlas models.

[20] Examples of models that fit a power law only to the tail of an empirical distribution are found, for example, in Embrechts et al (1997) and Goldberg et al. 2008.

[21] A discussion of equally weighted strategies is in DeMiguel et al. (2009).

[22] The properties of diversity weighting strategies are developed in Fernholz et al. (1998), Fernholz (1999), Fernholz (2002), and Fernholz (2005). Diversity weighting is a generalization of equal weighting, which is studied in Booth and Fama (1992) and Fernholz and Maguire (2007). Fernholz (1999) argues that diversity weighted portfolios will outperform cap weighted portfolios over the long run.

[23] See Harrison (2013), Chapter 1.

[24] We omit results for MSCI 25/50 rule and the UCITS 5/10/40 rule, which were rarely binding.

[25] While the median power law of the simulated cap-weighted index of 500 firms was 1.51, the corresponding exponent for the simulated full universe of 4,000 firms was 1.54, consistent with theory.

[26] Gabaix (1999) provides the necessary conditions for the existence of a stable, time-invariant distribution.

[27] Lowering the product NS_{min}, by lowering either the number of firms in the simulation or the reflecting barrier, drives the power law exponent closer to 1, increasing instability in estimates of concentration and turnover.