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Abstract

This paper demonstrates that the budget sets in the Contingent Markets (CM) and Financial Markets (FM) economies with infinite dimensional commodity spaces might not be norm-bounded and therefore might not be weakly (weakly*) compact. The lack of weak (weak*) compactness of these budget sets has serious implications. In particular, it is no longer guaranteed that weakly (weakly*) continuous utility functions attain their maximum on these budget sets. Thus, the individual and therefore total demand functions need not exist in CM and FM economies with infinite dimensional commodity spaces. The lack of existence of demand functions does not however imply the lack of existence of equilibrium in CM and FM economies with infinite dimensional commodity spaces.

Keywords: Demand Functions, Infinite Dimensional Commodity Spaces, GE, GEI.
JEL Classification: D50; D52; D53.

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1. INTRODUCTION

This paper demonstrates that budget sets in the Contingent Markets (CM) and Financial Markets (FM) economies with infinite dimensional commodity spaces might not be norm-bounded and therefore might not be weakly (weakly*) compact. The lack of weak (weak*) compactness of these budget sets has serious implications. In particular, it is no longer guaranteed that weakly (weakly*) continuous utility functions attain its maximum on these budget sets. Thus, the individual and therefore total demand functions need not exist in CM and FM economies with infinite dimensional commodity spaces. The lack of existence of demand functions does not, however, imply the lack of existence of equilibrium in CM and FM economies with infinite dimensional commodity spaces as is evident for example from Magill and Quinzii (1994) and (1996).

The paper is organized as follows. Section 2 demonstrates that budget sets in CM economies with infinite dimensional commodity spaces might not be norm-bounded and therefore might not be weakly (weakly*) compact. Section 3 demonstrates that the budget sets in FM economies with infinite dimensional commodity spaces might not be norm-bounded and therefore might not be weakly (weakly*) compact. Section 4 concludes.

2. THE LACK OF WEAK (WEAK*) COMPACTNESS OF BUDGET SETS IN CM ECONOMIES WITH INFINITE DIMENSIONAL COMMODITY SPACES

For the rest of this section let

$$\mathcal{E}_{\infty}\left(ET,\ \left(X,\ X'\right),\ e,\ \succeq\right)$$

be an infinite horizon CM economy, where I be the set of infinitely living consumers such that

$$|I| < \infty,$$

ET be the event-tree and L be the set of commodities traded on spot markets such that

$$\max\left[|ET|,\ |L|\right] = \infty.$$

The following sequence spaces defined below are the most natural candidates for the commodity space X.

DEFINITION: *For* $1 \leqslant \mathbf{p} < \infty$ *define the sequence space* $L_{\mathbf{p}}(ET \times L)$ *as follows*

$$L_{\mathbf{p}}(ET \times L) = \left\{ x = \{x(\xi,\ l)\}_{(\xi,\ l)\in ET\times L} \in \mathbb{R}^{\infty} \mid \parallel x \parallel_{\mathbf{p}} = \left[\sum_{(\xi,\ l)\in ET\times L} \mid x(\xi,\ l)\mid^{\mathbf{p}}\right]^{\frac{1}{p}} < \infty \right\}.$$

DEFINITION: *For* $\mathbf{p} = \infty$ *define the sequence space* $L_{\mathbf{p}}(ET \times L)$ *as follows*

$$L_{\infty}(ET \times L) = \left\{ x = \{x(\xi,\ l)\}_{(\xi,\ l)\in ET\times L} \in \mathbb{R}^{\infty} \mid \parallel x \parallel_{\infty} = \sup_{(\xi,\ l)\in ET\times L} \mid x(\xi,\ l)\mid < \infty \right\}.$$

DEFINITION: *Define the sequence space* $c(ET \times L)$ *as follows*

$$c(ET \times L) = \left\{ x = \{x(\xi,\ l)\}_{(\xi,\ l)\in ET\times L} \in L_{\infty}(ET \times L) \mid \lim_{(\xi,\ l)\longrightarrow\infty} x(\xi,\ l) = x \in \mathbb{R} \right\}.$$

DEFINITION: *Define the sequence space* $c_0(ET \times L)$ *as follows*

$$c_0(ET \times L) = \left\{ x = \{x(\xi,\ l)\}_{(\xi,\ l) \in ET \times L} \in c(ET \times L) \mid \lim_{(\xi,\ l) \longrightarrow \infty} x(\xi,\ l) = 0 \right\}.$$

DEFINITION: *Define the sequence space* $\varphi(ET \times L)$ *as follows*

$$\varphi(ET \times L) =$$
$$\{x = x(\xi,\ l) \in c_0(ET \times L) \mid \exists S \subset ET \times L \text{ s.t. } |S| < \infty \text{ and } x(\xi,\ l) = 0\ \forall (\xi,\ l) \in (ET \times L) \setminus S\}.$$

Recall also that

PROPOSITION 2.1. (p. 430 of Aliprantis and Border): *For* $1 \leqslant \mathbf{p} \leqslant \mathbf{q} < \infty$

$$\varphi(ET \times L) \subset L_{\mathbf{p}}(ET \times L) \subset L_{\mathbf{q}}(ET \times L) \subset c_0(ET \times L) \subset c(ET \times L) \subset L_\infty(ET \times L) \subset \mathbb{R}^\infty.$$

PROPOSITION 2.2. (p. 431 of Aliprantis and Border):

$$(L_{\mathbf{p}}(ET \times L),\ \|\cdot\|)' = L_{\mathbf{q}}(ET \times L),$$

where

$$1 \leqslant \mathbf{p} < \infty,$$
$$1 < \mathbf{q} \leqslant \infty,$$
$$\frac{1}{\mathbf{p}} + \frac{1}{\mathbf{q}} = 1.$$

PROPOSITION 2.3. (p. 431 of Aliprantis and Border):

$$(c_0(ET \times L),\ \|\cdot\|)' = L_{\mathbf{1}}(ET \times L).$$

PROPOSITION 2.4. (p. 431 of Aliprantis and Border):

$$(L_\infty(ET \times L),\ \|\cdot\|)' = ba(ET \times L).$$

PROPOSITION 2.5. (p. 431 of Aliprantis and Border):

$$\left(\mathbb{R}^{|ET \times L|},\ \tau\right)' = \varphi(ET \times L),$$

where τ *is the product topology on* X.

DEFINITION: *Let*

$$\mathcal{E}_\infty(ET,\ (X,\ X'),\ e,\ \succeq)$$

be an infinite horizon CM economy. Define the budget set for agent $i \in I$ *as*

$$B_\infty(P_i,\ e_i) = \{c_i \in X_+ \mid P_i \cdot c_i \leqslant P_i \cdot e_i\},$$

where

$$P_i = \{P_i(\xi) \mid \xi \in ET\} = \{\pi_i(\xi) \cdot p(\xi) \mid \xi \in ET\}.$$

Here we will investigates compactness of the budget set

$$B_\infty(P_i,\, e_i) \subset X$$

under various topologies.
When

$$|ET \times L| < \infty,$$

the budget set

$$B(P_i,\, e_i) = \left\{ c_i \in \mathbb{R}_+^{|ET \times L|} \mid P_i \cdot c_i \leqslant P_i \cdot e_i \right\},$$

will be $\|\cdot\|$-compact $\forall P_i \in \mathbb{R}_{++}^{|ET \times L|}$. When

$$|ET \times L| = \infty,$$

the situation is different.
Indeed, using the **Proposition 2.6.** below, we will show here that for some of the most natural candidates for the dual pair $(X,\, X')$ s.t.

$$(X, \;\; \|\cdot\|)' = X'$$

in the infinite horizon CM economies

$$\mathcal{E}_\infty \left(ET, \;\; (X, \;\; X') \,, \;\; e, \;\; \succeq \right),$$

the budget set

$$B_\infty(P_i,\, e_i) \subset X$$

is not $\sigma(X,\, X')$-compact.
PROPOSITION 2.6.: (Problem #40, p. 238 of Royden): *Let*

$$(X, \;\; \|\cdot\|)' = X'$$

and

$$S \subset X$$

be $\sigma(X,\, X')$-compact. Then S is $\|\cdot\|$-bounded.
Therefore, if we show that for some of the most natural candidates for the dual pair $(X,\, X')$ s.t.

$$(X, \;\; \|\cdot\|)' = X'$$

in the infinite horizon CM economies

$$\mathcal{E}_\infty \left(ET, \;\; (X, \;\; X') \,, \;\; e, \;\; \succeq \right),$$

the above budget sets are not $\|\cdot\|$-bounded $\forall P_i \in c_0 \left(ET \times L \right)$, we are done.
LEMMA 2.7.: *Let*

$$\mathcal{E}_\infty \left(ET, \;\; (X, \;\; X') \,, \;\; e, \;\; \succeq \right)$$

be an infinite horizon CM economy, where

$$X = L_{\mathbf{p}}\left(ET \times L\right) \text{ for } 1 \leqslant \mathbf{p} \leqslant \infty, \\ X' = \left(L_{\mathbf{p}}(ET \times L),\ \ \|\cdot\|\right)'.$$

Then the budget set

$$B_{\infty}(P_i,\, e_i) \subset X$$

is not $\|\cdot\|_p$*-bounded*

$$\forall (P_i,\, e_i) \in c_0\left(ET \times L\right)_+ \times \left[L_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]^{1}.$$

PROOF: Fix

$$(P_i,\, e_i) \in c_0\left(ET \times L\right)_+ \times \left[L_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right].$$

If

$$(P_i,\, e_i) \in \left[c_0\left(ET \times L\right)_+ \setminus c_0\left(ET \times L\right)_{++}\right] \times \left[L_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]$$

we are done.
If

$$(P_i,\, e_i) \in c_0\left(ET \times L\right)_{++} \times \left[L_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right],$$

then set

$$c_{\xi,\, l} = \left(0, ..., 0, \tfrac{P_i \cdot e_i}{P_i(\xi,\, l)}, 0, ..., 0\right) \in X.$$

Therefore,

$$c_{\xi,\, l} \in B_{\infty}(P_i,\, e_i).$$

Since

$$P_i \in c_0\left(ET \times L\right),$$

we can conclude that

$$\forall A > 0\ \exists\, (\xi,\ l) \in ET \times L \ s.t. \\ P_i(\xi,\, l) < \tfrac{P_i \cdot e_i}{A}.$$

So

$$\forall A > 0\ \exists\, (\xi,\ l) \in ET \times L \ s.t. \\ \tfrac{P_i \cdot e_i}{P_i(\xi,\, l)} > A.$$

Therefore,

[1] In the trivial case, where $e_i = 0$ we clearly have that the budget set $B_{\infty}(P_i,\, e_i) \subset X$ is not $\|\cdot\|_{\mathbf{p}}$-bounded $\forall P_i \in c_0\left(ET \times L\right)_+ \setminus c_0\left(ET \times L\right)_{++}$ and is $\|\cdot\|_{\mathbf{p}}$-bounded $\forall P_i \in c_0\left(ET \times L\right)_{++}$.

$$\|c_{\xi,\, l}\|_{\mathbf{p}} = \begin{cases} \sqrt[\mathbf{p}]{\left|\frac{P_i \cdot e_i}{P_i(\xi,\, l)}\right|^{\mathbf{p}}} > A & \text{for } 1 \leqslant \mathbf{p} < \infty \\ \frac{P_i \cdot e_i}{P_i(\xi,\, l)} > A & \text{for } \mathbf{p} = \infty \end{cases} .$$

Hence, the above budget set is not $\|\cdot\|_{\mathbf{p}}$-bounded $\forall (P_i,\, e_i) \in c_0\,(ET \times L)_+ \times \left[L_{\mathbf{p}}\,(ET \times L)_+ \setminus \{0\}\right]$. ■

THEOREM 2.8.: *Let*

$$\mathcal{E}_\infty\,(ET,\ \ (X,\ X')\,,\ e,\ \ \succeq)$$

be an infinite horizon CM economy, where

$$X = L_{\mathbf{p}}\,(ET \times L) \text{ for } 1 \leqslant \mathbf{p} \leqslant \infty,$$
$$X' = (L_{\mathbf{p}}(ET \times L),\ \ \|\cdot\|)'.$$

Then the budget set

$$B_\infty(P_i,\, e_i) \subset X$$

is not $\sigma(X,\, X')$-compact

$$\forall (P_i,\, e_i) \in c_0\,(ET \times L)_+ \times \left[L_{\mathbf{p}}\,(ET \times L)_+ \setminus \{0\}\right]^2.$$

PROOF: Since the above budget sets are not $\|\cdot\|_{\mathbf{p}}$-bounded, we can conclude by the above **Proposition 2.6. (Problem #40, p. 238 of Royden)** that they are not $\sigma(X,\, X')$-compact

$$\forall (P_i,\, e_i) \in c_0\,(ET \times L)_+ \times \left[L_{\mathbf{p}}\,(ET \times L)_+ \setminus \{0\}\right]. ■$$

THEOREM 2.9.: *Let*

$$\mathcal{E}_\infty\,(ET,\ \ (X,\ X')\,,\ e,\ \ \succeq)$$

be an infinite horizon CM economy, where

$$X = \mathbb{R}^{|ET \times L|},$$

τ is the product topology on X and

$$(X,\ \tau)' = X' = \varphi\,(ET \times L).$$

Then the budget set

$$B_\infty(P_i,\, e_i) \subset X$$

is not τ-compact $\forall P_i \in X' = \varphi\,(ET \times L)$.

PROOF: Fix

$$\left(\overline{\xi},\ \overline{l}\right) \in (ET \times L) \setminus S.$$

Therefore,

[2] In the trivial case, where $e_i = 0$, we clearly have that the budget set $B_\infty(P_i,\, e_i) \subset X$ is not $\sigma(X,\, X')$-compact $\forall P_i \in c_0\,(ET \times L)_+ \setminus c_0\,(ET \times L)_{++}$ but is $\sigma(X,\, X')$-compact $\forall P_i \in c_0\,(ET \times L)_{++}$.

$$P_i\left(\overline{\xi},\ \overline{l}\right) = 0$$

and so

$$\left[0,\ \frac{P_i \cdot e_i}{P_i\left(\overline{\xi},\ \overline{l}\right)}\right] \subset \mathbb{R}$$

is not $|\cdot|$-compact in $\mathbb{R}$. Hence,

$$\exists \left\{x_{i\lambda}\left(\overline{\xi},\ \overline{l}\right)\right\} \subset \left[0,\ \frac{P_i \cdot e_i}{P_i\left(\overline{\xi},\ \overline{l}\right)}\right],$$

which has no $|\cdot|$-convergent subnet. Define now net

$$\{c_{i\lambda}\} \subset B_\infty(P_i,\ e_i)$$

as follows

$$c_{i\lambda}\left(\xi,\ l\right) = \begin{cases} x_{i\lambda}\left(\xi,\ l\right) \text{ for } \left(\xi,\ l\right) = \left(\overline{\xi},\ \overline{l}\right) \\ 0 \text{ for } (ET \times L) \setminus \left\{\left(\overline{\xi},\ \overline{l}\right)\right\} \end{cases}.$$

Because the product topology τ on X is the topology of coordinatewise convergence and

$$\left\{x_{i\lambda}\left(\overline{\xi},\ \overline{l}\right)\right\} \subset \left[0,\ \frac{P_i \cdot e_i}{P_i\left(\overline{\xi},\ \overline{l}\right)}\right]$$

has no $|\cdot|$-convergent subnet, we can conclude that

$$\{c_{i\lambda}\} \subset B_\infty(P_i,\ e_i) \subset X$$

has no τ-convergent subnet. Therefore, $B_\infty(P_i,\ e_i)$ is not τ-compact $\forall P_i \in X'$. ■

We will now move to establishing the lack of weak* compactness of the above budget set. It should be noted that when the Banach space X is reflexive, i.e., when

$$X'' = X,$$

we have that

$$\sigma(X,\ X') = \sigma(X'',\ X').$$

That is, the weak topology on X coincides with the weak* topology on X''. Therefore, the weak compactness of a subset of X coincides with the weak* compactness of a subset of X''.

Next, we will consider the case when the Banach space X is not necessarily reflexive, i.e., when

$$X'' = X$$

does not necessarily hold.

Following exactly the case of weak compactness, we will show here using the **Corollary of Alaoglu's Theorem 2.11.** below that for some of the most natural candidates for the dual pair $(X,\ X')$ s.t.

$$(X,\ \|\cdot\|)' = X'$$

in the infinite horizon CM economies

$$\mathcal{E}_{\infty}\left(ET,\ \left(X',\ X\right),\ e,\ \succeq\right),$$

the budget set

$$B_{\infty}(P_i,\, e_i) \subset X'$$

is not $\sigma(X',\, X)$-compact $\forall P_i \in c_0\left(ET \times L\right)$.

ALAOGLU'S THEOREM 2.10. (Theorem #47, p. 237 of Royden): *Let*

$$\left(X,\ \left\|\cdot\right\|\right)' = X'.$$

Then

$$\left\{f \in X' \mid \left\|f\right\| \leqslant 1\right\} \subset X'$$

is $\sigma(X',\, X)$-compact.

COROLLARY OF ALAOGLU'S THEOREM 2.11.: *Let*

$$\left(X,\ \left\|\cdot\right\|\right)' = X'.$$

Then

$$S \subset X'$$

be $\sigma(X',\, X)$-compact iff S is $\sigma(X',\, X)$-closed and $\left\|\cdot\right\|$-bounded.

Therefore, if we show that for some of the most natural candidates for the dual pair $(X,\, X')$ the above budget set is not $\left\|\cdot\right\|$-bounded $\forall (P_i,\, e_i) \in c_0\left(ET \times L\right)_+ \times \left[L'_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]$, we are done.

LEMMA 2.12.: *Let*

$$\mathcal{E}_{\infty}\left(ET,\ \left(X',\ X\right),\ e,\ \succeq\right)$$

be an infinite horizon CM economy, where

$$X' = L_{\mathbf{p}}\left(ET \times L\right) \textit{ for } 1 \leqslant \mathbf{p} \leqslant \infty,$$
$$L_{\mathbf{p}}\left(ET \times L\right) = \left(X,\ \left\|\cdot\right\|\right)',$$
$$P_i \in c_0\left(ET \times L\right).$$

Then the budget set

$$B_{\infty}(P_i,\, e_i) \subset X'$$

is not $\left\|\cdot\right\|_{\mathbf{p}}$-bounded

$$\forall (P_i,\, e_i) \in c_0\left(ET \times L\right)_+ \times \left[L'_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]^{3}.$$

PROOF: The proof will go exactly the same way as in the previous case for the infinite horizon CM economy

$$\mathcal{E}_{\infty}\left(ET,\ \left(X,\ X'\right),\ e,\ \succeq\right).\ \blacksquare$$

[3] In the trivial case, where $e_i = 0$, we clearly have that the budget set $B_{\infty}(P_i,\, e_i) \subset X'$ is not $\left\|\cdot\right\|_{\mathbf{p}}$-bounded $\forall P_i \in c_0\left(ET \times L\right)_+ \setminus c_0\left(ET \times L\right)_{++}$ and is $\left\|\cdot\right\|_{\mathbf{p}}$-bounded $\forall P_i \in c_0\left(ET \times L\right)_{++}$.

THEOREM 2.13.: *Let*

$$\mathcal{E}_\infty\left(ET,\ \left(X',\ X\right),\ e,\ \succeq\right)$$

be an infinite horizon CM economy, where

$$X' = L_{\mathbf{p}}\left(ET \times L\right) \text{ for } 1 \leqslant \mathbf{p} \leqslant \infty,$$
$$L_{\mathbf{p}}\left(ET \times L\right) = \left(X,\ \left\|\cdot\right\|\right)'.$$

Then the budget set

$$B_\infty(P_i,\ e_i) \subset X'$$

is not $\sigma(X',\ X)$-compact

$$\forall (P_i,\ e_i) \in c_0\left(ET \times L\right)_+ \times \left[L'_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]^4.$$

PROOF: Since the above budget set is not $\left\|\cdot\right\|_{\mathbf{p}}$-bounded, we can conclude by the above **Corollary of Alaoglu's Theorem 2.11.** that it is not $\sigma(X',\ X)$-compact

$$\forall (P_i,\ e_i) \in c_0\left(ET \times L\right)_+ \times \left[L'_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]. \ \blacksquare$$

Note the condition that

$$P_i \in c_0\left(ET \times L\right)_+$$

is not too restrictive. Indeed, recall that for

$$0 \leqslant \mathbf{p} \leqslant \mathbf{q} < \infty$$

we have that

$$\varphi\left(ET \times L\right) \subset L_{\mathbf{p}}\left(ET \times L\right) \subset L_{\mathbf{q}}\left(ET \times L\right) \subset c_0\left(ET \times L\right) \subset c\left(ET \times L\right) \subset L_\infty\left(ET \times L\right) \subset \mathbb{R}^\infty.$$

The lack of $\sigma(X,\ X')$-compactness of these budget sets has serious implications. In particular, we are no longer guaranteed that $\sigma(X,\ X')$-continuous utility functions

$$U_i : X_i \longrightarrow \mathbb{R}$$

attain their maximum on these budget sets. Thus, in an infinite horizon CM economy

$$\mathcal{E}_\infty\left(ET,\ \left(X,\ X'\right),\ e,\ \succeq\right)$$

the individual demand for commodities $c_i(P_i,\ e_i)$ defined as

$$c_i \in \arg\max\left\{U_i(c_i) \mid c_i \in B_\infty(P_i,\ e_i)\right\} \ \forall i \in I,$$

might not exist for some

$$(i,\ P_i) \in I \times \mathbb{R}_+^{|ET \times L|}.$$

[4] In the trivial case, where $e_i = 0$, we clearly have that the budget set $B_\infty(P_i,\ e_i) \subset X'$ is not $\sigma(X',\ X)$-compact $\forall P_i \in c_0\left(ET \times L\right)_+ \setminus c_0\left(ET \times L\right)_{++}$ and is $\sigma(X',\ X)$-compact $\forall P_i \in c_0\left(ET \times L\right)_{++}$.

The following example below shows that demand functions need not exist in economies with infinite dimensional commodity spaces.

EXERCISE 6 (p. 176 of Aliprantis, Brown and Burkinshaw): *Let*

$$\mathcal{E}_{\infty}\left(ET,\ \left(X,\ X'\right),\ e,\ \succeq\right)$$

be an infinite horizon CM economy s.t.

A1. ET be s.t.

$$b\left(\xi\right) = \mid \xi^{+} \mid = 1\ \forall \xi \in ET.$$

A2.

$$\begin{array}{c} X = L_2(ET), \\ X' = L_2(ET). \end{array}$$

A3. Agent's preferences $\succeq_i$ *on*

$$X_i = L_2(ET)_+$$

are given by the utility function

$$\begin{aligned} U_i(c_i) &= \sum_{(\xi,\ l) \in ET \times L} \Pr_i(\xi) \cdot b_i^{T(\xi)} \cdot u_{i\ \xi}(c_i(\xi,\ l)) \\ &= \sum_{\xi \in ET} \Pr_i(\xi) \cdot b_i^{T(\xi)} \cdot u_{i\ \xi}(c_i(\xi,\ 1)) \\ &= \sum_{T=1}^{\infty} \left(\tfrac{2}{3}\right)^{\frac{T}{2}} c_{it+T}^{\frac{1}{2}}. \end{aligned}$$

Then, while $U_i(\cdot)$ *is strictly monotone, strictly concave and* $\|\cdot\|_2$*-continuous, we have that*

$$B_{\infty}(p,\ e_i) = \{c_i \in L_2(ET)_+ \mid p \cdot c_i \leqslant p \cdot e_i = 1\}$$

is not $\|\cdot\|_2$*-compact and*

$$\arg\max\left\{U_i(c_i) \mid c_i \in B_{\infty}(p,\ e_i)\right\} = \varnothing\ \forall i \in I.$$

PROOF: It is obvious that $U_i(\cdot)$ is strictly monotone and strictly concave. Let's show that $U_i(\cdot)$ is $\|\cdot\|_2$-continuous. Indeed, if we show that

$$\forall \left\{c_{in}\right\}_{n \in \mathbb{N}} \subset \left(l_2\right)_+$$

s.t.

$$c_{in} \stackrel{\|\cdot\|_2}{\longrightarrow} \bar{c}_i,$$

we have that

$$U_i(c_{in}) \longrightarrow U_i(\overline{c_i})$$

we are done.

So suppose

$$c_{in} \xrightarrow{\|\cdot\|_2} \bar{c}_i.$$

Therefore, since

$$\{c_{in}\}_{n\in N} \subset (l_2)_+$$

is a $\|\cdot\|_2$-converging sequence we can conclude that

$$\exists M > 0$$

s.t.

$$\| c_{in} \|_2 = \sqrt{\sum_{T=1}^{\infty} |c_{it+Tn}|^2} \leqslant M^2 \ \forall n \in \mathbb{N}.$$

Hence,

$$\sum_{T=1}^{\infty} |c_{it+Tn}|^2 \leqslant M^4 \ \forall n \in \mathbb{N}.$$

Thus,

$$|c_{it+Tn}| \leqslant M^2 \ \forall n \in \mathbb{N}$$

and so

$$\left|c_{it+Tn}^{\frac{1}{2}}\right| \leqslant M \quad \forall n \in \mathbb{N}$$

and

$$\left|\bar{c}_{it+T}^{\frac{1}{2}}\right| \leqslant M \quad \forall n \in \mathbb{N}.$$

Therefore,

$$\left(\tfrac{2}{3}\right)^{\frac{T}{2}} \cdot \left|c_{it+Tn}^{\frac{1}{2}}\right| \leqslant \left(\tfrac{2}{3}\right)^{\frac{T}{2}} \cdot M \quad \forall n \in \mathbb{N},$$
$$\left(\tfrac{2}{3}\right)^{\frac{T}{2}} \cdot \left|\bar{c}_{it+T}^{\frac{1}{2}}\right| \leqslant \left(\tfrac{2}{3}\right)^{\frac{T}{2}} \cdot M \quad \forall n \in \mathbb{N}.$$

Since

$$\sum_{T=1}^{\infty} \left(\tfrac{2}{3}\right)^{\frac{T}{2}} = \frac{\left(\frac{2}{3}\right)^{\frac{1}{2}}}{1-\left(\frac{2}{3}\right)^{\frac{1}{2}}} < \infty,$$

we can conclude that

$$2 \cdot M \cdot \sum_{T=1}^{\infty} \left(\tfrac{2}{3}\right)^{\frac{T}{2}} = 2M \cdot \left[\frac{\left(\frac{2}{3}\right)^{\frac{1}{2}}}{1-\left(\frac{2}{3}\right)^{\frac{1}{2}}}\right] < \infty.$$

Therefore,

$$\forall \varepsilon > 0 \ \exists \overline{T} \in \mathbb{N} \text{ s.t.}$$
$$2 \cdot M \cdot \sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \leqslant \frac{\varepsilon}{2}.$$

But by the Triangle Inequality we obtain

$$\sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \left|c_{t+Tn}^{\frac{1}{2}} - \bar{c}_{t+T}^{\frac{1}{2}}\right| \leqslant \sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \cdot \left|c_{it+Tn}^{\frac{1}{2}}\right| + \sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \cdot \left|\bar{c}_{it+T}^{\frac{1}{2}}\right| \leqslant \sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \cdot M + \sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \cdot M =$$
$$2 \cdot M \cdot \sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \quad \forall n \in \mathbb{N}$$

so we can conclude that

$$\forall \varepsilon > 0 \ \exists \overline{T} \in \mathbb{N} \text{ s.t.}$$
$$\sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \left|c_{t+Tn}^{\frac{1}{2}} - \bar{c}_{t+T}^{\frac{1}{2}}\right| \leqslant \frac{\varepsilon}{2} \quad \forall n \in \mathbb{N}.$$

Now, again by the Triangle Inequality we obtain

$$|U_i(c_{in}) - U_i(\bar{c}_i)| = \left|\sum_{T=1}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} c_{it+Tn}^{\frac{1}{2}} - \sum_{T=1}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \bar{c}_{it+T}^{\frac{1}{2}}\right| =$$
$$= \left|\sum_{T=1}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \left(c_{it+Tn}^{\frac{1}{2}} - \bar{c}_{it+T}^{\frac{1}{2}}\right)\right| \leqslant \sum_{T=1}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \left|c_{it+Tn}^{\frac{1}{2}} - \bar{c}_{it+T}^{\frac{1}{2}}\right| =$$
$$= \sum_{T=1}^{\overline{T}-1} \left(\frac{2}{3}\right)^{\frac{T}{2}} \left|c_{it+Tn}^{\frac{1}{2}} - \bar{c}_{it+T}^{\frac{1}{2}}\right| + \sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \left|c_{it+Tn}^{\frac{1}{2}} - \bar{c}_{it+T}^{\frac{1}{2}}\right|.$$

Now, since

$$c_n \xrightarrow{\|\cdot\|_2} \bar{c},$$

we can conclude that

$$\exists \overline{N} \in \mathbb{N}$$

s.t.

$$\sum_{T=1}^{\overline{T}-1} \left(\frac{2}{3}\right)^{\frac{T}{2}} \left|c_{it+Tn}^{\frac{1}{2}} - \bar{c}_{it+T}^{\frac{1}{2}}\right| \leqslant \frac{\varepsilon}{2}$$
$$\forall n \text{ s.t. } n \geqslant \overline{N}.$$

Hence,

$$|U_i(c_{in}) - U_i(\bar{c}_i)| \leqslant \sum_{T=1}^{\overline{T}-1} \left(\frac{2}{3}\right)^{\frac{T}{2}} \left|c_{it+Tn}^{\frac{1}{2}} - \bar{c}_{it+T}^{\frac{1}{2}}\right| + \sum_{T=\overline{T}}^{\infty} \left(\frac{2}{3}\right)^{\frac{T}{2}} \left|c_{it+Tn}^{\frac{1}{2}} - \bar{c}_{it+T}^{\frac{1}{2}}\right| \leqslant \varepsilon$$
$$\forall n \text{ s.t. } n \geqslant \overline{N}.$$

And so

$$U_i(c_{in}) \longrightarrow U_i(\bar{c}_i),$$

i.e., $U_i(\cdot)$ is $\|\cdot\|_2$-continuous.

Let us show now that $B_\infty(p, e_i)$ is not $\|\cdot\|_2$-compact. Indeed, we can conclude by Theorem 2.6. that the budget set

$$B_\infty(P_i, e_i) \subset X$$

is not $\sigma(X, X')$-compact $\forall P_i \in c_0(ET \times L)$. But clearly,

$$\sigma(X, X') \subset \tau_{\|\cdot\|_2}.$$

Therefore, the budget set

$$B_\infty(P_i, e_i) \subset X$$

is not $\|\cdot\|_2$-compact $\forall P_i \in c_0(ET \times L)$.

Let us show now that

$$\arg\max\{U_i(c_i) \mid c_i \in B_\infty(p, e_i)\} = \varnothing \ \forall i \in I.$$

Define

$$c_{it+T} = (0, ..., 0, 2^T, 0, 0, ...).$$

Clearly,

$$\{c_{it+T}\}_{T=1}^{\infty} \subset B_\infty(p, e_i).$$

Substituting $\{x_{it+T}\}_{T=1}^{\infty}$ into the utility function, we obtain

$$U_i(c_{it+T}) = \left(\tfrac{2}{3}\right)^{\frac{T}{2}} \sqrt{c_{it+T}} = \left(\tfrac{2}{3}\right)^{\frac{T}{2}} \sqrt{2^T} = \left(\tfrac{4}{3}\right)^{\frac{T}{2}}.$$

Set

$$U_i(c_{it+T}) = \left(\tfrac{4}{3}\right)^{\frac{T}{2}} \geqslant M.$$

Therefore,

$$\tfrac{T}{2} \ln\left(\tfrac{4}{3}\right) \geqslant \ln(M)$$

and

$$T \geqslant 2\frac{\ln(M)}{\ln\left(\frac{4}{3}\right)}.$$

So

$$\arg\max\{U_i(c_i) \mid c_i \in B_\infty(p, e_i)\} = \varnothing \ \forall i \in I. \ \blacksquare$$

3. THE LACK OF WEAK (WEAK*) COMPACTNESS IN FM ECONOMIES WITH INFINITE DIMENSIONAL COMMODITY SPACES

For the rest of this section let

$$\mathcal{E}_\infty\left(ET,\ \left(X,\ X'\right),\ e,\ \succeq,\ \mathcal{A}\right)$$

be an infinite horizon FM economy, where I be the set of infinitely living consumers such that

$$|I| < \infty,$$

ET be the event-tree, L be the set of commodities traded on spot markets such that

$$\max\left[|ET|,\ |L|\right] = \infty.$$

Following exactly the discussion in Section 2, where we investigated the compactness of the budget set

$$B_\infty(P_i,\, e_i) \subset X$$

in the infinite horizon CM economies

$$\mathcal{E}_\infty\left(ET,\ \left(X,\ X'\right),\ e,\ \succeq\right)$$

under various topologies, here we will investigate the compactness of budget sets with Implicit Debt Constraint, with Explicit Debt Constraint and with Transversality Condition.

When

$$|ET \times L| < \infty,$$

the budget set

$$\mathcal{B}(p,\ q,\ e_i,\ \mathcal{A}) = \\ = \left\{ c_i \in X_+ \;\middle|\; \begin{array}{c} \exists z_i \in \mathcal{Z} \ s.t.\ \forall \xi \in ET \\ p\cdot c_i - p\cdot e_i = W(p,\ q,\ d)\cdot z_i \end{array} \right\}.$$

will be $\|\cdot\|$-compact under NAC, i.e., when $\exists \pi_i \in \mathbb{R}^{|ET|}_{++}$ s.t. $\pi_i \cdot W(p,\ q,\ d) = 0$ and $\forall p \in \mathbb{R}^{|ET\times L|}_{++}$. When

$$|ET \times L| = \infty,$$

the situation is different.

DEFINITION: *Let*

$$\mathcal{E}_\infty\left(ET,\ \left(X,\ X'\right),\ e,\ \succeq,\ \mathcal{A}\right)$$

be an infinite horizon FM economy. Then we define the budget set with Implicit Debt Constraint as

$$\mathcal{B}^{IDC}_\infty(p,\ q,\ e_i,\ \mathcal{A}) = \left\{ c_i \in X_+ \;\middle|\; \begin{array}{c} \exists z_i \in \mathcal{Z} \ s.t. \\ q z_i \in L_\infty(ET) \ and \\ p\cdot c_i - p\cdot e_i = W(p,\ q,\ d)\cdot z_i \end{array} \right\} \subset X,$$

the budget set with Explicit Debt Constraint as

$$\mathcal{B}^{M}_{\infty}(p,\ \ q,\, e_i,\ \mathcal{A}) = \left\{ c_i \in X_+ \;\middle|\; \begin{array}{c} \exists z_i \in \mathcal{Z}\ s.t.\ q(\xi)\, z_i(\xi) \geqslant -M\ \forall \xi \in ET\ and \\ p \cdot c_i - p \cdot e_i = W(p,\ q,\ d) \cdot z_i \end{array} \right\} \subset X$$

and *the budget set with Transversality Condition (TC) as*

$$\mathcal{B}^{TC}_{\infty}(p,\ q,\ \pi_i,\, e_i,\ \mathcal{A}) = \left\{ c_i \in X_+ \;\middle|\; \begin{array}{c} \exists z_i \in \mathcal{Z}\ s.t. \\ \lim\limits_{T \longrightarrow \infty} \sum\limits_{\xi' \in ET_{t+T}(\xi)} \pi_i(\xi')\, q(\xi')\, z_i(\xi') = 0\ and \\ p \cdot c_i - p \cdot e_i = W(p,\ q,\ d) \cdot z_i \end{array} \right\} \subset X.$$

LEMMA 3.1.: *Let*

$$\mathcal{E}_{\infty}(ET,\ \ (X,\ X'),\ e,\ \ \succeq,\ \mathcal{A})$$

be an infinite horizon FM economy, where

$$X = L_{\mathbf{p}}(ET \times L)\ for\ 1 \leqslant \mathbf{p} \leqslant \infty,$$
$$X' = (L_{\mathbf{p}}(ET \times L),\ \ \|\cdot\|)'.$$

Then budget sets

$$\mathcal{B}^{IDC}_{\infty}(p,\ q,\ e_i,\ \mathcal{A}) \subset X,$$
$$\mathcal{B}^{M}_{\infty}(p,\ \ q,\, e_i,\ \mathcal{A}) \subset X,$$
$$\mathcal{B}^{TC}_{\infty}(p,\ q,\ \pi_i,\, e_i,\ \mathcal{A}) \subset X$$

are not $\|\cdot\|_p$-bounded

$$\forall (p,\, e_i) \in c_0(ET \times L)_+ \times \left[L_{\mathbf{p}}(ET \times L)_+ \setminus \{0\}\right]^{5}.$$

PROOF: Set

$$c_{\xi,\, l} = \left(0, ..., 0, \tfrac{p \cdot e_i}{p(\xi,\, l)}, 0, ..., 0\right) \in X,$$

where

$$(p,\, e_i) \in c_0(ET \times L)_+ \times \left[L_{\mathbf{p}}(ET \times L)_+ \setminus \{0\}\right]$$

and

$$z_i = 0.$$

Clearly,

$$\left\{ \begin{array}{l} pc_{\xi,\, l} = p(\xi,\, l) \cdot \frac{p \cdot e_i}{p(\xi,\, l)} = p \cdot e_i = p \cdot e_i + W(p,\, q,\, d) \cdot z_i \\ \|qz_i\|_{\infty} = \sup\limits_{\xi \in ET} |q(\xi)\, z_i(\xi)| = 0, \text{ i.e., } qz_i \in L_{\infty}(ET \times L) \\ q(\xi)\, z_i(\xi) = 0 > -M\ \forall \xi \in ET \\ \lim\limits_{T \longrightarrow \infty} \sum\limits_{\xi' \in ET_{t+T}(\xi)} \pi_i(\xi')\, q(\xi')\, z_i(\xi') = 0 \end{array} \right. .$$

Therefore,

$$c_{\xi,\, l} \in \mathcal{B}^{IDC}_{\infty}(p,\ q,\ e_i,\ \mathcal{A}) \bigcap \mathcal{B}^{M}_{\infty}(p,\ \ q,\, e_i,\ \mathcal{A}) \bigcap \mathcal{B}^{TC}_{\infty}(p,\ q,\ \pi_i,\, e_i,\ \mathcal{A}) \subset X$$

and so clearly the above budget sets are not $\|\cdot\|_p$-bounded

$$\forall (p,\, e_i) \in c_0(ET \times L)_+ \times \left[L_{\mathbf{p}}(ET \times L)_+ \setminus \{0\}\right]. \ \blacksquare$$

[5] In the trivial case, where $e_i = 0$ we clearly have that the above budget sets are not $\|\cdot\|_{\mathbf{p}}$-bounded $\forall p \in c_0(ET \times L)_+ \setminus c_0(ET \times L)_{++}$ and is $\|\cdot\|_{\mathbf{p}}$-bounded $\forall p \in c_0(ET \times L)_{++}$.

THEOREM 3.2.: *Let*

$$\mathcal{E}_\infty \left(ET,\ \left(X,\ X'\right),\ e,\ \succeq,\ \mathcal{A}\right)$$

be an infinite horizon FM economy, where

$$X = L_{\mathbf{p}} \left(ET \times L\right) \text{ for } 1 \leqslant \mathbf{p} \leqslant \infty, \\ X' = \left(L_{\mathbf{p}}(ET \times L),\ \|\cdot\|\right)'.$$

Then budget sets

$$\mathcal{B}_\infty^{IDC}(p,\ q,\ e_i,\ \mathcal{A}) \subset X, \\ \mathcal{B}_\infty^{M}(p,\ q,\ e_i,\ \mathcal{A}) \subset X, \\ \mathcal{B}_\infty^{TC}(p,\ q,\ \pi_i,\ e_i,\ \mathcal{A}) \subset X$$

are not $\sigma(X',\ X)$-compact

$$\forall (p,\ e_i) \in c_0\left(ET \times L\right)_+ \times \left[L_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]^6.$$

PROOF: Since the above budget sets are not $\|\cdot\|_{\mathbf{p}}$-bounded, we can conclude by the above **Proposition 2.6. (Problem #40, p. 238 of Royden)** that they are not $\sigma(X,\ X')$-compact

$$\forall (p,\ e_i) \in c_0\left(ET \times L\right)_+ \times \left[L_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]. \ \blacksquare$$

THEOREM 3.3: *Let*

$$\mathcal{E}_\infty \left(ET,\ \left(X,\ X'\right),\ e,\ \succeq\right)$$

be an infinite horizon CM economy, where

$$X = \mathbb{R}^{|ET \times L|},$$

τ is the product topology on X and

$$\left(X,\ \tau\right)' = X' = \varphi\left(ET \times L\right).$$

Then budget sets

$$\mathcal{B}_\infty^{IDC}(p,\ q,\ e_i,\ \mathcal{A}) \subset X, \\ \mathcal{B}_\infty^{M}(p,\ q,\ e_i,\ \mathcal{A}) \subset X, \\ \mathcal{B}_\infty^{TC}(p,\ q,\ \pi_i,\ e_i,\ \mathcal{A}) \subset X$$

are not τ-compact

$$\forall p \in X' = \varphi\left(ET \times L\right).$$

PROOF: Fix

$$\left(\bar{\xi},\ \bar{l}\right) \in \left(ET \times L\right) \setminus S.$$

[6] In the trivial case, where $e_i = 0$ we clearly have that the above budget sets are not $\sigma(X,\ X')$-compact $\forall p \in c_0\left(ET \times L\right)_+ \setminus c_0\left(ET \times L\right)_{++}$ and is $\sigma(X,\ X')$-compact $\forall p \in c_0\left(ET \times L\right)_{++}$.

Therefore,

$$p\left(\bar{\xi},\ \bar{l}\right) = 0$$

and so

$$\left[0,\ \frac{p \cdot e_i}{p\left(\bar{\xi},\ \bar{l}\right)}\right] \subset \mathbb{R}$$

is not $|\cdot|$-compact in $\mathbb{R}$. Hence,

$$\exists \left\{x_{i\lambda}\left(\bar{\xi},\ \bar{l}\right)\right\} \subset \left[0,\ \frac{p \cdot e_i}{p\left(\bar{\xi},\ \bar{l}\right)}\right],$$

which has no $|\cdot|$-convergent subnet. Define now net

$$\left\{c_{i\lambda}\right\} \subset \mathbb{R}^{|ET \times L|}$$

as follows

$$c_{i\lambda}\left(\xi,\ l\right) = \begin{cases} x_{i\lambda}\left(\xi,\ l\right) \text{ for } \left(\xi,\ l\right) = \left(\bar{\xi},\ \bar{l}\right) \\ 0 \text{ for } \left(ET \times L\right) \setminus \left\{\left(\bar{\xi},\ \bar{l}\right)\right\} \end{cases}.$$

Define also net

$$\left\{z_{i\lambda}\right\} \in \mathcal{Z}$$

as follows

$$z_{i\lambda} = 0\ \forall \lambda.$$

Clearly,

$$\begin{cases} pc_{i\lambda} = p(\xi,\ l) \cdot \frac{p \cdot e_i}{p(\xi,\ l)} = p \cdot e_i = p \cdot e_i + W(p,\ q,\ d) \cdot z_{i\lambda} \\ \left\|qz_{i\lambda}\right\|_\infty = \sup\limits_{\xi \in ET} \left|q\left(\xi\right) z_{i\lambda}\left(\xi\right)\right| = 0, \text{ i.e., } qz_{i\lambda} \in L_\infty\left(ET \times L\right) \\ q\left(\xi\right) z_{i\lambda}\left(\xi\right) = 0 > -M\ \forall \xi \in ET \\ \lim\limits_{T \longrightarrow \infty} \sum\limits_{\xi' \in ET_{t+T}(\xi)} \pi_i\left(\xi'\right) q\left(\xi'\right) z_{i\lambda}\left(\xi'\right) = 0 \end{cases} \forall \lambda.$$

Therefore,

$$\left\{c_{i\lambda}\right\} \subset \mathcal{B}_\infty^{IDC}(p,\ q,\ e_i,\ \mathcal{A}) \bigcap \mathcal{B}_\infty^{M}(p,\ q,\ e_i,\ \mathcal{A}) \bigcap \mathcal{B}_\infty^{TC}(p,\ q,\ \pi_i,\ e_i,\ \mathcal{A}) \subset X.$$

Because the product topology τ on X is the topology of coordinatewise convergence and

$$\left\{x_{i\lambda}\left(\bar{\xi},\ \bar{l}\right)\right\} \subset \left[0,\ \frac{p \cdot e_i}{p\left(\bar{\xi},\ \bar{l}\right)}\right]$$

has no $|\cdot|$-convergent subnet, we can conclude that

$$\left\{c_{i\lambda}\right\} \subset \mathcal{B}_\infty^{IDC}(p,\ q,\ e_i,\ \mathcal{A}) \bigcap \mathcal{B}_\infty^{M}(p,\ q,\ e_i,\ \mathcal{A}) \bigcap \mathcal{B}_\infty^{TC}(p,\ q,\ \pi_i,\ e_i,\ \mathcal{A}) \subset X$$

has no τ-convergent subnet. Therefore, the above budget sets are not τ-compact $\forall p \in X'$. ■

We will now move to establishing the lack of weak* compactness of the above budget set. It should be noted that when the Banach space X is reflexive, i.e., when

$$X'' = X,$$

we have that

$$\sigma(X,\ X') = \sigma(X'',\ X').$$

That is, the weak topology on X coincides with the weak* topology on X''. Therefore, the weak compactness of a subset of X coincides with the weak* compactness of a subset of X''.

Next, we will consider the case when the Banach space X is not necessarily reflexive, i.e., when

$$X'' = X$$

does not necessarily hold.

LEMMA 3.4.: *Let*

$$\mathcal{E}_\infty\left(ET,\ \left(X',\ X\right),\ e,\ \succeq,\ \mathcal{A}\right)$$

be an infinite horizon FM economy, where

$$X = L_{\mathbf{p}}\left(ET \times L\right) \text{ for } 1 \leqslant \mathbf{p} \leqslant \infty, \\ X' = (L_{\mathbf{p}}(ET \times L),\ \left\|\cdot\right\|)'.$$

Then budget sets

$$\mathcal{B}_\infty^{IDC}(p,\ q,\ e_i,\ \mathcal{A}) \subset X', \\ \mathcal{B}_\infty^{M}(p,\ q,\ e_i,\ \mathcal{A}) \subset X', \\ \mathcal{B}_\infty^{TC}(p,\ q,\ \pi_i,\ e_i,\ \mathcal{A}) \subset X'$$

are not $\left\|\cdot\right\|_p$-bounded

$$\forall(p,\ e_i) \in c_0\left(ET \times L\right)_+ \times \left[L'_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]^7.$$

PROOF: The proof will go exactly the same way as the proof of Lemma 2.7. ■

THEOREM 3.5.: *Let*

$$\mathcal{E}_\infty\left(ET,\ \left(X',\ X\right),\ e,\ \succeq,\ \mathcal{A}\right)$$

be an infinite horizon FM economy, where

$$X = L_{\mathbf{p}}\left(ET \times L\right) \text{ for } 1 \leqslant \mathbf{p} \leqslant \infty, \\ X' = (L_{\mathbf{p}}(ET \times L),\ \left\|\cdot\right\|)'.$$

Then budget sets

[7] In the trivial case, where $e_i = 0$, we clearly have that the above budget sets are not $\left\|\cdot\right\|_{\mathbf{p}}$-bounded $\forall p \in c_0\left(ET \times L\right)_+ \setminus c_0\left(ET \times L\right)_{++}$ and is $\left\|\cdot\right\|_{\mathbf{p}}$-bounded $\forall p \in c_0\left(ET \times L\right)_{++}$.

$$\mathcal{B}_{\infty}^{IDC}(p,\ q,\ e_i,\ \mathcal{A}) \subset X',$$
$$\mathcal{B}_{\infty}^{M}(p,\ q, e_i,\ \mathcal{A}) \subset X',$$
$$\mathcal{B}_{\infty}^{TC}(p,\ q,\ \pi_i, e_i,\ \mathcal{A}) \subset X'$$

are not $\sigma(X,\, X')$-compact

$$\forall (p,\, e_i) \in c_0\left(ET \times L\right)_+ \times \left[L'_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]^8.$$

PROOF: Since the above budget sets are not $\|\cdot\|_{\mathbf{p}}$-bounded, we can conclude by the above **Corollary of Alaoglu's Theorem 2.11.** that they are not $\sigma(X',\, X)$-compact

$$\forall (p,\, e_i) \in c_0\left(ET \times L\right)_+ \times \left[L'_{\mathbf{p}}\left(ET \times L\right)_+ \setminus \{0\}\right]. \blacksquare$$

4. CONCLUSION

This paper demonstrates that budget sets in the Contingent Markets (CM) and Financial Markets (FM) economies with infinite dimensional commodity spaces might not be norm-bounded and therefore might not be weakly (weakly*) compact. The lack of weak (weak*) compactness of these budget sets has serious implications. In particular, it is no longer guaranteed that weakly (weakly*) continuous utility functions attain their maximum on these budget sets. Thus, the individual and therefore total demand functions need not exist in CM and FM economies with infinite dimensional commodity spaces. The lack of existence of demand functions does not, however, imply the lack of existence of equilibrium in CM and FM economies with infinite dimensional commodity spaces.

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[8] In the trivial case, where $e_i = 0$, we clearly have that the above budget sets are not $\sigma(X',\, X)$-compact $\forall p \in c_0\left(ET \times L\right)_+ \setminus c_0\left(ET \times L\right)_{++}$ and is $\sigma(X',\, X)$-compact $\forall p \in c_0\left(ET \times L\right)_{++}$.