The Dispersion Bias

Correcting a large source of error in minimum variance portfolios

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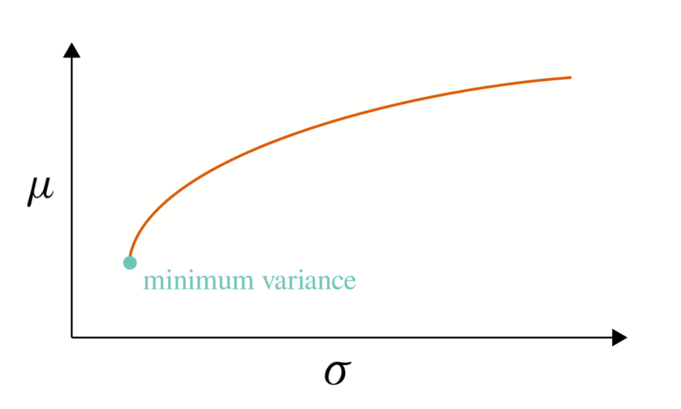
Optimized portfolios and the impact of estimation error

Optimized portfolios

Since (Markowitz 1952), quantitative investors have constructed portfolios with mean-variance optimization.



The impact of estimation error

In practice, optimization relies on an estimate of the mean and covariance matrix ($\widehat{\boldsymbol{\Sigma}}$ estimates $\boldsymbol{\Sigma}$).

Estimation error leads to two types of errors:

- You get the wrong portfolio: Estimation error distorts portfolio weights so optimized portfolios are never optimal.
- And it's probably risker than you think it is: A risk-minimizing optimization tends to materially underforecast portfolio risk.

We measure both errors in simulation.

Measuring the impact of estimation error in simulation

Measuring errors in weights

(Squared) tracking error of an optimized portfolio $\widehat{w}$ measures its distance from the optimal portfolio w_*:

$$\mathscr{T}_{\widehat{w}}^2 = (\widehat{w} - w_*)^\top \boldsymbol{\Sigma} (\widehat{w} - w_*)$$

Tracking error is the width of the distribution of return differences between w_* and $\widehat{w}$. Ideally, tracking error should be as close to 0.

Measuring errors in risk forecasts

Variance forecast ratio measures the error in the risk forecast as:

$$\mathscr{R}_{\widehat{w}} = \frac{\widehat{w}^\top \widehat{\mathbf{\Sigma}} \widehat{w}}{\widehat{w}^\top \mathbf{\Sigma} \widehat{w}}$$

Ideally, the variance forecast ratio should be as close to 1.

Error metrics in simulation

In simulation,

- generate returns, estimate $\widehat{\mathbf{\Sigma}}$, compute $\widehat{w}$;
- compute w_* using $\mathbf{\Sigma}$ (accessible in simulation);
- measure the errors.

$$\mathscr{T}_{\widehat{w}}^2 = (\widehat{w} - w_*)^\top \mathbf{\Sigma} (\widehat{w} - w_*)$$

$$\mathscr{R}_{\widehat{w}} = \frac{\widehat{w}^\top \widehat{\mathbf{\Sigma}} \widehat{w}}{\widehat{w}^\top \mathbf{\Sigma} \widehat{w}}$$

Minimum variance

Why minimum variance?

Theory

Error amplification: Highly sensitive to estimation error.

Error isolation: Impervious to errors in expected return.

Insight into a general problem: Informs our understanding of how estimation error distorts portfolios and points to a remedy.

Practice

Large investments: For example, the Shares Edge MSCI Min Vol USA ETF had net assets of roughly $14 billion on Sept. 8, 2017.

True and optimized minimum variance portfolios

The true minimum variance portfolio w_* is the solution to:

$$\min_{x \in \mathbb{R}^N} x^\top \boldsymbol{\Sigma} x$$
$$x^\top \mathbf{1_N} = \mathbf{1}\,.$$

In practice, we construct an estimated minimum variance portfolio, $\widehat{w}$, that solves the same problem with $\widehat{\boldsymbol{\Sigma}}$ replacing $\boldsymbol{\Sigma}$.

Factor models

Factor models and equity markets

Beginning with the development of the Capital Asset Pricing Model (CAPM) in (Treynor 1962) and (Sharpe 1964), factor models have been central to the analysis of equity markets.

In a fundamental model, human analysts identify factors. Fundamental models have been widely used by equity portfolio managers since (Rosenberg 1984) and (Rosenberg 1985).

In a statistical model (such as PCA, factor analysis, etc), machines identify factors. An enormous academic literature on PCA models has descended from (Ross 1976).

PCA is the focus of our analysis.

A one-factor model

The return generating process is specified by

$$R = \phi\beta + \epsilon$$

where ϕ is the return to a market factor, β is the N-vector of factor exposures, δ^2 is the N-vector of diversifiable specific returns.

When the ϕ and ϵ are uncorrelated, the security covariance matrix can be expressed as

$$\mathbf{\Sigma} = \sigma^2\beta\beta^\top + \mathbf{\Delta},$$

where σ^2 is the variance of the factor and the diagonal entries of $\mathbf{\Delta}$ are specific variances, δ^2.

Estimation error in factor models

In practice, we have only estimates: $\hat{\sigma}^2$, $\hat{\beta}$ and $\hat{\delta}^2$.

$$\widehat{\boldsymbol{\Sigma}} = \hat{\sigma}^2 \hat{\beta} \hat{\beta}^\top + \widehat{\boldsymbol{\Delta}}$$

We measure the errors in estimated parameters, of course.

But our focus is how errors in parameter estimates affect portfolio metrics: tracking error and variance forecast ratio.

Motivating example

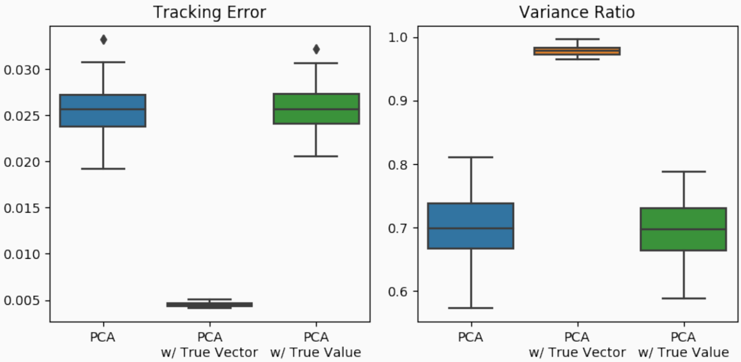
Parameter errors $\rightarrow$ portfolio metric errors

A large literature on random matrix theory identifies and corrects biases in estimated eigenvalues.

It turns out, however, that in a simple PCA model, portfolio metrics for a minimum variance portfolio are insensitive to errors in eigenvalues.

But errors in the dominant eigenvector make a difference.

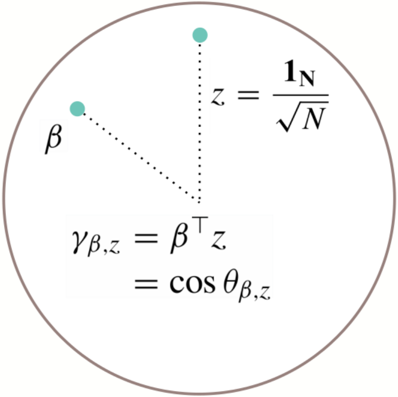
Selective error correction in minimum variance



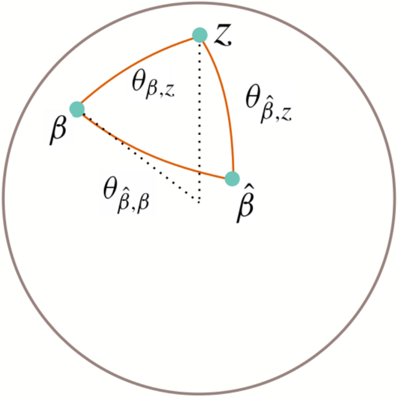
Results communicated by Stephen Bianchi

The dispersion bias

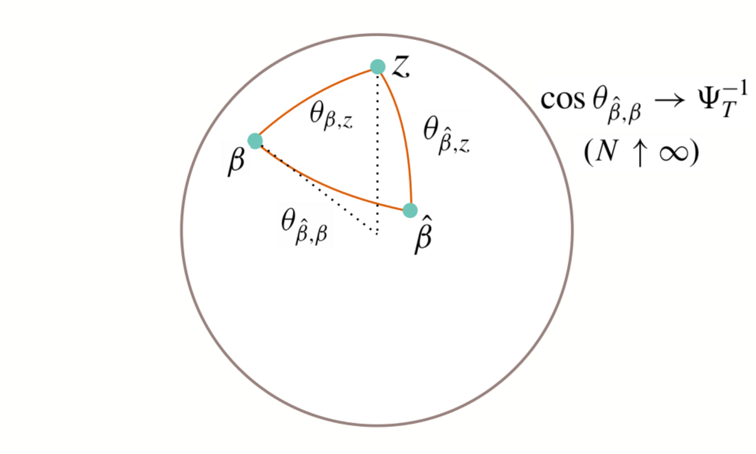
The dispersionless vector



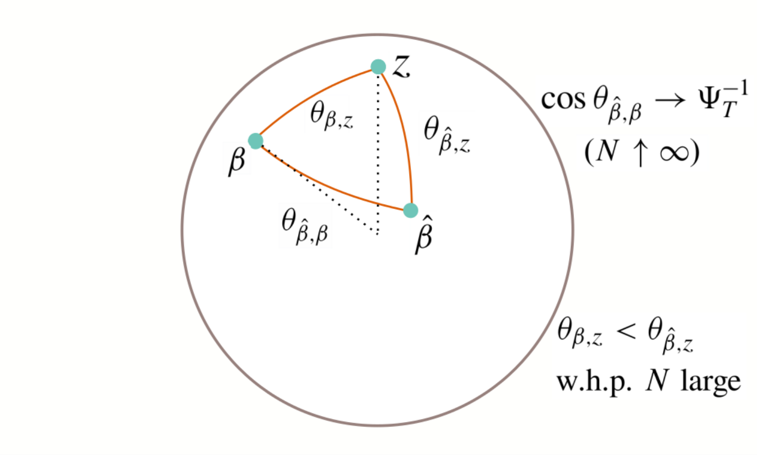
Geometry of the problem



Sample eigenvector behavior



The dispersion bias



Estimation error in minimum variance portfolio metrics

For fixed T and large N, let $r = \frac{\gamma_{\beta,z}}{\gamma_{\hat{\beta},z}}$.

$$\mathscr{T}^2_{\widehat{w}} \asymp \frac{\sigma_N^2}{N}(r - \gamma_{\beta,\hat{\beta}})^2$$

$$\mathscr{R}_{\widehat{w}} \asymp \frac{\hat{\delta}^2}{\sigma_N^2(r - \gamma_{\beta,\hat{\beta}})^2 + \delta^2}$$

PCA estimator has $\mathscr{R}_{\widehat{w}} \to 0$ and $\mathscr{T}^2_{\widehat{w}}$ positive as N becomes large.

Decreasing the magnitude of $r - \gamma_{\beta,\hat{\beta}}$ lowers tracking error and raises variance forecast ratio (both desirable), and it amounts to decreasing $\theta_{\beta,\hat{\beta}}$.

Correcting the dispersion bias

Dispersion bias correction for a standard one-factor model

Almost surely, some shrinkage of $\hat{\beta}$ toward z along the geodesic on the sphere connecting the two points lowers tracking error and raises variance forecast ratio of a minimum variance portfolio.

The oracle estimate of β is given by:

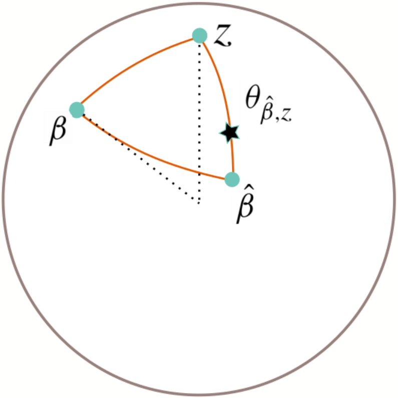
$$\hat{\beta}^* \propto \hat{\beta} + \rho^* z$$

where

$$\rho^* = \frac{\gamma_{\beta,z} - \gamma_{\beta,\hat{\beta}}\gamma_{\hat{\beta},z}}{\gamma_{\beta,\hat{\beta}} - \gamma_{\beta,z}\gamma_{\hat{\beta},z}}.$$

For the oracle, $r - \gamma_{\beta,\hat{\beta}}$ is 0, and ρ^* has a useful limit as $N \to \infty$.

Moving in the right direction



Eigenvalue bias correction for a standard one-factor model

For a large N minimum variance portfolio, (bias in) the largest eigenvalue does not affect tracking error and variance forecast ratio...

... as predicted by the Bianchi experiment, which shows that replacing an estimated eigenvector with a true eigenvector improves portfolio metrics even when the estimated eigenvalue is not corrected.

But eigenvalue correction is important for other portfolios, so we do it.

$$\hat{\sigma}^2_{\rho^*} = \left(\frac{\gamma_{\hat{\beta},z}}{\gamma_{\hat{\beta}^*,z}} \right)^2 \hat{\sigma}^2$$

Impact of bias correction theorem for a standard one-factor model

For fixed T with $N \to \infty$

Model	Tracking error	Variance forecast ratio
PCA	bounded away from 0	$\to 0$
Oracle	$\to 0$	$\to 1$
Target	0	1

A bona fide shrinkage estimator

Our estimate begins with the asymptotic (large N) formula for the oracle estimator

$$\rho^* = \frac{\gamma_{\beta,z}}{1 - \gamma_{\beta,z}^2} \left(\Psi - \Psi^{-1} \right),$$

where Ψ is a positive random variable that is expressed in terms of χ_T and asymptotic estimates for eigenvalues.

We rely on (Yata & Aoshima 2012) for the latter.

Stand on the shoulders of giants

The emphasis of the fixed T large N regime dates back to (Connor & Korajczyk 1986) and (Connor & Korajczyk 1988).

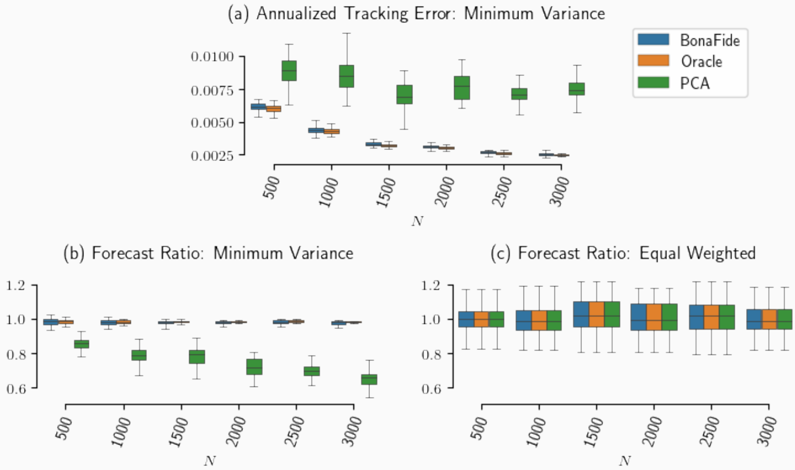
Our proofs rely heavily on (Shen, Shen, Zhu & Marron 2016) and (Wang & Fan 2017).

Numerical results

Calibrating the one-factor model

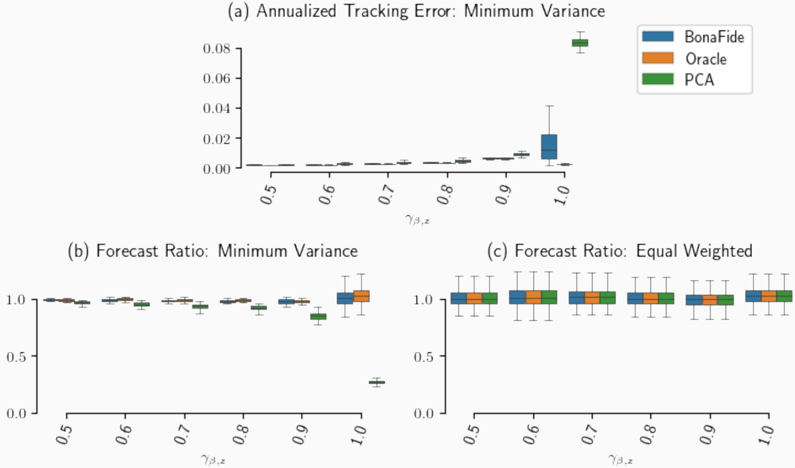
Parameter	Value	Comment
β	normalized so $\|\|\beta\|\|_2 = 1$	factor exposure
$\gamma_{\beta,z}$	0.5–1.0	controls dominant factor dispersion
σ^2	dominant eigenvalue of Σ	annualized factor volatility of 16%
δ^2	specific variances	annualized specific volatilities drawn from $[10\%, 64\%]$

Numerical results from a one-factor model, $\gamma_\beta = 0.90$



Simulation based on 50 samples

Numerical results from a one-factor model, $N = 500$



Simulation based on 50 samples

Beta shrinkage has been used by practitioners since the 1970s

The innovators

In the 1970s, Oldrich Vasicek and Marshall Blume observed excess dispersion in betas estimated from time series regressions, and they proposed adjustments.

(Vasicek 1973) shrinks estimated betas toward their cross-sectional mean using a Bayesian formula.

(Blume 1975) uses the empirically observed average shrinkage of betas on individual stocks in the current period relative to a previous period to adjust forecast betas for the next period.

An ultra-simplifed version of the Blume adjustment is on the exam taken by aspiring Chartered Financial Analysts (CFA)s.

The CFA Level II Exam

Quizlet

CFA Level 2 Equity

Multifactor Models		
Build-up Model	r= RF + equity risk premium + size premium + specific-company premium	☆
Blume Adjustment	adjusted beta = (2/3 raw beta + 1/3)	☆
WACC	MV debt / Total Assets **(1-T)** rd + MV equity / Total Assets * re	☆

Chat about Blume on AnalystForum.com

(/profile/chowder) chowder (/profile/chowder)
Jun 7th, 2009 2:35pm
205 AF Points

wtf is Blume method?

(/profile/bpdulog) bpdulog (/profile/bpdulog)
Jun 7th, 2009 2:36pm
CFA Charterholder 3,696 AF Points

I don't know but this is what I did:

2/3* provided beta + .33

I think I read that in the Secret Sauce

(/profile/topher) topher (/profile/topher)
Jun 7th, 2009 2:36pm
1,649 AF Points

I didn't know what Blume was either, but they asked for adjusted beta which is:

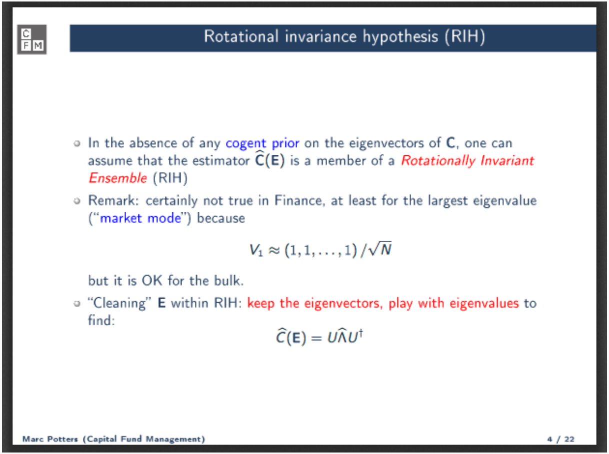
(2/3)Beta + 1/3*1 = 1.27

(/comment/91005043#comment-91005043)

Covariance Shrinkage methods

Capital Fund Management

A flat market mode acknowledged as a target by Bouchaud, Bun, & Potters.



Honey, I shrunk the sample covariance matrix

Seminal work (Ledoit & Wolf 2004) introduced a linear shrinkage correction to the sample covariance utilizing a constant correlation target

$$C = \alpha E + (1 - \alpha)[(1 - \rho)I + \rho e e^T].$$

Our vector z represents a constant correlation mode so there are parallels.

We also take aim at a specific misbehaving artifact in factor models and minimum variance portfolios and leverage a different asymptotic theory.

Summary and ongoing research

Summary

We identified a large, damaging dispersion bias in PCA-estimated factor models.

We determined an oracle correction that elevates portfolio construction and risk forecasting for a minimum variance portfolios for fixed T as $N \to \infty$.

And we have developed a bona fide (data-driven) correction that has proven effective in empirically-calibrated simulation.

Our results can be viewed as an extension and formalization of ideas that have been known by practitioners since the 1970s.

Ongoing research

Extension to multi-factor models.

Application to sparse low rank factor extractions.

Investigation of a wider class of portfolios.

Empirical studies.

Thank you

Photograph by Jim Block

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