CENTER FOR
RISK MANAGEMENT
RESEARCH

Working Paper # 2014 - 03

On a Convex Measure of Drawdown Risk

Lisa R. Goldberg, University of California at Berkeley

Ola Mahmoud, University of St. Gallen

April 29, 2014

University of California
Berkeley

On a Convex Measure of Drawdown Risk

Lisa R. Goldberg[*] Ola Mahmoud [†]

April 29, 2014[‡]

Abstract

Maximum drawdown, the largest cumulative loss from peak to trough, is one of the most widely used indicators of risk in the fund management industry, but one of the least developed in the context of probabilistic risk metrics. We formalize drawdown risk as Conditional Expected Drawdown (CED), which is the tail mean of maximum drawdown distributions. We show that CED is a degree one positive homogenous risk measure, so that it can be attributed to factors; and convex, so that it can be used in quantitative optimization. We develop an efficient linear program for minimum CED optimization and empirically explore the differences in risk attributions based on CED, Expected Shortfall (ES) and volatility. An important feature of CED is its sensitivity to serial correlation. In an empirical study that fits AR(1) models to US Equity and US Bonds, we find substantially higher correlation between the autoregressive parameter and CED than with ES or with volatility.

Key terms: leverage; drawdown; maximum drawdown distribution; Conditional Expected Drawdown; volatility; Expected Shortfall; tail mean; liquidity trap; illiquidity; coherent risk measure; deviation measure; risk attribution; risk contribution; risk concentration; generalized correlation; marginal contribution to risk; optimization; portfolio construction; serial correlation

[*]Department of Statistics and Center for Risk Management Research, University of California, Berkeley, CA 94720-3880, USA, email: `lrg@stat.berkeley.edu`.

[†]Faculty of Mathematics and Statistics, University of St Gallen, CH-9000, Switzerland and Center for Risk Management Research, University of California, Berkeley, CA 94720-3880, USA, email: `olamahmoud@berkeley.edu`.

[‡]We are grateful to Robert Anderson for insightful comments on the material discussed in this article, and to Vladislav Dubikovsky, Michael Hayes, and Márk Horváth for their contributions to an earlier version of this article.

Contents

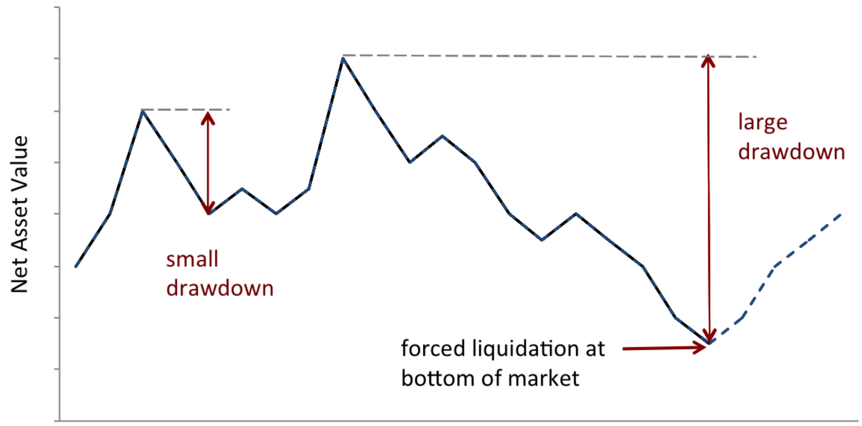


Figure 1: Simulation of a portfolio's net asset value over a finite path. A large drawdown may force liquidiation at the bottom of the market, and the proceeding market recovery is never experienced.

1 Introduction

A levered investor is liable to get caught in a liquidity trap: unable to secure funding after an abrupt market decline, he may be forced to sell valuable positions under unfavorable market conditions. This experience was commonplace during the 2007-2009 financial crisis and it has refocused the attention of both levered and unlevered investors on an important liquidity trap trigger, a drawdown, which is the maximum decline in portfolio value over a fixed horizon (see Figure 1).

In the event of a large drawdown, common risk diagnostics, such as volatility, Value-at-Risk, and Expected Shortfall, at the end of the intended investment horizon are irrelevant. Indeed, within the universe of hedge funds and commodity trading advisors (CTAs), one of the most widely quoted measures of risk is maximum drawdown. However, a generally accepted mathematical methodology for forming expectations about future potential drawdowns does not seem to exist. Drawdown in the context of measures of risk as developed in Artzner et al. (1999) has failed to attract the same kind of research devoted to other more conventional risk measures.

Our purpose is to formulate a mathematically sound and practically useful measure of drawdown risk. To this end, we develop a probabilistic measure of risk capturing drawdown in the spirit of Artzner et al. (1999). Our formalization of drawdown risk is achieved by modeling the uncertain payoff along a finite path as a time-ordered random vector $X_{T_n} = (X_{t_1}, \dots, X_{t_n})$ representing return paths, to which a certain real-valued functional, the *Conditional Expected Drawdown*, is applied. Mathematically, the random variables X_{t_i} are first transformed to the random variable $\mu(X_{T_n})$, representing the maximum drawdown within a path of some fixed length n. At confidence level $\alpha \in [0, 1]$, the *Conditional Expected Drawdown* CED_α is then defined to be the expected maximum drawdown given that some maximum drawdown threshold DT_α, the α-quantile of the maximum drawdown distribution, is breached:

$$\mathrm{CED}_\alpha(\mathrm{X}_{\mathrm{T_n}}) = \mathbb{E}\left(\mu(\mathrm{X}_{\mathrm{T_n}}) | \mu(\mathrm{X}_{\mathrm{T_n}}) > \mathrm{DT}_\alpha\right).$$

In the context of risk measures, CED is not a *monetary* risk metric, in the sense that it fails to satisfy the translation invariance and monotonicity axioms. It is, however, convex, which means that it promotes diversification and can be used in an optimizer. It is also homogenous of degree one, so that it supports risk attribution. Moreover, CED is a deviation measure in the sense of Rockafellar et al. (2002, 2006).

Because Conditional Expected Drawdown is defined as the tail mean of a distribution of maximum drawdowns, it is analogous to Expected Shortfall, which is the tail mean of a return distribution. Hence, much of the theory surrounding Expected Shortfall carries over when moving from returns to maximum drawdowns. We will show, however, that drawdown is inherently path dependent and accounts for serial correlation, whereas Expected Shortfall does not account for consecutive losses.

1.1 Synopsis

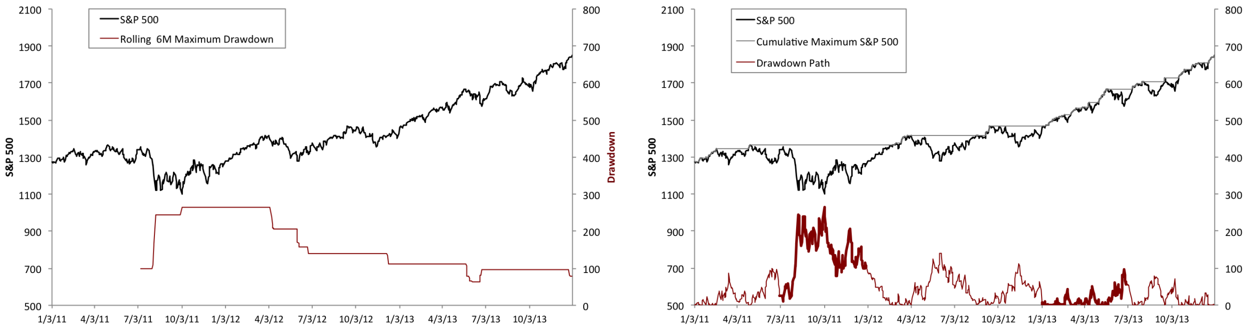
Drawdown risk is formalized in Section 2, which includes mathematical definitions of maximum drawdown and Conditional Expected Drawdown (CED). In Section 3, we axiomatize Conditional Expected Drawdown by using the framework of probabilistic measures of risk as developed by Artzner et al. (1999) as a guide. CED is shown to be positive homogenous and convex, but not *monetary* in the sense of Artzner et al. (1999). Moreover, we show that CED satsifies the axioms of deviation measures developed by Rockafellar et al. (2002, 2006). Section 4 illustrates how CED can be attributed to linear factors, since positive homogeneity ensures that the overall drawdown risk of a portfolio can be decomposed into additive subcomponents representing the individual factor contributions to drawdown risk. In Section 5, we provide a computationally efficient linear programming algorithm for a CED optimization problem. This enables investors to allocate funds in such a way that minimizes drawdown risk. Section 6 contains an empirical study analyzing drawdown risk and drawdown risk concentrations. We provide empirical support for a positive relationship between serial correlation and CED and further show that serial correlation in the assets manifests itself more in the drawdown risk concentrations than in those of Expected Shortfall or volatility. Concluding remarks and suggestions for further empirical research are given in Section 7.

2 Measuring Drawdown Risk

We represent cumulative returns by real-valued random variables $X\colon \Omega \to \mathbb{R}$ over a fixed probability space $(\Omega, \mathcal{F}, P)$. A time series of returns will be represented by a real-valued discrete stochastic processes $(X_T) = \{X_t\}_{t\in T}$ (for T a totally ordered discrete set representing time). For simplicity, the time intervals between t_i and t_{i+1} are assumed to be equal for all $i \in \mathbb{N}$. For every finite ordered subset $T_n \subset T = \{t_1, \dots, t_n\}$ of length n, we denote by X_{T_n} the random vector $(X_{t_1}, \dots, X_{t_n})$ taking values in $\mathbb{R}^n$ and representing a finite path of the discrete process (X_T). At time t_j, the j-th element of the return path X_{T_n} is the simple return between t_1 and t_j, that is $X_{t_j} = P_{t_j}/P_{t_1} - 1$, where for all $i \leq n$, P_{t_i} denotes the price level (or net asset value) at time t_i. Our main object of interest is the maximum drawdown along a path of realizations $(x_{t_1}, \dots, x_{t_N})$ of the random vector X_{T_n} representing return paths. Formally, maximum drawdown will be defined via a random variable transformation, which is a vector-valued functional applied to the outcomes of the random vector X_{T_n}.

2.1 Maximum Drawdown

The maximum drawdown within a path is defined as the maximum drop from peak to trough within that path. Formally, for $X_{T_n} = (X_{t_1}, \dots, X_{t_n})$ a random vector representing return paths of length n of, the maximum drawdown is the random variable obtained through the transformation $\mu : \mathbb{R}^n \to \mathbb{R}$



(a) (b)

Figure 2: Daily prices of the S&P 500 from 1 January 2011 to 31 December 2013, together with (a) the backward-looking maximum drawdown within rolling six month periods (that is 125 business days) and (b) the corresponding daily rolling 6-month drawdown path.

defined by

$$\mu\left(X_{T_n}\right) = \max_{1\le i<n}\max_{i<j\le n}\left\{X_{t_j} - X_{t_i}, 0\right\}.$$

Figure 2(a) shows the daily price series of the S&P 500 over the three-year period 2011–2013, together with the corresponding daily 6-month rolling maximum drawdown.

Remark 1 (Drawdown paths). An alternative way of looking at maximum drawdown is by first defining a *drawdown path*. Let $d : \mathbb{R}^n \to \mathbb{R}^n$ be the random variable transformation defined by $d(X_{T_n}) = (d_1(X_{T_n}), \dots, d_n(X_{T_n}))$, where for $1 \le j \le n$, $d_j : \mathbb{R}^n \to \mathbb{R}$ with $d_j(X_{T_n})$ given by

$$d_j(X_{T_n}) = \max_{1\le i\le j}\{X_{t_i}\} - X_{t_j} \quad (1 \le j \le n).$$[1]

Note that the j-th entry is the maximum drawdown with endpoint at time t_j, and so

$$d_j(X_{T_n}) = \mu\left(X_{t_1}, \dots, X_{t_j}\right).$$

The maximum drawdown within the path X_{T_n} is then simply the largest amongst all drawdowns:

$$\mu\left(X_{T_n}\right) = \max\left\{d(X_{T_n}), 0\right\} = \max_{1\le i\le n}\left\{d_{t_i}, 0\right\}.$$

We make use this alternative definition in some of the proofs later on in this article. Figure 2(b) shows the daily price series of the S&P 500 over a three-year period, together with the corresponding daily drawdown path over the same period.[2] The highlighted sub-paths illustrate two different drawdown paths, each of a 6-month length. Note the difference in the magnitudes within the two drawdown paths. The larger (July to December 2011) occurred during a turbulent period; the smaller (January to June 2013) occurred in the midst of rising equity markets.

[1] We suppress explicit reference to X_{T_n} if it is known from the context and will simply denote $d_j(X_{T_n})$ by d_{t_j}.

[2] Drawdowns are often quoted in terms of percentage cumulative loss rather than in absolute terms as depicted here.

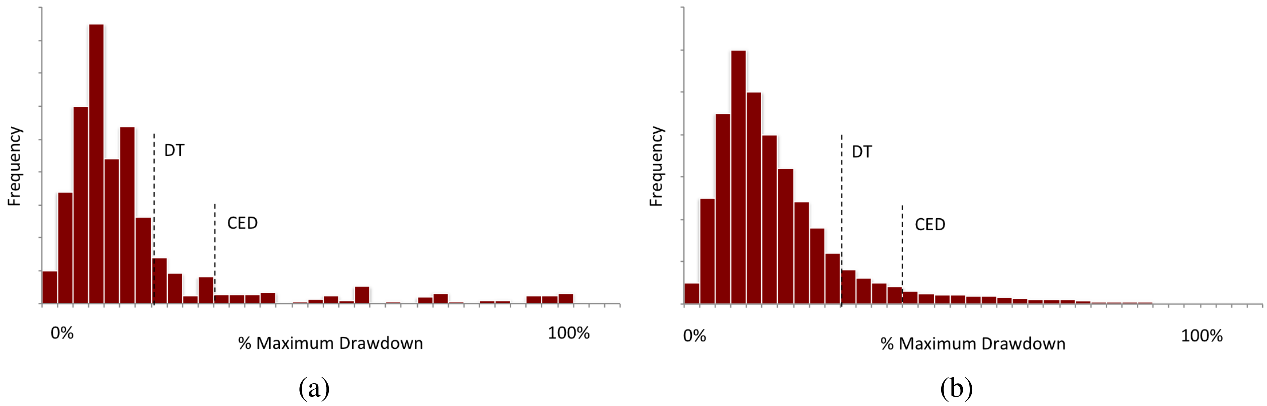


Figure 3: (a) Empirical distribution of the realized 6-month maximum drawdowns for the daily S&P 500 over the period 1 January 1950 to 31 December 2013, together with the 90% quantile (the *drawdown threshold* DT) and tail-mean (CED) of the distribution. (b) Distribution of 6-month maximum drawdowns for an idealized standard normally distributed random variable, together with the 90% quantile and tail-mean of the distribution.

2.2 Maximum Drawdown Distributions

Even though, in a given horizon, only a single maximum drawdown is realized along any given path, it is beneficial to consider the distribution from which the maximum drawdown is taken. By looking at the maximum drawdown distribution, one can form reasonable expectations about the size and frequency of maximum drawdowns for a given portfolio over a given investment horizon.

Figure 3 shows (a) the empirical maximum drawdown distribution (for paths of length 125 business) of the daily S&P 500 time series over the period 1950 to 2013, and (b) the simulated distribution for an idealized Gaussian random variable. Both distributions are positively skewed, which implies that very large drawdowns occur less frequently than smaller ones.

Using Monte Carlo simlations, Burghardt et al. (2003) show that maximum drawdown distributions are highly sensitive to the length of the track record (increases in the length of the track record shift the entire distribution to the right), mean return (for larger mean returns, the distribution is less skewed to the right, since large means tend to produce smaller maximum drawdowns, volatility of returns (higher volatility increases the likelihood of large drawdowns), and data frequency (a drawdown based on longer horizon data would ignore the flash crash).

The tail of the maximum drawdown distribution, from which the likelihood of a drawdown of a given magnitude can be distilled, could be of particular interest in practice. Our drawdown risk metric, defined next, is a tail-based risk metric on the maximum drawdown distribution.

2.3 Conditional Expected Drawdown

Our proposed drawdown risk metric, the *Conditional Expected Drawdown* (Definition 3), measures the average of worst case maximum drawdowns exceeding a quantile of the maximum drawdown distribution. Hence, it is analogous to the return-based Expected Shortfall (ES). Both ES and CED are given by the *tail mean* of an underlying distribution, namely that of the losses and maximum drawdowns, respectively.

Definition 2 (Tail mean)**.** For a confidence level $\alpha \in (0, 1)$, the *lower α-quantile* of a random variable X is defined by $q_\alpha(X) = \inf\{x \in \mathbb{R} : P(X \leq x) \geq \alpha\}$. Assuming $\mathbb{E}[X] < \infty$, the *α-tail mean* of X is given by:

$$\mathrm{TM}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 q_\theta(X)d\theta,$$

which, for continuous distributions, is equivalent to the tail conditional expectation given by $\mathrm{TM}_\alpha = \mathbb{E}[X \mid X \geq q_\alpha(X)]$.

Analogous to the return-based Value-at-Risk (VaR), we define for confidence level $\alpha \in [0, 1]$ the *maximum drawdown threshold* DT_α to be a quantile of the maximum drawdown distribution:

$$\mathrm{DT}_\alpha\left(\mu(\mathrm{X}_{\mathrm{T_n}})\right) = \inf\left\{\mu \mid \mathbb{P}\left(\mu(\mathrm{X}_{\mathrm{T_n}}) > \mu\right) \leq 1-\alpha\right\}$$

It is thus the smallest maximum drawdown μ such that the probability that the maximum drawdown $\mu(X_{T_n})$ exceeds μ is at most $(1-\alpha)$. For example, the 95% maximum drawdown threshold separates the 5% worst maximum drawdowns from the rest. It is both a worst case for drawdown in an ordinary period and a best case among extreme scenarios.

Definition 3 (Conditional Expected Drawdown)**.** At confidence level $\alpha \in [0, 1]$, the *Conditional Expected Drawdown* CED_α is the function mapping the random variable $\mu(X_{T_n})$, representing maximum drawdown within the path X_{T_n}, to the expected maximum drawdown given that the maximum drawdown threshold at α is breached. More formally,

$$\mathrm{CED}_\alpha\left(X_{T_n}\right) = \mathrm{TM}_\alpha\left(\mu(X_{T_n})\right) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{DT}_u\left(\mu(\mathrm{X}_{\mathrm{T_n}})\right)\mathrm{d}u.$$

If the distribution of $\mu(X_{T_n})$ is continuous, then CED_α is equivalent to the tail conditional expectation:

$$\mathrm{CED}_\alpha\left(\mathrm{X}_{\mathrm{T_n}}\right) = \mathrm{TM}_\alpha\left(\mu(\mathrm{X}_{\mathrm{T_n}})\right) = \mathbb{E}\left(\mu(\mathrm{X}_{\mathrm{T_n}}) \mid \mu(\mathrm{X}_{\mathrm{T_n}}) > \mathrm{DT}_\alpha\left(\mu(\mathrm{X}_{\mathrm{T_n}})\right)\right).$$

The definition of CED is analogous to the definition of the return-based Expected Shortfall (ES). Both CED and ES inuitively measure downside risk, as by averaging over the worst events (maximum drawdowns and returns, respectively), they look deep into the underlying distribution. In theory, one advantage of looking at maximum drawdown rather than return distributions lies in the fact that drawdown is path dependent. We will investigate the effect of serial correlation in the context of risk concentrations in Section 6.

Remark 4. There is some overlap between CED and Conditional Drawdown, which was developed in Chekhlov et al. (2003, 2005), but there are three important distinctions. First, Chekhlov et al. (2003, 2005) define Conditional Drawdown as the tail-mean (the average of the worst X%) within a given path of drawdowns; that is the path of maximum losses incurred up to every point in time within that path. CED on the other hand is the tail mean of a distribution, namely that representing maximum drawdown. From a mathematical perspective, the distributional approach governing CED allows us to form expectations about drawdown risk in a way that is analogous to Expected Shortfall risk. From a practitioner's perspective, it makes drawdown risk amenable to the investment process, as we will show in Sections 4, 5, and 6 of this article. Second, the generalization of Conditional Drawdown to a multivariate setting looks at the average worst drawdowns within the union of all paths. This does not take coincident drawdown events into account. On the other hand, by defining marginal contributions to drawdown risk, we are able to infer a notion of generalized drawdown correlation, which accounts for drawdown events coinciding for two assets. Third, while Chekhlov et al. (2003, 2005) introduce an optimization algorithm with constraints for Conditional Drawdown within a given path, we show how to minimize CED directly.

3 Axiomatizing Drawdown Risk

We use the axiomatic theory of probabilistic risk measurement of Artzner et al. (1999) and McNeil et al. (2005) as a guide to derive theoretical properties of CED, most notably convexity and positive homogeneity.

3.1 Theory of Risk Measures

In classical risk assessment, uncertain portfolio outcomes over a fixed time horizon are represented as random variables on a probability space. A risk measure maps each random variable to a real number summarizing the overall position in risky assets of a portfolio.

Definition 5 (Risk Measure)**.** For the probability space $(\Omega, \mathcal{F}, P)$, let $L^0(\Omega, \mathcal{F}, P)$ be the set of all random variables on $(\Omega, \mathcal{F})$. A *risk measure* is a real-valued function $\rho\colon \mathcal{M} \to \mathbb{R}$, where $\mathcal{M}$ is a convex cone.[3]

Practitioners in risk management are often concerned with the profit-and-loss (P&L) distribution, which is the distribution of the change in net asset value of a portfolio. Because the main concern is the probability of large losses, random variables in the convex cone $\mathcal{M}$ are traditionally interpreted as portfolio *loss* L at a given horizon. Note, however, that Conditional Expected Drawdown is defined over *return paths* and not over one-period portfolio losses. To incorporate CED in the risk measurement framework, we will introduce the notion of *path-dependent risk measure* in the following Section.

One of the most widely used measures of risk is volatility, or the standard deviation of portfolio return, which was introduced in Markowitz (1952). However, Markowitz, himself, was not satisfied with volatility, since it penalizes gains and losses equally, and he proposed semideviation, which penalizes only losses, as an alternative. Over the past two decades, risk measures that focus on losses, such as Value-at-Risk (VaR) and Expected Shortfall (ES), have increased in popularity, both in the context of regulatory risk reporting and in downside-safe portfolio construction.

An axiomatic approach to (loss-based) risk measures was initiated by Artzner et al. (1999). They specified a number of properties that a good risk measure should have, with particular focus on applications in financial risk management. Their main focus is the class of *monetary* such measures, which can translate into capital requirement, hence making risk directly useful to regulators. Here, the risk $\rho(L)$ of a financial position L is interpreted as the minimal amount of capital that should be added to the portfolio positions (and invested in a risk-free manner) in order to make them acceptable:

[3] The requirement that the set of random variables $\mathcal{M}$ is a convex cone means that $L_1, L_2 \in \mathcal{M}$ implies $L_1 + L_2 \in \mathcal{M}$ and $\lambda L \in \mathcal{M}$ for every $\lambda > 0$ and $L \in \mathcal{M}$.

Definition 6 (Monetary Risk Measure)**.** A risk measure $\rho : \mathcal{M} \to \mathbb{R}$ is called *monetary* if it satisfies the following two axioms:

(A1) Translation invariance: For all $L \in \mathcal{M}$ and all constant almost surely $C \in \mathcal{M}$, $\delta(L + C) = \delta(X) - C$.
(A2) Monotonicity: For all $L_1, L_2 \in \mathcal{M}$ such that $L_1 \leq L_2$, $\rho(L_1) \geq \rho(L_2)$.[4]

A monetary risk measure is *coherent* if it is convex and positive homogenous:

Definition 7 (Coherent Risk Measure)**.** A risk measure $\rho : \mathcal{M} \to \mathbb{R}$ is called *coherent* if it is monetary and satisfies the following two axioms:

(A3) Convexity: For all $L_1, L_2 \in \mathcal{M}$ and $\lambda \in [0, 1]$, $\rho(\lambda L_1 + (1 - \lambda)L_2) \leq \lambda\rho(L_1) + (1 - \lambda)\delta(L_2)$.
(A4) Positive homogeneity: For all $L \in \mathcal{M}$ and $\lambda > 0$, $\rho(\lambda L) = \lambda\rho(L)$.

Since coherent measures of risk were introduced by Artzner et al. (1999), several other classes of risk measures were proposed, most notably *convex* measures[5] (Föllmer and Schied (2002, 2010, 2011)) and *deviation* measures (Rockafellar et al. (2002, 2006)). We briefly discuss the latter in Section 3.5.

3.2 CED as a Path-Dependent Risk Measure

Unlike traditional risk metrics, Conditional Expected Drawdown is defined over random variables representing return paths rather than one-period portfolio return or loss. We formalize the space of return paths as a convex cone, on top of which path-dependent risk measures can be defined as real-valued functionals.

Fix a time horizon t, and define $\mathcal{V}_t \subset L^0(\Omega, \mathcal{F}, P)$ to be the space of random variables $X_t : \Omega \to \mathbb{R}$ representing the return at time t in excess of some risk-free return at the same horizon, and note that $\mathcal{V}_t$ is a convex cone. For a path of length $n \in \mathbb{N}$, we can construct the direct sum (or equivalently the coproduct $\coprod$) $\mathcal{V}_{T_n} = \bigoplus_{i=1}^{n} \mathcal{V}_{t_i}$ using the finite ordered indexing set $T_n = \{t_1, \ldots, t_n\} \subset T$. Then $\mathcal{V}_{T_n}$ is also a convex cone containing random vectors $X_{T_n} = (X_{t_1}, \ldots, X_{t_n})$ representing return paths, where $X_{t_i} \in \mathcal{V}_{t_i}$.

Definition 8 (Path-dependent risk measure)**.** Fix a finite ordered set $T_n = \{t_1, \ldots, t_n\} \subset T$. A *path-dependent risk measure* is a real-valued function $\rho_{T_n} : \mathcal{V}_{T_n} \to \mathbb{R}$.

Analogous to traditional one-period risk measures, a path-dependent risk measure ρ_{T_n} is convex if for all $X_{T_n}, Y_{T_n} \in \mathcal{V}_{T_n}$ and $\lambda \in [0, 1]$, $\rho_{T_n}(\lambda X_{T_n} + (1 - \lambda)Y_{T_n}) \leq \lambda\rho_{T_n}(X_{T_n}) + (1 - \lambda)\rho_{T_n}(Y_{T_n})$. Similarly, ρ_{T_n} is positive homogenous of degree one if for all $X_{T_n} \in \mathcal{V}_{T_n}$ and $\lambda > 0$, $\rho_{T_n}(\lambda X_{T_n}) = \lambda\rho_{T_n}(X_{T_n})$.

By definition, Conditional Expected Drawdown is a path-dependent risk measure mapping $X_{T_n} \in \mathcal{V}_{T_n}$ to $\mathbb{E}\left(\mu(X_{T_n}) \mid \mu(X_{T_n}) > \mathrm{DT}_\alpha\left(\mu(X_{T_n})\right)\right) \in \mathbb{R}$. Since this is the tail-mean $\mathrm{TM}_\alpha\left(\mu(X_{T_n})\right)$ of the maximum

[4] All equalities and inequalities between random variables and processes are understood in the almost sure sense with respect to the probability measure P. For example, for processes X_T and Y_T, $X_T \leq Y_T$ means that for P-almost all $\omega \in \Omega$, $X_t(\omega) \leq Y_t(\omega)$ for all $t \in T$.

[5] In the larger class of convex risk measures, the conditions of subadditivity and positive homogeneity are relaxed. The positive homogeneity axiom, in particular, has received some criticism since its introduction. For example, it has been suggested that for large values of the multiplier λ concentration risk should be penalized by enforcing $\rho(\lambda X) > \lambda\rho(X)$.

drawdown distribution $\mu(X_{T_n})$, we can rewrite CED as the composite of the tail-mean functional TM_α and the function μ mapping return paths to maximum drawdowns. The domain of the former and range of the latter function is the space $\mathcal{D}_{T_n} \subset L^0(\Omega, \mathcal{F}, P)$, whose random variables represent maximum drawdowns within return paths at $T_n = \{t_1, \dots, t_n\}$. Then we have that

$$\mathrm{CED}_\alpha = \mathrm{TM}_\alpha \circ \mu,$$

where $\mu : \mathcal{V}_{T_n} \to \mathcal{D}_{T_n}$ maps the random vector X_{T_n} to $\max_{1\leq i<n} \max_{i<j\leq n} \left\{X_j - X_i, 0\right\}$, as defined in Section 2.1.

We know that the tail mean is a coherent measure of risk, independent of the underlying distribution (see Acerbi and Tasche (2002a,b)). To investigate properties of CED with respect to the return universe $\mathcal{V}_{T_n}$, we will need to characterize the transformation μ. Consider the convexity axiom: convexity of CED along the return space $\mathcal{V}_{T_n}$ would ensure that adding a position to an existing portfolio does not increase the overall drawdown risk.

Before proving convexity and positive homogeneity, we first show that CED is not a risk measure in the monetary sense of Definition 5, since it satisfies neither the translation invariance nor the monotonicity axiom.

Lemma 9. *For all $X_{T_n} \in \mathcal{V}_{T_n}$ and all constant almost surely $C \in \mathcal{V}$, $\mathrm{CED}_\alpha(X_{T_n} + C) = \mathrm{CED}_\alpha(X_{T_n})$ (for all $\alpha \in (0, 1)$).*

Proof. The drawdown functional d is invariant under constant deterministic shifts because $d_j(X_{T_n} + C) = \max_{1\leq i\leq j}\{X_{t_i} + C\} - X_{t_j} - C = \max_{1\leq i\leq j}\{X_{t_i}\} - X_{t_j} = d_j(X_{T_n})$. And so $\mathrm{CED}_\alpha(X_{T_n} + C) = \mathrm{CED}_\alpha(\max(d(X_{T_n} + C))) = \mathrm{CED}_\alpha(\max(d(X_{T_n}))) = \mathrm{CED}_\alpha(X_{T_n})$. □

Lemma 9 essentially states that by (deterministically) shifting the path of the portfolio value up or down, the drawdown within that path remains the same. Non-monotonicity is similarly easy to derive.

Lemma 10. *It is not generally the case that for $X_{T_n}, Y_{T_n} \in \mathcal{V}_{T_n}$ such that $X_{t_i} \leq Y_{t_i}$ (for all $i \leq n$),* $\mathrm{CED}_\alpha(X_{T_n}) \geq \mathrm{CED}_\alpha(Y_{T_n})$.

Proof. For a fixed $j \leq n$, $\max_{1\leq i\leq j}\{X_i\} \leq \max_{1\leq i\leq j}\{Y_i\}$, and therefore $d_j(X_{T_n}) \not\geq d_j(Y_{T_n})$, where $d = (d_1, \dots, d_n)$ is the drawdown transformation. □

Both translation invariance and monotonicity were originally introduced as desirable properties for risk measures ρ under the assumption that the risk $\rho(X)$ of a position X represents the amount of capital that should be added to the position X so that it becomes acceptable to the regulator. From a regulatory viewpoint, translation invariance means that adding the value of any guaranteed (that is, deterministic) position C to an existing portfolio portfolio simply decreases the capital required by the amount C, and vice versa. Moroever, monotonicity essentially states that positions that lead to higher losses should require more risk capital. We have developed the drawdown risk measure CED not with the regulatory reporting framework in mind, but with the purpose of mathematically formalizing drawdown in a way that is amenable to risk analysis and management. This is of particular interest to the asset management community. Hence, the impact of non-monotonicity and non-translation-invariance of CED in practice is limited.[6]

We now proceed to derive two theoretically and practically important properties of CED, namely convexity and postive homogeneity.

3.3 Convexity of CED

In Föllmer and Schied (2002, 2010, 2011), the essence of diversification is encapsulated in the convexity axiom. Suppose we have two random variables P_1 and P_2 representing returns to two portfolios. Rather than holding either only P_1 or only P_2, an investor could *diversify* by allocating a fraction $\lambda \in [0, 1]$ of his capital to, say, P_1, and the remainder $1 - \lambda$ to P_2. Under a convex risk measure ρ, we are ensured that diversification would not increase overall risk $\rho(P_1 + P_2)$.

Lemma 11. *The maximum drawdown transformation* $\mu : \mathcal{V}_{T_n} \to \mathcal{D}_{T_n}$ *is convex.*

Proof. Recall that we can define maximum drawdown as the maximum within a path of drawdowns, that is by $\mu(X_{T_n}) = \max(d(X_{T_n})) = \max_{1\leq i\leq n}\{d_{t_i}\}$. We show that each real-valued component $d_j : \mathbb{R}^n \to \mathbb{R}$ of the vector-valued transformation $d : \mathbb{R}^n \to \mathbb{R}^n$ given by $d(X_{T_n}) = (d_1(X_{T_n}), \ldots, d_n(X_{T_n}))$ is convex. For the j-th real-valued component d_j, we have for $\lambda \in [0, 1]$ and paths $X_{T_n} = (X_{t_1}, \ldots, X_{t_n})$ and $Y_{T_n} = (Y_{t_1}, \ldots, Y_{t_n})$

$$
\begin{aligned}
d_j(\lambda X_{T_n} + (1-\lambda)Y_{T_n}) &= \max_{1\leq i\leq j}\{\lambda X_{t_i} + (1-\lambda)Y_{t_i}\} - \left(\lambda X_{t_j} + (1-\lambda)Y_{t_j}\right) \\
&\leq \lambda \max_{1\leq i\leq j}\{X_{t_i}\} + (1-\lambda)\max_{1\leq i\leq j}\{Y_{t_i}\} - \lambda X_{t_j} - (1-\lambda)Y_{t_j} \\
&= \lambda\left(\max_{1\leq i\leq j}\{X_{t_i}\} - X_{t_j}\right) + (1-\lambda)\left(\max_{1\leq i\leq j}\{Y_{t_i}\} - Y_{t_j}\right) \\
&= \lambda d_j(X_{T_n}) + (1-\lambda)d_j(Y_{T_n})
\end{aligned}
$$

Since the max functional is convex and monotonically increasing, the composite transformation $\mu = max \circ d$ is also convex.[7] □

Because the tail-mean functional is also convex and monotonic, its composite with μ is also convex, and so we immediately obtain the following:

Proposition 12 (Convexity of CED)**.** *The path-dependent risk measure* $\mathrm{CED}_\alpha : \mathcal{V}_{T_n} \to \mathbb{R}$ *is convex.*

3.4 Positive Homogeneity of CED

Degree-one positive homogenous risk measures are characterized by Euler's homogenous function theorem, and hence play a prominent role in portfolio risk analysis. More precisely, for a portfolio return $P = \sum_i w_i X_i$ in $\mathcal{M}$, we know that a risk measure $\rho : \mathcal{M} \to \mathbb{R}$ is postive homogenous of

[6] Note also that the fact that CED is not a monetary measure of risk implies that it is not coherent in the strict sense of Definition 6, even though we will show that it is convex and positive homogenous.

[7] More precisely, $\max_{1\leq j\leq n} d_j(\lambda X_{T_n} + (1-\lambda)Y_{T_n}) \leq \max_{1\leq j\leq n}\left\{\lambda d_j(X_{T_n}) + (1-\lambda)d_j(Y_{T_n})\right\}$ because each d_j is convex and max is increasing, from which we get $\lambda \max_{1\leq j\leq n} d_j(X_{T_n}) + (1-\lambda)\max_{1\leq j\leq n} d_j(Y_{T_n})$ since max is convex.

degree one if and only if $\sum_i w_i \left(\partial\rho(P)\right)/(\partial w_i) = \rho(P)$.[8] The risk $\rho(P)$ of the portfolio $P = \sum_i w_i X_i$ can therefore be linearly attributed along its factors X_i.

The tail-mean functional is positive homogenous. To see that the drawdown transformation $\mu : \mathcal{V}_{T_n} \to \mathcal{D}_{T_n}$ is also positive homogenous, note that the components $d_j(X_{T_n}) = \max_{1\le i\le j}\{X_{t_i}\} - X_{t_j}$ of the drawdown transformation $d = (d_1, \ldots, d_n)$ are invariant under multiplication, since $\lambda d_j(X_{T_n}) = \lambda\left(\max_{1\le i\le j}\{X_{t_i}\} - X_{t_j}\right) = \max_{1\le i\le j}\{\lambda X_{t_i}\} - \lambda X_{t_j} = d_j(\lambda X_{T_n})$. Since positive homogeneity is preserved under composition, we have the following:

Proposition 13 (Positive homogeneity of CED)**.** *The path-dependent risk measure* $\mathrm{CED}_\alpha : \mathcal{V}_{T_n} \to \mathbb{R}$ *is positive homogenous.*

3.5 CED as a Deviation Measure

We show that Conditional Expected Drawdown is a *generalized deviation measure*, as developed by Rockafellar et al. (2002, 2006).

Definition 14 (Generalized Deviation Measure)**.** A *deviation measure* on the space $L^0(\Omega, \mathcal{F}, P)$ is a real-valued function $\delta\colon \mathcal{M} \to \mathbb{R}^+$ (with $\mathcal{M}$ a convex cone) satisfying the following four axioms

(D0) For all constant deterministic $C \in \mathcal{M}$, $\delta(C) = 0$.
(D1) For all $X \in \mathcal{M}$, $\delta(X) \ge 0$.
(D2) For all $X \in \mathcal{M}$ and all constant deterministic $C \in \mathcal{M}$, $\delta(X + C) = \delta(X)$.
(D3) For all $X \in \mathcal{M}$ and $\lambda > 0$, $\delta(\lambda X) = \lambda\delta(X)$.
(D4) For all $X_1, X_2 \in \mathcal{M}$ and $\lambda \in [0, 1]$, $\delta(\lambda X_1 + (1-\lambda)X_2) \le \lambda\delta(X_1) + (1-\lambda)\delta(X_2)$.

Positive homogeneity and convexity of CED imply that CED satisfies axioms (D3) and (D4). From Lemma 9 we know that, for confidence level $\alpha \in (0, 1)$, $\mathrm{CED}_\alpha(X_{T_n} + C) = \mathrm{CED}_\alpha(X_{T_n})$, hence CED satisfies axiom (D2). Note also that any portfolio of zero value and, more generally, of constant deterministic value is not exposed to drawdown risk, and so for all constant deterministic $C \in \mathcal{V}$, we have $\mathrm{CED}_\alpha(C) = 0$ (axiom (D0)). Finally, CED is always non-negative because the maximum drawdown is by definition non-negative, and so CED satisfies (D1).

Proposition 15. *Conditional Expected Drawdown is a generalized deviation measure.*[9]

Deviation measures obey axioms broadly taken from the properties of measures such as standard deviation and semideviation, which are not coherent measures of risk.[10] Unlike coherent risk, in the framework of Rockafellar et al. (2002, 2006) a quantification of risk is applied to a loss relative to expectation rather than to a negative outcome.[11] Moreover, a deviation measure is insensitive to translation in portfolio value, while a coherent risk measure is translation invariant.

[8] This formula and the topic of risk attribution is discussed in more detail in Section 4.

[9] Technically for that statement to hold, we would need to generalize the axioms of deviation measures to the path-dependent universe $\mathcal{V}_{T_n}$ first.

[10] The obstacles are the positive homogeneity and translation invariance axioms.

[11] In Artzner et al. (1999) and most of the subsequent literature on coherent and convex risk measures, the term *loss* is defined as an outcome below zero, whereas in practice there may be situations where there is interest in treating the extent to which a random variable falls short of a certain threshold, such as its expected value, differently from the extent to which it exceeds it.

The connection between deviation and coherent risk is, however, close. Axioms (D3) and (D4) coincide with the positive homogeneity (A3) and convexity axioms (A4) of coherent risk measures. However, there is a subtle but crucial distinction. A coherent risk measures focuses on loss rather than gain. If one is interested in the extent to which a position X drops below a threshold C, then one needs to replace the random variable X by $X - C$. Indeed, under some conditions,

Theorem 16 (Rockafellar et al. (2002, 2006))**.** *Lower range dominant deviation measures (i.e. those satisfying $\delta(X) \leq E[X]$ for all $X \geq 0$) correspond bijectively to coherent, strictly expectation bounded risk measures (i.e. coherent measures satisfying $\rho(X) > E[-X]$) under the relations $\delta(X) = \rho(X - E[X])$ and $\rho(X) = E[-X] + \delta(X)$.*

4 Drawdown Risk Attribution

With the theoretical framework of drawdown risk measurement in place, the next step is to understand how Conditional Expected Drawdown can be integrated in the investment process.

We show how to systematically analyze the sources of drawdown risk within a portfolio and how these sources interact. In practice, investors may be interested in attributing risk to individual securities, asset classes, sectors, industries, currencies, or style factors of a particular risk model. In what follows, we assume a generic such risk factor model.

4.1 Rudiments of Risk Contributions

Fix an investment period and let X_i denote the return of factor i over this period ($1 \leq i \leq n$). Then the portfolio return over the period is given by the sum

$$P = \sum_{i=1}^{n} w_i X_i \quad ,$$

where w_i is the portfolio exposure to factor i and the summand representing idiosyncratic risk is not included for simplicity. Because portfolio risk is not a weighted sum of source risks, there is no direct analog to this decomposition for risk measures. However, there is a parallel in terms of *marginal risk contributions* (MRC), which are interpreted as a position's percent contribution to overall portfolio risk. They provide a mathematically and economically sound way of decomposing risk into additive subcomponents.

For a risk measure ρ, the marginal contribution to risk of a factor is the approximate change in overall portfolio risk when increasing the factor exposure by a small amount, while keeping all other exposures fixed. Formally, marginal risk contributions can be defined for any differentiable risk measure ρ.

Definition 17. For a factor X_i in the portfolio $P = \sum_i w_i X_i$, its *marginal risk contribution* MRC_i is the derivative of the underlying risk measure ρ along its exposure w_i:

$$\mathrm{MRC}_i^{\rho}(P) = \frac{\partial \rho(P)}{\partial w_i}.$$

If ρ is homogenous of degree one, the overall portfolio risk can be decomposed using Euler's homogoneous function theorem as follows:

$$\sum_i w_i \mathrm{MRC}_i^{\rho}(P) = \sum_i \mathrm{RC}_i^{\rho}(P) = \rho(P),$$

where $\mathrm{RC}_i^\rho(P) = w_i \mathrm{MRC}_i^\rho(P)$ is the i-th total *risk contribution* to ρ. Finally, *fractional risk contributions*

$$\mathrm{FRC}_i^\rho(P) = \frac{\mathrm{RC}_i^\rho(P)}{\rho(P)}$$

denote the fractional contribution of the i-th factor to portfolio risk.

Risk contributions have become part of the standard toolkit for risk management, often under the labels of *risk budgeting* and *capital allocation*. Tasche (2000) showed that the first partial derivative is the only definition for risk contribution that is suitable for performance measurement. In Kalkbrener (2005), and with a more game-theoretic approach in Denault (2001), an axiomatic system for capital allocation is uniquely and completely satisfied by risk contributions. Qian (2006) investigated the financial significance of risk contributions as predictors of each component's contribution to ex-post losses.

Remark 18 (Generalized risk correlations). Risk contributions implicitly define a notion of correlation that is general enough to be defined for any risk measure. Consider volatility. The linear correlation between a portfolio and one of its assets can be recast in terms of marginal contribution to volatility risk as follows:

$$\mathrm{Corr_i}^\sigma = \frac{\mathrm{Cov}_i(P)}{\sigma(P)\sigma(X_i)} = \frac{\mathrm{MRC}_i^\sigma(P)}{\sigma(X_i)}$$

This leads to the more general definition of *generalized risk-based correlation* $\mathrm{Corr}_\mathrm{i}^\rho$ for a generic risk measure $\rho : \mathcal{M} \to \mathbb{R}$ between the portfolio and the ith asset X_i:

$$\mathrm{Corr}_\mathrm{i}^\rho = \frac{\mathrm{MRC}_\mathrm{i}^\rho(\mathrm{P})}{\rho(\mathrm{X_i})}.$$

Generalized correlations are monotonically decreasing in position weight. Factoring out the ith marginal risk $\rho(X_i)$ from the ith risk contribution $\mathrm{RC}_i(P)$, we obtain the generalized form of the "X-Sigma-Rho" decomposition of Menchero and Poduri (2008):

$$\mathrm{RC}_i^\rho(P) = w_i \rho(X_i) \frac{\mathrm{MRC}_i^\rho(P)}{\rho(X_i)} = w_i \rho(X_i) \mathrm{Corr}_\mathrm{i}^\rho \quad .$$

We refer the reader Goldberg et al. (2010) for a more detailed development of generalized correlations.

4.2 Drawdown Risk Contributions

Menchero and Poduri (2008) and Goldberg et al. (2010) developed a standard toolkit for analyzing portfolio risk using a framework centered around marginal risk contributions. By integrating drawdown risk into this framework, investors can estimate how a trade would impact the overall drawdown risk of the portfolio. Because Conditional Expected Drawdown is positive homogenous, the individual factor contributions to drawdown risk add up to the overall drawdown risk within a path $P_{T_n} = (P_{t_1}, \ldots, P_{t_n})$ of returns to a portfolio with values at time t_j ($j \leq n$) given by $P_{t_j} = \sum_i w_i X_{i,t_j}$[12]:

$$\mathrm{CED}_\alpha(P_{T_n}) = \sum_i w_i \mathrm{MRC}_i^{\mathrm{CED}_\alpha}(P_{T_n}), \quad \alpha \in [0, 1]. \tag{1}$$

[12]The path of the i-th factor X_i is written as X_{i,T_n} with the j-th entry within that path given by X_{i,t_j}.

Recall that a marginal risk contribution is a partial derivative, and so practitioners can implement Formula 1 using numerical differentiation. However, this tends to introduce noise. We next show that an individual marginal contribution to drawdown risk can be expressed as an integral, and this reduces noise, since integration is a smoothing operator.[13]

The individual marginal contribution $\mathrm{MRC}_i^{\mathrm{CED}_\alpha}$ of the i-th factor to overall portfolio drawdown risk $\mathrm{CED}_\alpha(P_{T_n})$ is given by the expected drop of the i-th factor in the interval $[t_{j*}, t_{k*}]$ where the overall portfolio maximum drawdown $\mu(P_{T_n})$ occurs, given that the maximum drawdown of the overall portfolio exceeds the drawdown threshold. This definition is analogous to the marginal contribution to shortfall, and we formalize it next.

Proposition 19. *Marginal contributions to drawdown risk are given by:*

$$\mathrm{MRC}_i^{\mathrm{CED}_\alpha}(P_{T_n}) = \mathbb{E}\left[\left(X_{i,t_{k*}} - X_{i,t_{j*}}\right) \mid \mu(P_{T_n}) > \mathrm{DT}_\alpha(P_{T_n})\right], \tag{2}$$

where $\mathrm{CED}_\alpha(P_{T_n})$ *is the overall portfolio CED,* $\mu(P_{T_n})$ *is the maximum drawown random variable,* $\mathrm{DT}_\alpha(\mathrm{P}_{\mathrm{T}_\mathrm{n}})$ *is the portfolio maximum drawdown threshold at* α*, and* $j^* < k^* \leq n$ *are such that:*

$$\mu(P_{T_n}) = \max_{1\leq j<n} \max_{j<k\leq n} \left\{P_{t_k} - P_{t_j}, 0\right\} = P_{t_{k*}} - P_{t_{j*}},$$

and we assume that the maximum drawdown of $P_{T_n} = \sum_i w_i X_{i,T_n}$ *is strictly positive.*

Proof. Most of the following is based on Goldberg et al. (2010) and McNeil et al. (2005), who show that the i-th marginal contribution to Expected Shortfall ES_α at confidence level $\alpha \in (0, 1)$ of a random variable $P = \sum_i w_i Y_i$ representing portfolio loss is given by

$$\mathrm{MRC}_i^{\mathrm{ES}_\alpha}(P) = \mathbb{E}\left[Y_i \mid P > \mathrm{Var}_\alpha(P)\right], \tag{3}$$

where $\mathrm{Var}_\alpha(P)$ denotes the Value-at-Risk of P at α, that is the α-quantile of the loss distribution P.

Because the maximum drawdown functional μ is not additive, Formula 3 cannot be immediately applied to a random variable representing maximum drawdown. More precisely, unlike losses, we do not generally have that $\mu(P_{T_n})$ equals $\sum_i w_i \mu(X_{i,T_n})$. But we are able to derive an analog to Formula 3 nevertheless..

Assuming that the maximum drawdown of $P_{T_n} = \sum_i w_i X_{i,T_n}$ is strictly positive, let

$$\mu(P_{T_n}) = \max_{1\leq j<n} \max_{j<k\leq n} \left\{P_{t_k} - P_{t_j}, 0\right\} = P_{t_{k*}} - P_{t_{j*}}$$

for some $j^* < k^* \leq n$. Then the i-th marginal contribution $\mathrm{MRC}_i^{\mathrm{CED}_\alpha}(P_{T_n})$ to overall portfolio draw-

[13] This is analogous to marginal contributions to Expected Shortfall, which can also be expressed as integrals.

down risk $\mathrm{CED}_\alpha(P_{T_n})$ is given by

$$\begin{aligned}
\mathrm{MRC}_i^{\mathrm{CED}_\alpha}(P_{T_n}) &= \frac{\partial}{\partial w_i}\left(\mathrm{TM}_\alpha\left(\mu(P_{T_n})\right)\right) \\
&= \frac{\partial}{\partial w_i}\mathbb{E}\left[\mu(P_{T_n}) \mid \mu(P_{T_n}) > \mathrm{DT}_\alpha(P_{T_n})\right] \\
&= \frac{\partial}{\partial w_i}\mathbb{E}\left[(P_{t_{k*}} - P_{t_{j*}}) \mid \mu(P_{T_n}) > \mathrm{DT}_\alpha(P_{T_n})\right] \\
&= \frac{\partial}{\partial w_i}\mathbb{E}\left[\left(\sum_{i=1}^{n} w_i X_{i,t_{k*}} - \sum_{i=1}^{n} w_i X_{i,t_{j*}}\right) \mid \mu(P_{T_n}) > \mathrm{DT}_\alpha(P_{T_n})\right] \\
&= \frac{\partial}{\partial w_i}\mathbb{E}\left[w_i \sum_{i=1}^{n}\left(X_{i,t_{k*}} - X_{i,t_{j*}}\right) \mid \mu(P_{T_n}) > \mathrm{DT}_\alpha(P_{T_n})\right] \\
&= \frac{\partial}{\partial w_i}\left(\sum_{i=1}^{n} w_i \mathbb{E}\left[\left(X_{i,t_{k*}} - X_{i,t_{j*}}\right) \mid \mu(P_{T_n}) > \mathrm{DT}_\alpha(P_{T_n})\right]\right) \qquad (4)
\end{aligned}$$

Using the fact that the partial derivative with respect to the quantile DT_α is zero, as discussed by Bertsimas et al. (2004), Formula 4 simplifies to:

$$\mathrm{MRC}_i^{\mathrm{CED}_\alpha}(P_{T_n}) = \mathbb{E}\left[\left(X_{i,t_{k*}} - X_{i,t_{j*}}\right) \mid \mu(P_{T_n}) > \mathrm{DT}_\alpha(P_{T_n})\right].$$

□

5 Drawdown Risk Optimization

A rigorous mathematical theory of risk measurement can go beyond risk analysis purposes. In particular, it can be used for making risk-based asset allocation decisions. Before incorporating a risk measure other than variance – around which much of risk and portfolio management continues to revolve in practice – into the portfolio construction process, one needs to address the issues of feasibility and potential use. Consider minimum variance optimization, the goal of which is to allocate funds to a selection of assets in such a way that portfolio risk is minimized. Recent work has shown the theoretical feasibility (Rockafellar and Uryasev (2000, 2002)) and practical applicability (Goldberg et al. (2013)) of downside-safe portfolio construction based on Expected Shortfall.

We extend this line of research by showing that one can allocate assets to trade off drawdown risk against portfolio return. There are two crucial ingredients for carrying out any optimization in practice. Convexity of the objective function to be minimized ensures that the minimum, if it exists, is a global one. In the context of risk measures, convexity ensures that diversification of a portfolio does not increase risk, as formalized by Föllmer and Schied (2002, 2010, 2011). The second ingredient is the feasibility and efficiency of the optimization algorithm.[14] Having established the convexity of Conditional Expected Drawdown, thereby making it amenable to traditional optimization tools, we

[14] A third crucial ingredient is having a reliable risk model feeding the optimizer with realistic and useful scenarios. This being beyond the scope of the present article, we have focused on the two main theoretical requirements in the present article.

next show that its minimization for the purpose of active portfolio construction is indeed feasible and can be implemented efficiently via a linear programming algorithm.

We consider a general asset allocation problem with m assets and weights $\mathbf{w} = (w_1, \ldots, w_m)$.[15,16] We assume that our optimization is based on a given vector of T' maximum drawdown scenarios $\hat{\mu}_i$. For example, these scenarios may be historical time series or simulations based on a parametric distribution. An estimate $\widehat{\mathrm{CED}}$ of the Conditional Expected Drawdown at, say, a 90% threshold is then given by the average of the 10% worst maximum drawdown scenarios. For $\alpha \in [0, 1]$, we have

$$\widehat{\mathrm{CED}}_\alpha = \frac{1}{K} \sum_{i=1}^{K} \hat{\mu}_{(i)}, \tag{5}$$

where $K = \lfloor T' (1 - \alpha) \rfloor$. The drawdown minimization can than be written as:

$$\min_{\mathbf{w}} \qquad \frac{1}{K} \sum_{i=1}^{K} \hat{\mu}_{(i)}. \tag{6}$$

This is referred to as a *sample drawdown optimization*, as the input depends on the underlying sample of maximum drawdown scenarios. The optimal weight vector $\mathbf{w}$ does not appear explicitly in Formula 6, but is implicit in the maximum drawdown μ. Each scenario $\hat{\mu}_i$ depends on both portfolio returns and weights. We can, however, reformulate this optimization into a linear programming (LP) problem, making it computationally feasible.

Theorem 20 (LP formulation of CED optimization)**.** *The drawdown optimization problem of Formula 1 is equivalent to the following linear programming optimization:*

$$\begin{aligned} \min_{\mathbf{w},t,z,u} \qquad & t + \frac{1}{K} \sum_{i=1}^{T'} z_i \\ \text{s.t.} \qquad & z_i + t \geq u_{i,j} - \mathbf{w}'\mathbf{r}_{i,j}; \\ & z_i \geq 0; \quad u_{i,0} = 0; \quad u_{i,j} \geq 0; \quad i \leq T'; \quad j \leq N \end{aligned} \tag{7}$$

where $i \leq T'$ indexes over the maximum drawdown scenarios (and return paths), $j \leq N$ over the returns within a path of length N, and $\mathbf{r}_{i,j} = \left(r_{i,j}^{(1)}, \ldots, r_{i,j}^{(m)}\right)$ is a vector of returns to m assets.

Proof. Using the LP formulation of shortfall optimization developed by Rockafellar and Uryasev (2000), the drawdown optimization problem of Formula 6 can be transformed into the following linear optimization problem:

$$\begin{aligned} \min_{\mathbf{w},t,z} \qquad & t + \frac{1}{K} \sum_{i=1}^{T'} z_i \\ \text{s.t.} \qquad & z_i + t \geq \hat{\mu}_i; \quad z_i \geq 0; \quad i \leq T' \end{aligned} \tag{8}$$

Inputs to this optimization are T', N-day paths of portfolio returns $\mathbf{P}_i = (P_{i,1}, \ldots, P_{i,N})$, for $i \leq T'$, to a portfolio of, m assets. From these paths, the maximum drawdown estimates $\hat{\mu}_i$ are calculated. At a

[15]This can be reformulated in terms of linear asset exposures to some factors.

[16]Note that since CED is convex in the return paths, it is also convex in weights.

given point in time j within this path, each portfolio return $P_{i,j}$ is given by the sum of the product of asset weights $\mathbf{w} = (w_1, \ldots, w_m)$ and asset returns $\mathbf{r}_{i,j} = \left(r_{i,j}^{(1)}, \ldots, r_{i,j}^{(m)}\right)$, that is

$$P_{i,j} = \mathbf{w}'\mathbf{r}_{i,j}, \quad i \leq T', j \leq N.$$

Because the optimal weight vector $\mathbf{w}$ still does not appear explicitly in the LP formulation of Formula 8, it remains unclear how this LP optimization can be solved as it stands. Our reformulation, which makes the problem tractable while retaining linearity, goes as follows.

Recall that each maximum drawdown μ_i of the i-th path of portfolio returns is defined as $\mu_i = \max_{1 \leq j \leq N}\{d_{i,j}\}$, where $\mathbf{d} = (d_{i,1}, \ldots, d_{i,N})$ is the corresponding path of drawdowns. We can therefore replace each constraint $z_i + t \geq \mu_i$ with N equivalent constraints $z_i + t \geq d_{i,j}$, for $j \leq N$. Optimization problem 8 can then be rewritten as:

$$\begin{aligned} \min_{\mathbf{w},t,z} \quad & t + \frac{1}{K}\sum_{i=1}^{T'} z_i \\ \text{s.t.} \quad & z_i + t \geq d_{i,j}; \quad z_i \geq 0; \quad i \leq T'; \quad j \leq N. \end{aligned} \tag{9}$$

We have added $T'(N-1)$ additional constraints in this way, each of which is linear in its arguments.[17] Next, observe that the drawdown $d_{i,j}$ at point in time i within a given j-th path can be redefined recursively as:

$$d_{i,j} = \max\{d_{i,j-1} - P_{i,j}, 0\}.$$

So we can replace each constraint:

$$z_i + t \geq d_{i,j} \quad i \leq T', \quad j \leq N$$

by:

$$z_i + t \geq u_{i,j}; \quad u_{i,j} \geq u_{i,j-1} - \mathbf{P}_{i,j}; \quad u_{i,0} = 0; \quad u_{i,j} \geq 0$$

or equivalently:

$$z_i + t \geq u_{i,j}; \quad u_{i,j} \geq u_{i,j-1} - \mathbf{w}'\mathbf{r}_{i,j}; \quad u_{i,0} = 0; \quad u_{i,j} \geq 0 \quad .$$

Optimization problem (4) can then be written as:

$$\begin{aligned} \min_{\mathbf{w},t,z,u} \quad & t + \frac{1}{K}\sum_{i=1}^{T'} z_i \\ \text{s.t.} \quad & z_i + t \geq u_{i,j} - \mathbf{w}'\mathbf{r}_{i,j}; \\ & z_i \geq 0; \quad u_{i,0} = 0; \quad u_{i,j} \geq 0; \quad i \leq T'; \quad j \leq N \end{aligned} \tag{10}$$

which has $T' + 1$ new constraints. □

[17]These additional constraints do not significantly slow down the optimization algorithm, since the time complexity is still of linear magnitude. We refer the reader to Dantzig and Thapa (1997, 2003) for a review of linear programming algorithms and their complexity.

	Volatility	$ES_{0.9}$	$CED_{0.9}$ (6M-paths)	$CED_{0.9}$ (1Y-paths)	$CED_{0.9}$ (5Y-paths)
US Equity	18.35%	2.19%	47%	51%	57%
US Bonds	5.43%	0.49%	29%	32%	35%
50/50	9.53%	1.30%	31%	32%	35%
60/40	11.12%	1.35%	33%	35%	38%
70/30	12.92%	1.40%	36%	40%	44%

Table 1: Summary statistics for daily US Equity and US Bond Indices and three fixed-mix portfolios over the period 1 January 1982 to 31 December 2013. Expected Shortfall and Conditional Expected Drawdown are calculated at the 90% confidence level. Three drawdown risk metrics are calculated by considering the maximum drawdown within return paths of different fixed lengths (6 months, 1 year and 5 years).

6 Empirical Analysis of Drawdown Risk

We analyze historical values of Conditional Expected Drawdown based on daily data for two asset classes: US Equity and US Government Bonds. The US Government Bond Index we use[18] includes fixed income securities issued by the US Treasury (excluding inflation-protected bonds) and US government agencies and instrumentalities, as well as corporate or dollar-denominated foreign debt guaranteed by the US government, with maturities greater than 10 years. These include government agencies such as the Federal National Mortgage Association (Fannie Mae) and the Federal Home Loan Mortgage Corporation (Freddie Mac), which are private companies, without an explicit guarantee. In comparison to US *Treasury* Bond Indices, US Government Bond Indices were highly volatile and correlated with US Equities during the financial crisis of 2008. The effect of this will be seen in our empirical analysis.[19] Summary risk statistics for the two assets and three fixed-mix portfolios are shown in Table 1.

6.1 Time-varying Drawdown Risk Concentrations

Using the definition of marginal contributions to Conditional Expected Drawdown (derived in Proposition 18), we look at the time varying contributions to CED. Figure 4 displays the daily 6-month rolling fractional contributions to drawdown risk $CED_{0.9}$ (at the 90% threshold of the 6-month maximum drawdown distribution) of the two assets (US Equity and US Bonds) in the balanced 60/40 allocation.[20,21] Over most of the period, the contributions of US Equity to overall drawdown risk fluctuated between 80% and 100%. Note that this includes two of the three turbulent market regimes that occurred during this 30-year window, namely the 1987 stock market crash and the burst of the internet bubble in the early millennium. During the credit crisis of 2008, however, we see, unexpectedly, that bonds contribute almost as much as equities to portfolio drawdown risk.

Our analysis shows little connection between market turbulence and drawdown risk concentration in

[18]See Appendix A for details on the data and their source.

[19]We thank Robert Anderson for pointing out the important distinction between US Government Bond and US Treasury Bond Indices.

[20]See Appendix B for details on the risk estimation and portfolio construction methodologies used.

[21]Similar effects can be seen in other fixed-mix portfolios, such as the equal-weighted 50/50 portfolio and the 70/30 allocation. In the following empirical analyses, we will be focusing exclusively on the traditional 60/40 allocation.

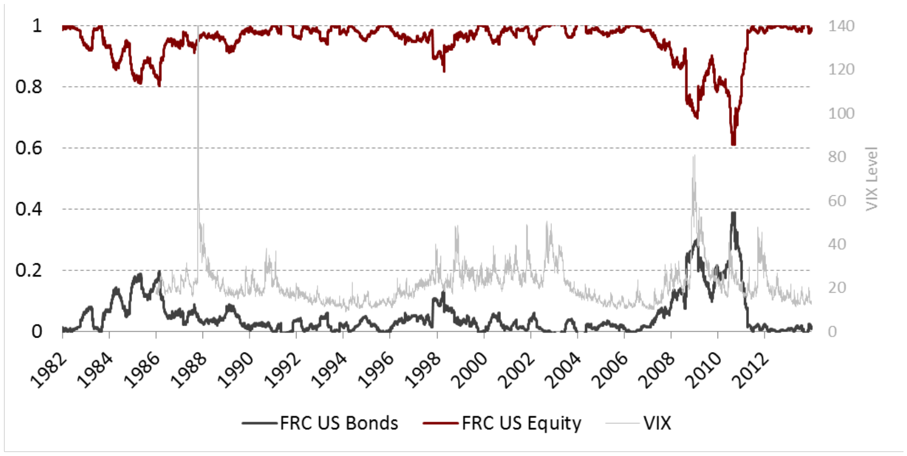


Figure 4: Daily 6-month rolling Fractional Risk Contributions (FRC) along the 90% Conditional Expected Drawdown (CED) of US Equity and US Bonds to the balanced 60/40 portfolio. Also displayed is the daily VIX series over the same period, with the right-hand axis indicating its level.

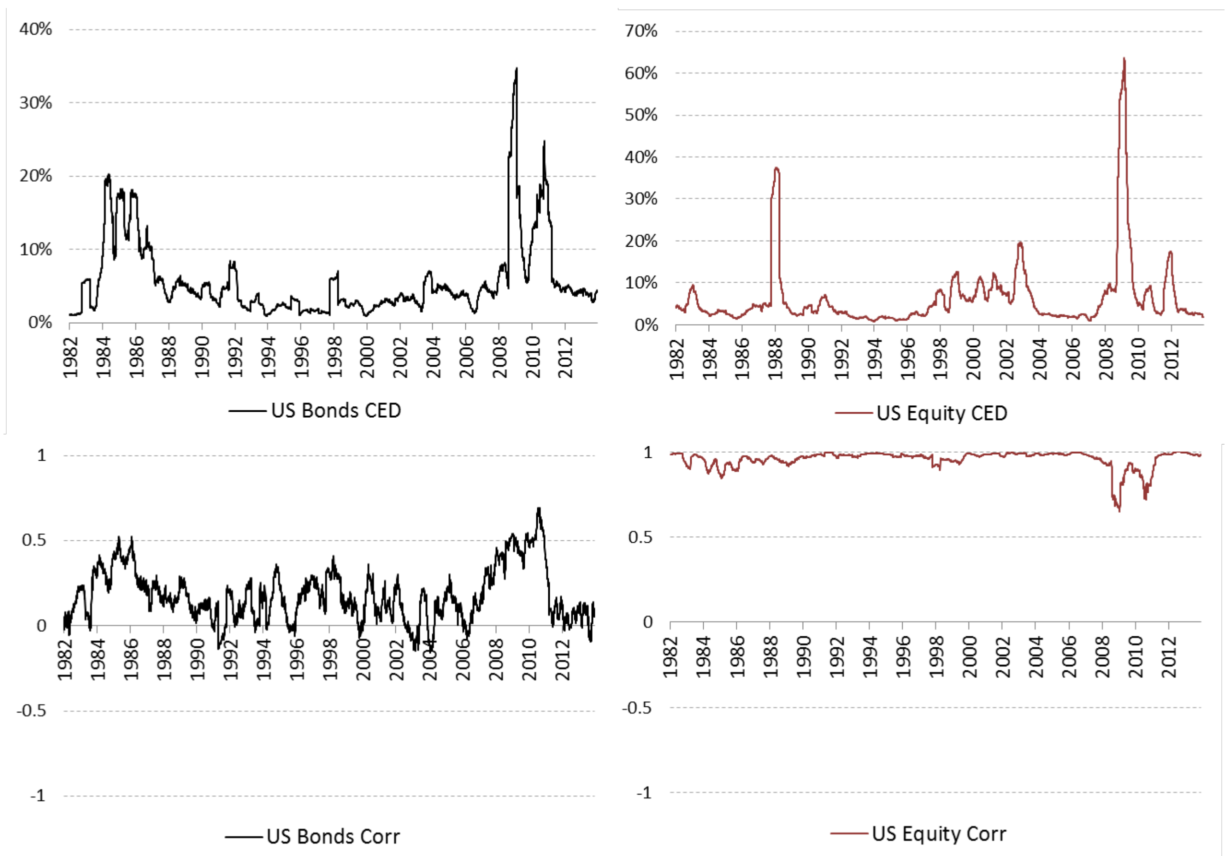


Figure 5: Decomposition of the individual contributions to drawdown risk $\mathrm{RC}_i^{\mathrm{CED}}(P) = w_i \mathrm{CED}(X_i)\mathrm{Corr}_{\mathrm{i}}^{\mathrm{CED}}$ for the 60/40 allocation to US Equity and US Bonds. The top two panels show the daily 6-month rolling standalone 90% Conditional Expected Drawdown (CED) of the two assets, while the bottom two panels show the daily 6-month rolling generalized correlations of the individual assets along CED.

the 60/40 fixed mix of US Equity and US Bonds. Notably, the most equitable attribution of drawdown risk occurred during the 2008 financial crisis. This can be explained by the inclusion of bonds issued by Fannie Mae and Freddie Mac in the US Government Bond Index. In calm regimes, these Agency Bonds tend to be correlated with US Treasury bonds, but during the financial crisis, Agency Bonds were more correlated with US Equity. For comparison, we provide the same analysis when the underlying Bond Index used is the US Treasury Bond Index (see Figures 10 and 11 in Appendix B). In this case, as one would expect, the least equitable attribution of drawdown risk occurred during turbulent market periods.

To understand the sources of the risk contributions, particularly during the credit crisis of 2008 where the concentrations of US Equity and US Government Bonds approached parity, we carry out the "X-Sigma-Rho" decomposition of Menchero and Poduri (2008). Recall from Section 4.1 that risk contribution is proportional to the product of standalone risk and generalized correlation. In the case of Conditional Expected Drawdown, this means that:

$$\mathrm{RC}_i^{\mathrm{CED}}(P) = w_i \mathrm{CED}(X_i)\mathrm{Corr}_{\mathrm{i}}^{\mathrm{CED}}.$$

Because we are working with a fixed-mix portfolio, the exposures w_i are constant: 0.6 and 0.4 for US Equity and US Bonds, respectively. This means that the time-varying risk contributions of Figure 4 depend on the time-varying drawdowns ($\mathrm{CED}(X_i)$) and correlations ($\mathrm{Corr}_{\mathrm{i}}^{\mathrm{CED}}$). Figure 5 displays these for each of the two assets in our 60/40 portfolio. Observe that during the 2008 financial crisis, both the drawdown risk contribution of US Bonds and its generalized correlation were elevated relative to the subsequent period. On the other hand, the generalized correlation of US Equity during the 2008 crisis decreased. The combination of these effects may have driven the changes in the drawdown contributions of US Bonds and US Equity during the 2008 crisis. [22]

In Section 6.2, we give a statistical analysis that supports the economic explanation of the increased CED values for US Government Bonds. In practice, investors can efficiently control such regime-dependent fluctuations in drawdown risk concentrations since Conditional Expected Drawdown is a convex risk measure; that is both the return path and the drawdown path are convex functions of asset weights. Hence, they are convex functions of factors that are linear combinations of asset weights. This implies that reducing the portfolio exposure to an asset or factor in a linear factor model decreases its marginal contribution to overall portfolio drawdown.

It is possible for a portfolio to have equal risk contributions with respect to one measure while harboring a substantial concentration with respect to another.[23] Figure 6 illustrates such a case. Four portfolios are constructed to be maximally diversified along the following risk measures: volatility, Expected Shortfall, and Conditional Expected Drawdown. The underlying assets are US Equity and US Government Bonds as before.[24] We refer to these as being in *parity* with respect to the underlying risk measure. The confidence level for both ES and CED is fixed at 90%. Figure 6 shows fractional risk contribution of the equity component to each of three risk measures in three types of risk parity portfolios. Concentrations in terms of drawdown risk, in particular, are revealed. For instance, even

[22] For comparison, we include in Figure 12 of Appendix C the risk decomposition along Expected Shortfall.

[23] Risk parity portfolios, which are constructed to equalize risk contributions, have been popular investment vehicles in the wake of the 2008 financial crisis (see Anderson et al. (2012) and Anderson et al. (2014)). This is in spite of the fact that there may be no theoretical basis for the construction.

[24] See Appendix B for details on the data, risk estimation, and portfolio construction methodologies used.

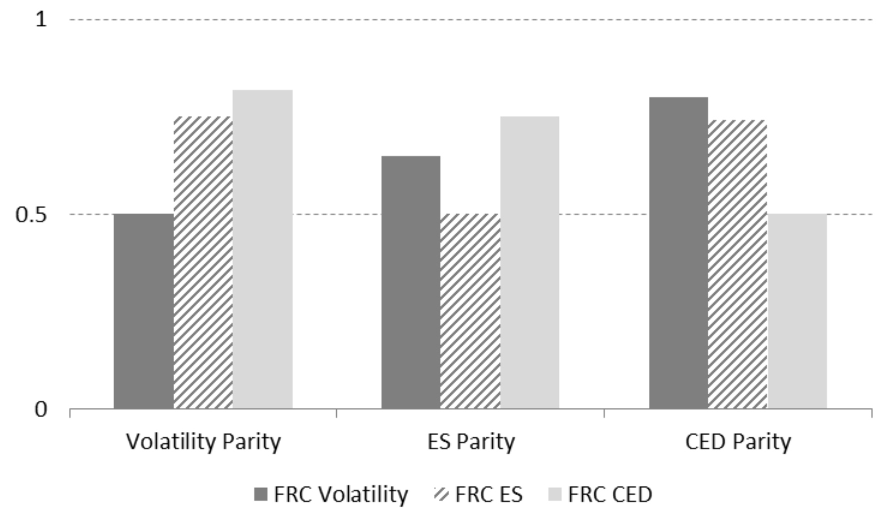


Figure 6: Fractional Risk Contributions (FRC) of US Equity measured along three different risk measures (volatility, 90% Expected Shortfall and 90% Conditional Expected Drawdown) for the following two-asset portfolios consisting of US Equity and US Bonds: Volatility Parity, ES Parity and CED Parity. Each parity portfolio is constructed to have equal risk contributions along its eponymous risk measure.

though the ES Parity portfolio, which has equal contributions to Expected Shortfall, is constructed to minimize downside risk concentrations, it turns out to have 75% of its drawdown risk concentrated in US Equity.

6.2 Drawdown Risk and Serial Correlation

One advantage of looking at maximum drawdown rather than return distributions, and thus Conditional Expected Drawdown rather than Expected Shortfall, lies in the fact that drawdown is inherently path dependent. In other words, drawdown measures the degree to which losses are sustained, as small but persistent cumulative losses may still lead to large drops in portfolio net asset value, and hence may force liquidation. On the other hand, volatility and Expected Shortfall fail to distinguish between intermittent and consecutive losses. We show that, to a greater degree than these two risk measures, Conditional Expected Drawdown captures temporal dependence. Moreover, the effect of serial correlation on drawdown risk can be seen in the drawdown risk contributions.

An increase in serial correlation increases drawdown risk. To see how temporal dependence affects risk measures, we use Monte Carlo simulation to generate an autoregressive AR(1) model:

$$r_t = \kappa r_{t-1} + \epsilon_t,$$

with varying values for the autoregressive parameter κ (while ϵ is Gaussian with variance 0.01), and calculate volatility, Expected Shortfall, and Conditional Expected Drawdown of each simulated autoregressive time series. Figure 7 displays the results. All three risk measures are affected by the increase in the value of the autoregressive parameter, but the increase is steepest by far for CED. We next use maximum likelihood to fit the AR(1) model to the daily time series of US Equity and US Government Bonds on a 6-month rolling basis to obtain time series of estimated κ values for

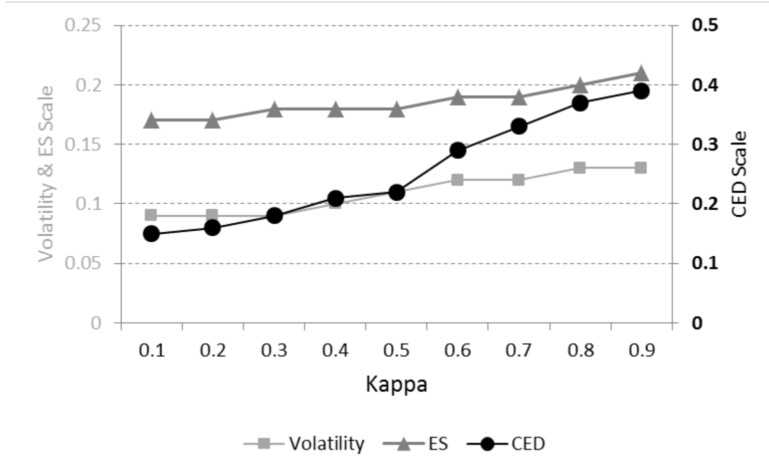


Figure 7: Volatility, 90% Expected Shortfall (ES), and 90% Conditional Expected Drawdown (CED) of a Monte Carlo simulated AR(1) model (with 10,000 data points) for varying values of the autoregressive parameter κ.

	Volatility	$ES_{0.9}$	$CED_{0.9}$
US Equity	0.47	0.52	0.75
US Bonds	0.32	0.39	0.69

Table 2: For the daily time series of each of US Equity and US Government Bonds, correlations of estimates of the autoregressive parameter κ in an AR(1) model with the values of the three risk measures (volatility, 90% Expected Shortfall and 90% Conditional Expected Drawdown) estimated over the entire period (1982–2013).

each asset. The correlations of the time series of κ with the time series of 6-month rolling volatility, Expected Shortfall, and Conditional Expected Drawdown are shown in Table 2. The correlations are substantially higher for US Equity across all three risk measures. Note that for both assets, the correlation with the autoregressive parameter is highest for CED. Figure 8 contains the scatter plots of the time series of estimated κ parameters for US Equity and US Bonds against their time series of their CED.

An increase in serial correlation increases drawdown risk concentrations. We now show how temporal dependence is manifest in the drawdown risk contributions. Figure 9a shows the fractional risk contributions over the entire period 1982–2013 of US Equity to the balanced 60/40 portfolio for three risk measures, volatility, ES, and CED, based on daily data. The fractional contributions of US Equity to volatility and ES are large (over 90%) and close in magnitude. For CED, however, the concentration is less pronounced, which means that the contribution of US Bonds to drawdown risk exceeds its contribution to volatility and shortfall risk. A candidate explanation is temporal dependence: while bonds systematically have lower volatility and shortfall risk than do equities, they do occasionally suffer from extended periods of consecutive losses.

To test this hypothesis, we simulate the returns r_E and r_B to two assets E and B representing equities

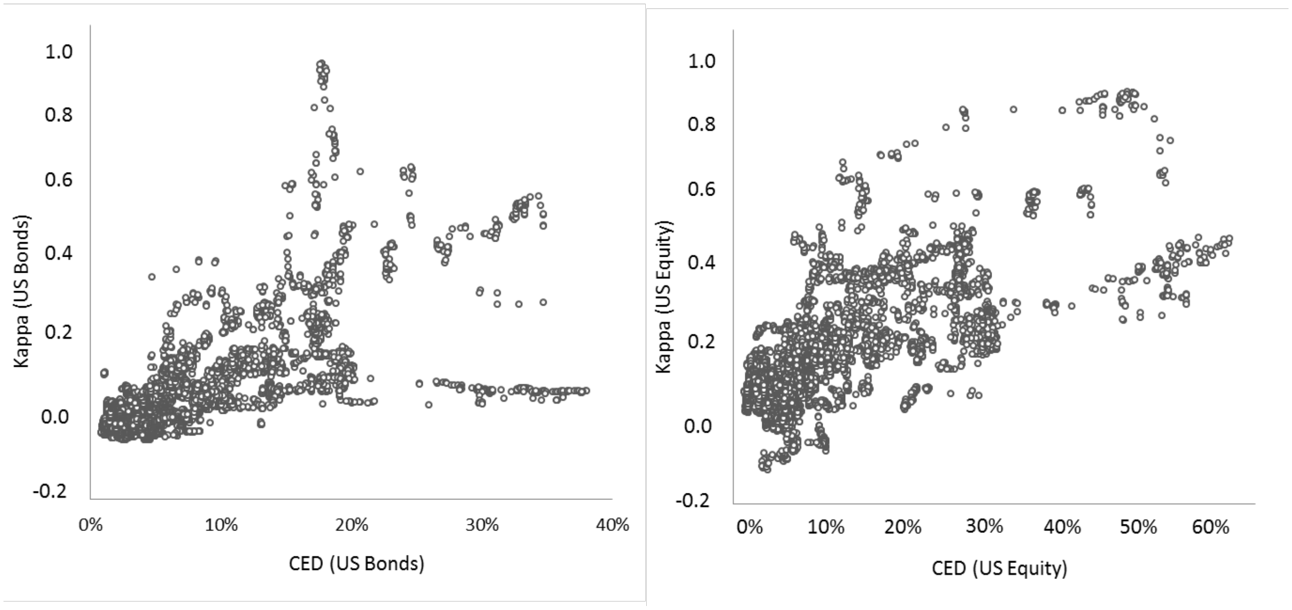


Figure 8: For each of US Equity and US Government Bonds, scatter plots of the daily time series of 6-month rolling estimates of the autoregressive parameter κ with the 6-month rolling estimates of 90% Conditional Expected Drawdown.

and bonds, respectively, with an autoregressive AR(1) model:

$$r_{E,t} = \kappa_E r_{E,t-1} + \epsilon_{E,t},$$

and

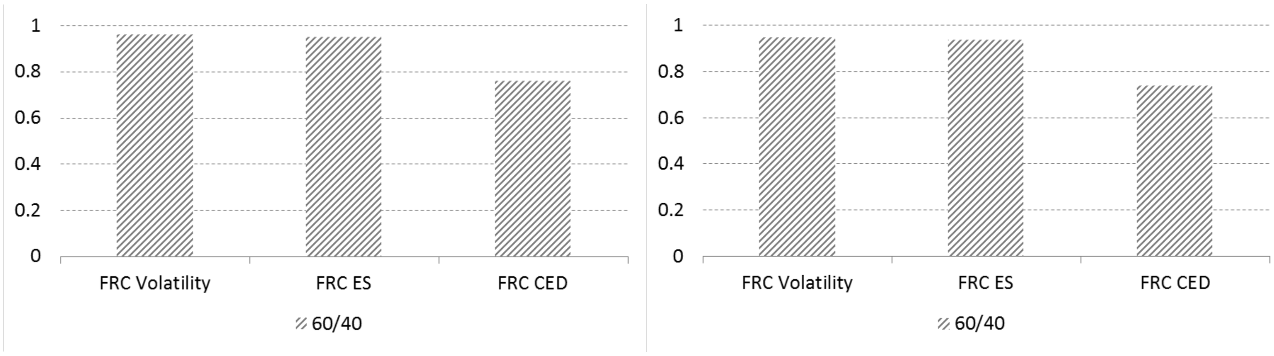
$$r_{B,t} = \kappa_B r_{B,t-1} + \epsilon_{B,t},$$

and we construct a simulated 60/40 fixed-mix portfolio. The AR(1) model parameters are obtained by calibrating to daily time series of US Equity and US Bonds. The estimated autoregressive parameters are $\kappa_E = 0.43$ and $\kappa_B = 0.35$. We assume the ϵ variable is Gaussian, with volatility of 18.4% for asset E (based on the volatility of US Equity) and 5.5% for asset B, (based on the volatility of US Bonds). From the simulated data, we fit AR(1) models and their fractional contributions to volatility, ES and CED. When using only the residuals, we obtain statistically equal risk contributions since the innovations are Gaussian. However, without removing the autoregressive component, contributions to CED once again differ from contributions to volatility and ES. Figure 9b displays the corresponding fractional contributions of the more volatile asset, E, to the three risk measures. Note that the two panels in Figure 9 are visually indistinguishable even though one is based on historical data, whereas the other is simulated.

In Goldberg and Mahmoud (2014), these simulations are expanded further and the effects of serial correlation on various risk measures is studied more systematically.

7 Drawdown: From Practice to Theory and Back Again

Financial practitioners rely on maximum drawdown as an indicator of investment risk. However, due to its inherent path dependency, drawdown has tended to fall outside of probabilistic treatments of



(a) (b)

Figure 9: (a) Fractional contributions over the entire period 1982–2013 of the US Equity asset to volatility, 90% Expected Shortfall and 90% Conditional Expected Drawdown in the 60/40 portfolio, based on daily data. (b) Fractional contributions of the simulated high-volatility AR(1) asset to volatility, 90% Expected Shortfall and 90% Conditional Expected Drawdown in the 60/40 portfolio.

investment risk, which focus on return and loss distributions at fixed horizons. As a result, drawdown has been excluded from standard portfolio analysis toolkits that attribute risk to factors or asset classes, and that use risk forecasts as counterweights to expected return in portfolio construction routines.

In this article, we develop a new probabilistic measure of drawdown risk, Conditional Expected Drawdown (CED), which is the tail-mean of a drawdown distribution at a fixed horizon. Since CED is perfectly analogous to the familiar return-based risk measure, Expected Shortfall (ES), CED is easy for practitioners to interpret and it enjoys desirable theoretical properties of tail-means such as positive degree-one homogeneity and convexity. Thus, the development of a consistent theory for drawdown facilitates an extension of its current practical applications.

The path dependency of Conditional Expected Drawdown makes it more sensitive to serial correlation than Expected Shortfall or volatility. We demonstrate this using a simulated AR(1) model. All else equal, CED increases much more rapidly as a function of the autoregressive parameter κ than do Expected Shortfall or volatility. In an empirical study, we find relatively high correlations between serial correlation and estimated CED (.75 for US Equity, .69 for US Bonds) compared to Expected Shortfall (.52 for US Equity, .39 for US Bonds) and volatility (.47 for US Equity, .32 for US Bonds).

Since it is positive degree-one homogenous, CED (like ES and volatility) can be decomposed into a sum of risk contributions, and the relative sensitivity of CED to serial correlation is manifest in risk concentrations. In an empirical study of a balanced 60/40 portfolio of US Equity and US Bonds over the period 1982–2013, US Equity accounted for roughly 75% of CED, but more than 90% of ES and volatility. A plausible explanation is the relatively high level of serial correlation in US Bonds. We support this hypothesis with another simulation: we replicate the empirically observed concentrations of CED, ES and volatility using a simulated 60/40 balanced portfolio based on AR(1) models calibrate to US Equity and US Bonds over the study period.

Since CED is convex, it can serve as a counterweight to expected return in a quantitative optimization. Exploiting the parallels between Expected Shortfall as a tail-mean of a return distribution and Conditional Expected Drawdown as a tail-mean of a drawdown distribution, we formulate optimization against CED as a linear program using the ideas developed in Rockafellar and Uryasev (2000, 2002).

This article lays the foundation needed to incorporate Conditional Expected Drawdown in the investment process. Further empirical exploration of the properties of CED, research into the incremental information it adds beyond what is in return-based risk measures, and the study of its impact on quantitative portfolio construction, are the next steps.

A Data and Estimation Methodologies

A.1 Data

The data were obtained from the Global Financial Data database (`www.globalfinancialdata.com`). We took the daily time series for the S&P 500 Index and the USA 10-year Government Bond Total Return Index.

A.2 Portfolio Construction

Rather than provide thorough realistic empirical analyses of portfolio risk and return, our goal behind the simulated portfolios is to illustrate this article's theoretical development in relation to drawdown risk. For simplicity, we therefore do not account for transaction costs or market frictions in all hypothetical portfolios constructed throughout this study. Moreover, we assume that all portfolios are fully invested and long only.

Fixed-mix portfolios. In the fixed-mix portfolios, rebalancing to the fixed weights is done on a monthly basis. When comparing to other popular rebalancing schemes (quarterly, bi-annually and yearly), similar results were obtained.

Risk parity portfolios. In risk parity strategies, assets are weighted so their ex post risk contributions are equal. As mentioned in Section 5, parity portfolios are not restricted to volatility only, but can be constructed along other risk measures, such as Expected Shortfall and Conditional Expected Drawdown. Asset weights in the strategies depend on estimates of the underlying risk measures (see Section A.3), which are calculated using a 3-year rolling window of trailing returns. Varying the estimation methodology by changing the length of the rolling window or the weighting scheme applied to the returns within this window did not alter our results substantially. Similar to the fixed-mix portfolios, risk parity portfolios are rebalanced monthly, with other rebelancing schemes yielding similar results.

A.3 Risk Estimation

Volatility. Portfolio volatility is calculated as the annualized standard deviation of the daily time series over the entire period under consideration. To obtain the volatility risk contributions for a n-asset portfolio $P = \sum_i w_i X_i$, note that the i-th total contribution RC_i^{σ} to portfolio volatility

$$\sigma(P) = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_{i,j}$$

is

$$\mathrm{RC}_i^{\sigma} = w_i^2\sigma_i^2 + \sum_{j\neq i} w_i w_j \sigma_{i,j},$$

where σ_i^2 is the variance of X_i and $\sigma_{i,j}$ is the covariance of X_i and X_j. Then, the i-th fractional contribution to volatility is given by

$$\mathrm{FRC}_i^{\sigma}(P) = \frac{w_i^2\sigma_i^2 + \sum_{j\neq i} w_i w_j \sigma_{i,j}}{\sigma(P)} \quad .$$

Expected Shortfall. For confidence level $\alpha \in (0,1)$, an estimate for the Expected Shortfall of a portfolio is calculated by ordering the daily return time series over the whole period according to the magnitude of the returns, then averaging over the worst $(1-\alpha)$ percent outcomes, more specifically:

$$\widehat{\mathrm{ES}}_{\alpha} = \frac{1}{K}\sum_{i=1}^{K} r_{(i)},$$

where T is the length of the daily time series, $K = \lfloor T(1-\alpha)\rfloor$, and $r_{(i)}$ is the i-th return of the *magnitude-ordered* time series. To obtain the contributions to shortfall risk, recall that under a continuity assumption, the Expected Shortfall of an asset $X \in \mathcal{M}$ can be expressed as $\mathrm{ES}_{\alpha}(X) = \mathbb{E}\,(X \mid X \geq \mathrm{VaR}_{\alpha}(X))$, or the expected loss in the event that its Value-at-Risk at α is exceeded.[25] As usual, let $P = \sum_i w_i X_i$ be the portfolio in consideration. Assuming differentiability of the risk measure VaR, the marginal contribution of X_i to portfolio shortfall $\mathrm{ES}_{\alpha}(P)$ is given by

$$\mathrm{MRC}_i^{\mathrm{ES}_{\alpha}}(P) = \frac{\partial \mathrm{ES}_{\alpha}(P)}{\partial w_i} = \mathbb{E}(X_i \mid P \geq \mathrm{VaR}_{\alpha}(P)) \quad .$$

An estimate for the i-th marginal contribution to shortfall risk is then obtained by averaging over all the returns of asset X_i that coincide with portfolio returns exceeding the portfolio's Value-at-Risk at threshold α.

Conditional Expected Drawdown. The first step in calculating an estimate for the Conditional Expected Drawdown is to obtain the empirical maximum drawdown distribution. From the historical time series of returns, we generate return paths of fixed length n using a one-day rolling window. This means that consecutive paths overlap. The advantage is that for a return time series of length T, we obtain a maximum drawdown series of length $T-n$, which for large T and small n is fairly large, too. From these $T-n$ return paths we calculate the maximum drawdown as defined in Section 2. An estimate for the Conditional Expected Drawdown at confidence level $\alpha \in (0,1)$ is then calculated as the average of the largest $(1-\alpha)$ percent maximum drawdowns. To obtain an estimate for the i-th contribution to drawdown risk CED, we take the average over all the drawdowns of the i-th asset in the path $[t_{j*}, t_{k*}]$ that coincide with the overall portfolio's maximum drawdowns that exceed the portfolio's drawdown threshold DT_{α} at confidence level α. (Recall that $j^* < k^* \leq n$ are such that $\mu(P_{T_n}) = P_{t_{k*}} - P_{t_{j*}}$.)

[25] See for example McNeil et al. (2005).

B Drawdown risk decomposition along a balanced portfolio of US Equity and US Treasury Bonds

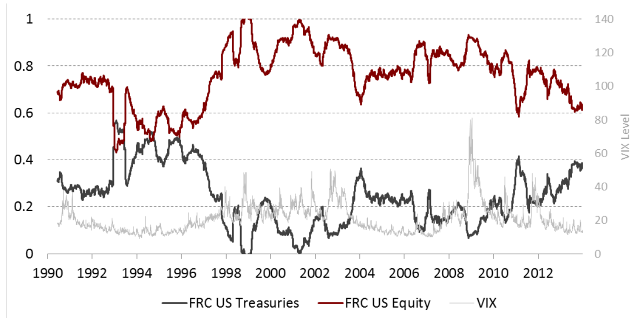


Figure 10: Daily 6-month rolling Fractional Risk Contributions (FRC) along 90% Conditional Expected Drawdown (CED) of US Equity and US Treasury Bonds to the balanced 60/40 portfolio. Also displayed is the daily VIX series over the same period, with the right-hand axis indicating its level.

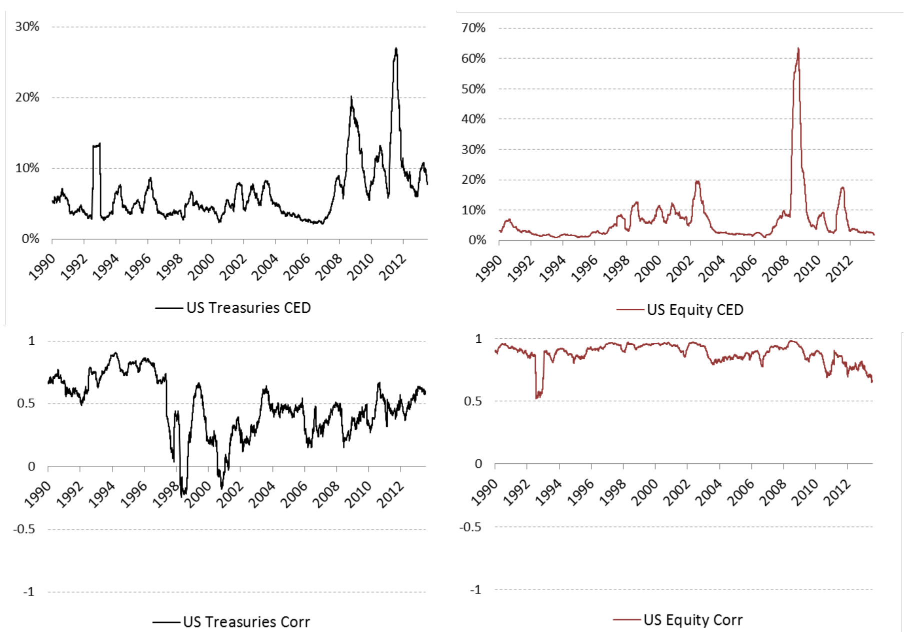
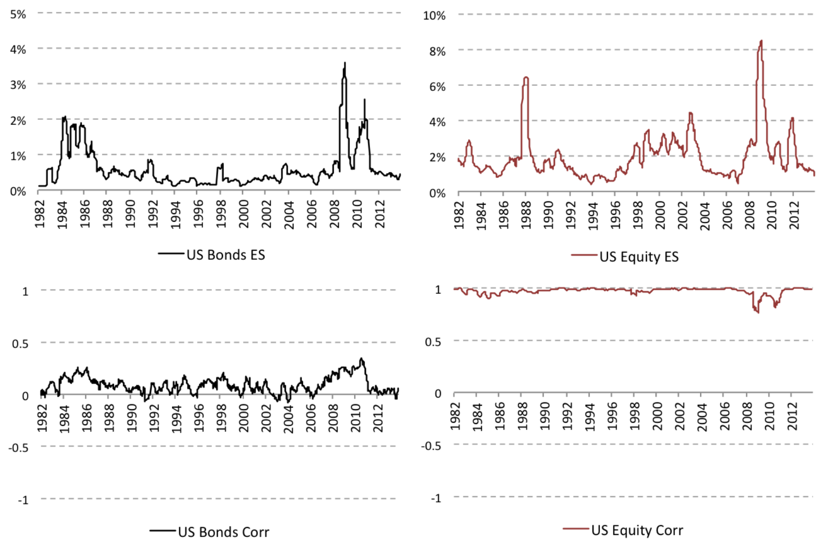
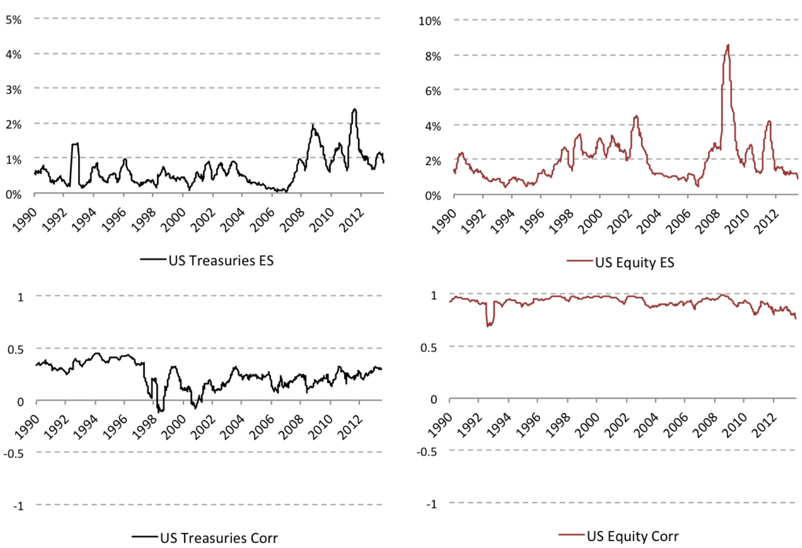


Figure 11: Decomposition of drawdown risk contributions $\mathrm{RC}_i^{\mathrm{CED}}(P) = w_i \mathrm{CED}(X_i)\mathrm{Corr}_{\mathrm{i}}^{\mathrm{CED}}$ for the 60/40 allocation to US Equity and US Treasury Bonds. The top two panels show the daily 6-month rolling standalone 90% Conditional Expected Drawdown (CED) of the two assets, while the bottom two panels show the 6-month rolling generalized correlations of the individual assets along CED.

C Risk decomposition along Expected Shortfall



(a)



(b)

Figure 12: Decomposition of contributions $\mathrm{RC}_i^{\mathrm{ES}}(P) = w_i \mathrm{ES}(X_i)\mathrm{Corr}_{\mathrm{i}}^{\mathrm{ES}}$ to 90% Expected Shortfall (ES) for the 60/40 allocation to (a) US Equity and US Government Bonds, and (b) US Equity and US Treasury Bonds.

References

Acerbi, C. and Tasche, D. (2002a). Expected shortfall: a natural coherent alternative to value at risk. *Economic Notes*, 31(2):379–388.

Acerbi, C. and Tasche, D. (2002b). On the coherence of expected shortfall. *Journal of Banking and Finance*, 26(7):1487–1503.

Anderson, R. M., Bianchi, S. W., and Goldberg, L. R. (2012). Will my risk parity strategy outperform? *Financial Analysts Journal*, 68(6):75–93.

Anderson, R. M., Bianchi, S. W., and Goldberg, L. R. (2014). Determinants of levered portfolio performance. Forthcoming in *Financial Analysts Journal*.

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3):203–228.

Bertsimas, D., Lauprete, G. J., and Samarov, A. (2004). Shortfall as a risk measure: Properties, optimization and applications. *Journal of Economic Dynamics Control*, 28:1353–1381.

Burghardt, G., Duncan, R., and Liu, L. (2003). Deciphering drawdown. *Risk Magazine*.

Chekhlov, A., Uryasev, S., and Zabarankin, M. (2003). Portfolio optimization with drawdown constraints. In Scherer, B., editor, *Asset and Liability Management Tools*, pages 263–278. Risk Books, London.

Chekhlov, A., Uryasev, S., and Zabarankin, M. (2005). Drawdown measure in portfolio optimization. *International Journal of Theoretical and Applied Finance*, 8(1):13–58.

Dantzig, G. B. and Thapa, M. N. (1997). *Linear Programming 1: Introduction*. Springer Verlag.

Dantzig, G. B. and Thapa, M. N. (2003). *Linear Programming 2: Theory and Extensions*. Springer Verlag.

Denault, M. (2001). Coherent allocation of risk capital. *Journal of Risk*, 4(1):7–21.

Föllmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, 6:429–447.

Föllmer, H. and Schied, A. (2010). Coherent and convex risk measures. In Cont, R., editor, *Encyclopedia of Quantitative Finance*, pages 355–363. John Wiley and Sons.

Föllmer, H. and Schied, A. (2011). *Stochastic Finance: An Introduction in Discrete Time*. Walter de Gruyter, Berlin.

Goldberg, L. R., Hayes, M. Y., and Mahmoud, O. (2013). Minimizing shortfall. *Quantitative Finance*, 13(10):1533–1545.

Goldberg, L. R. and Mahmoud, O. (2014). Risk measures and serial correlation. Working Paper, University of California at Berkeley.

Goldberg, L. R., Menchero, J., Hayes, M., and Mitra, I. (2010). Extreme risk analysis. *The Journal of Performance Measurement*, 14(3).

Kalkbrener, M. (2005). An axiomatic approach to capital allocation. *Mathematical Finance*, 15(3):425–437.

Markowitz, H. M. (1952). Portfolio selection. *The Journal of Finance*, 7(1):77–91.

McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton University Press.

Menchero, J. and Poduri, V. (2008). Custom factor attribution. *Financial Analysts Journal*, 64(2):81–92.

Qian, E. (2006). On the financial interpretation of risk contributions: Risk budgets do add up. *Journal of Investment Management*, 4(4):41–51.

Rockafellar, R. T. and Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk*, 2:21–41.

Rockafellar, R. T. and Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking and Finance*, 26:1443–1471.

Rockafellar, R. T., Uryasev, S. P., and Zabarankin, M. (2002). Deviation measures in risk analysis and optimization. Technical report, University of Florida, Department of Industrial and Systems Engineering.

Rockafellar, R. T., Uryasev, S. P., and Zabarankin, M. (2006). Generalized deviations in risk analysis. *Finance and Stochastics*, 10(1):51–74.

Tasche, D. (2000). Risk contributions and performance measurement. Technical report, Research paper, Zentrum Mathematik (SCA).