Does the Leverage Effect Affect the Distribution of Return?

Dangxing Chen

Consortium for Data Analytics in Risk
UC Berkeley

September 3, 2019

Interpretations of the leverage effect

- The leverage effect refers to the observed tendency of an asset's volatility to be negatively correlated with the asset's return.
- Original possible interpretations by Black (1976):
  - change of return $\rightarrow$ change of volatility: financial leverage, operating leverage
  - change of volatility $\rightarrow$ change of return: volatility feedback effect
- Related works in these interpretations, see e.g., Christie (1982); Figlewski and Wang (2000); French, Schwert, and Stambaugh (1987); Campbell and Hentschel (1992)

Estimation of the leverage effect

- There are debates about the sources or explanations for the presence of the leverage effect correlation. But, there is broad agreement in the literature that the effect should be present.
- Various discrete-time models with a leverage effect has been estimated by Yu (2005).
- High frequency nonparametric estimation by Bandi and Reno (2012), Wang and Mykland (2009), and Ait-Sahalia, Fan, and Li (2013).

Does the leverage effect affect the distribution of return?

- Silva, Yakovenko (2002): fit three major stock-market indices (Nasdaq, S&P 500, and Dow-Jones). Fitting is **insensitive** to the leverage effect.
- Chorro, Guégan, Ielpo, Lalaharison (2016): they consistently find a **weak** contribution of leverage effects over the past 25 years of S&P 500 returns.
- **Question**: Does the leverage effect affect the distribution of the return?

Stochastic Volatility Model

- We assume the price S_t of a security follows a Stochastic Differential Equation (SDE) of the form

$$dS_t = rS_t \, dt + \sqrt{V_t} S_t \, d\widetilde{B}_t,$$
$$dV_t = \mu(V_t) \, dt + \sigma(V_t) \, dW_t,$$

  where $dW_t d\widetilde{B}_t = \rho \, dt, \rho < 0$.
- The leverage effect in this framework refers to ρ.
- Assumptions:
  - Stationarity:

$$\lim_{t \to \infty} f(V_t | V_0) = f(V).$$

  - Ergodicity: The statistical properties of the time series can be deduced from a single, sufficiently long, random sample of the process.

Marginal distribution of the log return

- Gram-Schmidt:

$$dS_t = rS_t\, dt + \rho\sqrt{V_t}\, dW_t + \sqrt{1-\rho^2}\sqrt{V_t}\, dB_t,$$

where $B_t \perp W_t$.

- Log return $X_t = \ln(S_t)$:

$$dX_t = \left(r - \frac{V_t}{2}\right)\, dt + \rho\sqrt{V_t}\, dW_t + \sqrt{1-\rho^2}\sqrt{V_t}\, dB_t.$$

- Marginal distribution: distribution of the $X_{t+\Delta t} - X_t$.
- Δt can be days, weeks, months ...

Marginal distribution of the log return - continued

- Temporal direction:

$$
\begin{array}{ccccccccc}
X_0 & \rightarrow & X_{\Delta t} & \rightarrow & X_{2\Delta t} & \rightarrow & \dots & \rightarrow & X_{N\Delta t} \\
 & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\
V_0 & \rightarrow & V_{\Delta t} & \rightarrow & V_{2\Delta t} & \rightarrow & \dots & \rightarrow & V_{N\Delta t}
\end{array}
$$

- Spatial direction:

$$
\begin{array}{cccccccccc}
X_0^{(1)} & \rightarrow & X_{\Delta t}^{(1)}, & X_0^{(2)} & \rightarrow & X_{\Delta t}^{(2)}, & \dots, & X_0^{(M)} & \rightarrow & X_{\Delta t}^{(M)} \\
 & \nearrow & & & \nearrow & & & & \nearrow & \\
V_0^{(1)} & \sim & V, & V_0^{(2)} & \sim & V, & \dots, & V_0^{(M)} & \sim & V
\end{array}
$$

- Under our assumptions, two data generating process gives the same distribution,

$$
\text{(temporal) } f(X_{t+\Delta t} - X_t) \sim f(X_{\Delta t} | V_0 \sim V) \text{ (spatial).}
$$

Heston model - example

- Heston model:

$$dX_t = \left(r - \frac{1}{2}V_t\right)\, dt + \rho\sqrt{V_t}\, dW_t + \sqrt{1-\rho^2}\sqrt{V_t}\, dB_t,$$
$$dV_t = \kappa(\theta - V_t)\, dt + \sigma\sqrt{V_t}\, dW_t. \qquad \text{(CIR process)}$$

- Stationary distribution of the variance:

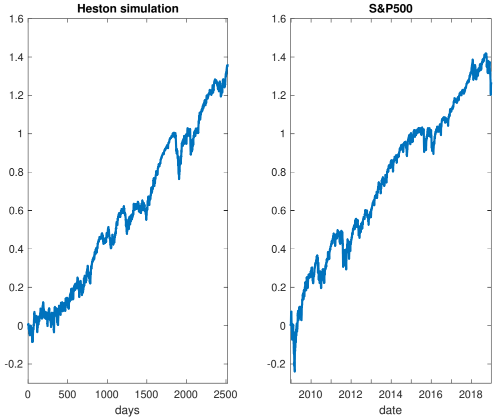
$$\lim_{t\to\infty} V_t | V_0 \sim \frac{\sigma^2}{4\kappa}\Gamma\left(\frac{4\kappa\theta}{\sigma^2}\right)$$

- Fourier transform of X_t can be explicitly calculated from the reference: Drăgulescu, Adrian A., and Victor M. Yakovenko. "Probability distribution of returns in the Heston model with stochastic volatility." Quantitative Finance 2.6 (2002): 443-453.

Heston model example - continued

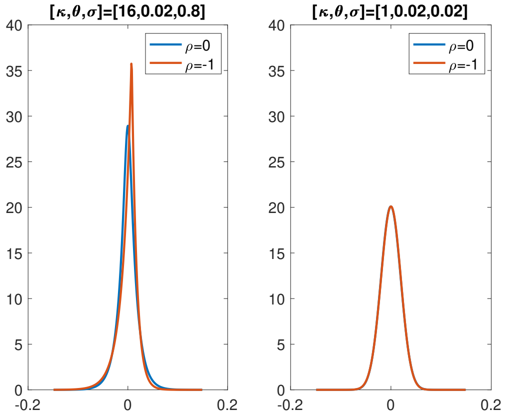
Parameters: $[\kappa, \theta, \sigma, \rho] = [1, 0.02, 0.2, -1]$.

Figure: The sample trajectory of the log return in Heston model and its comparison with the S&P 500 data



Impact of the leverage effect

Figure: Marginal return distribution of the Heston model at 5-day horizon



Question: Why is the impact of the leverage effect relatively weak in the second example?

What affects the impact strength of the leverage effect?

- Asymptotic cases:
  - $\rho \to 0$: no leverage effect
  - $V_t \perp W_t$: the mean-reversion effect of variance is very strong

  For both cases, the marginal distribution of the log return becomes mixed Gaussian:

$$X_t \Big| \int_0^t V_s \, ds \sim \mathcal{N}\left(rt - \frac{1}{2}\int_0^t V_s \, ds, \int_0^t V_s \, ds\right).$$

- Idea: Incorporate the mean-reversion effect into the leverage effect
- The interaction effect: The mixture of the leverage effect and the mean-reversion effect
- The variance is latent and the Brownian motion that drives it is not observable. Hence, we need an alternative indirect way to measure it.

Decomposition of the marginal variance

- Marginal variance:

$$\begin{aligned}\mathsf{Var}[X_t] &= \mathbb{E}\left[\int_0^t V_s\ ds\right] + \frac{1}{4}\mathsf{Var}\left[\int_0^t V_s\ ds\right] \\ &- \rho\mathbb{E}\left[\left(\int_0^t V_s\ ds\right)\left(\int_0^t \sqrt{V_s}\ dW_s\right)\right]. \\ &= \mathsf{EIV}_t + \mathsf{VIV}_t + \mathbf{EMIV}_t.\end{aligned}$$

- $\mathsf{EIV}_t = \frac{t}{s}\mathsf{EIV}_s$
- $\mathsf{VIV}_t \ll \mathsf{EIV}_t$
- $\mathbf{EMIV}_t$ comes from the contribution from the interaction effect.

Heston model

- SDEs:

$$dX_t = \left(r - \frac{1}{2}V_t\right)\ dt + \rho\sqrt{V_t}\ dW_t + \sqrt{1-\rho^2}\sqrt{V_t}\ dB_t,$$
$$dV_t = \kappa(\theta - V_t)\ dt + \sigma\sqrt{V_t}\ dW_t.$$

- Marginal moments:

$$\mathsf{EIV}_t = \theta t,$$
$$\mathsf{VIV}_t = \theta\frac{\sigma^2}{\kappa^2}\left[t + \frac{e^{-\kappa t}-1}{\kappa}\right],$$
$$\mathsf{EMIV}_t = -\theta\frac{\rho\sigma}{\kappa}\left[t + \frac{e^{-\kappa t}-1}{\kappa}\right].$$

Asymptotic of moments by Heston model at short horizon

- For $t \to 0$,

$$\begin{aligned}
\mathsf{EIV}_t &\to \theta t, \\
\mathsf{VIV}_t &\to \theta \frac{\sigma^2}{2\kappa} t^2, \\
\mathsf{EMIV}_t &\to -\theta \frac{\rho\sigma}{2} t^2.
\end{aligned}$$

- The EMIV_t is not observable at short horizon.
- At short horizon, we observe EIV_t. Then automatically we know EIV_t at any horizons.

Asymptotic of moments by Heston model at longer horizon

- For $t \to \infty$,

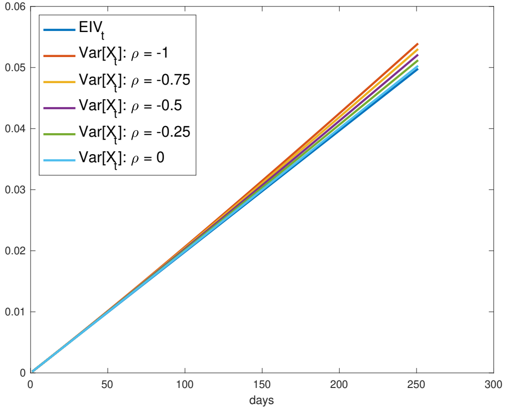
$$\begin{aligned}
\mathsf{EIV}_t &\to \theta t, \\
\mathsf{VIV}_t &\to \theta t \frac{\sigma^2}{\kappa^2}, \\
\mathsf{EMIV}_t &\to -\theta t \frac{\rho\sigma}{\kappa}.
\end{aligned}$$

- The EMIV_t is observed at the long horizon.
- In Heston model, the constant $\frac{\sigma}{\kappa}$ serves as the mean-reversion factor.

Heston model example

Parameters: $[\kappa, \theta, \sigma] = [1, 0.05, 0.2]$.

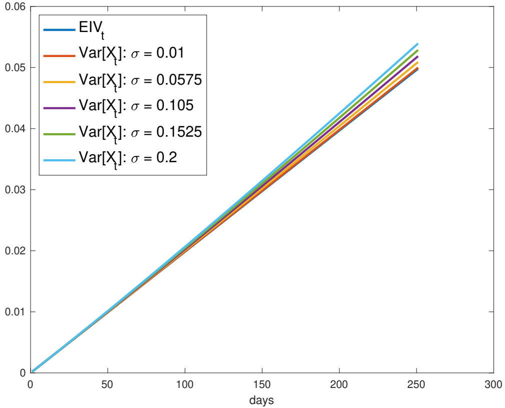
Figure: The comparison of the Var$[X_t]$ with EIV$_t$ for different ρ



Heston model example

Parameters: $[\kappa, \theta, \rho] = [1, 0.05, -1]$.

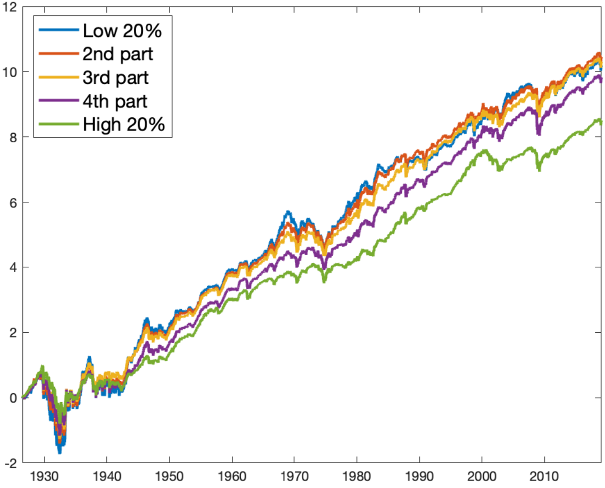
Figure: The comparison of the Var$[X_t]$ with EIV$_t$ for different σ



Portfolios formed by the size

- Criteria: The portfolios are constructed by the Prof. French. They are formed depending on the size (market equity), price times shares outstanding. It is evenly split into 5 parts.
- Include all NYSE, AMEX, and NASDAQ stocks
- Period: 1926 July - 2019 January.
- Frequency: daily.

Size portfolios' trajectories



The interaction effect for the portfolios constructed by size

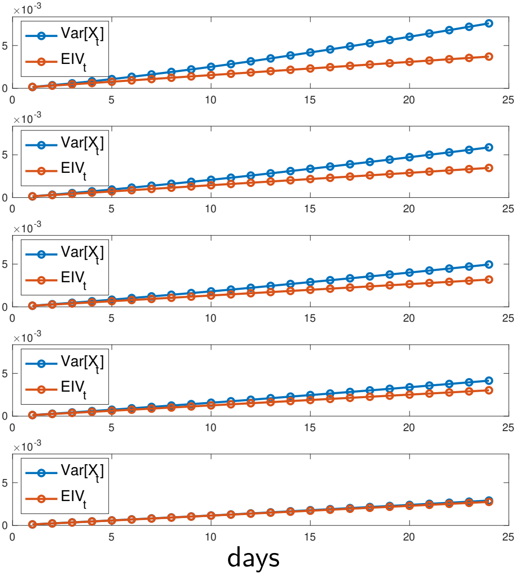
Low 20%

2nd part

3rd part

4th part

High 20%



Summary statistics

- Notation:

$$\mathsf{RIV}_t = \frac{\mathsf{Var}[X_t] - \mathsf{EIV}_t}{\mathsf{Var}[X_t]}.$$

- Summary statistics:

Portfolios	EIV_{25}	RIV_{25}
$0\% - 20\%$	0.037	0.51
$20\% - 40\%$	0.035	0.41
$40\% - 60\%$	0.032	0.36
$60\% - 80\%$	0.030	0.27
$80\% - 100\%$	0.028	0.053

- Observation: The interaction effect is stronger for small firms than large firms.

Convergence of return distribution to long-run Gaussian distribution

- Check the mean-reversion effect alone
- Under some mild assumptions, can show that

$$\lim_{t\to\infty} \frac{X_t - \mathbb{E}[X_t]}{\sqrt{\mathrm{Var}[X_t]}} \xrightarrow{d} \mathcal{N}(0,1).$$

- Compare the centralized and scaled return distribution with the standard Gaussian distribution

$$e_t = \left\| f\left(\frac{X_t - \mathbb{E}[X_t]}{\sqrt{\mathrm{Var}[X_t]}}\right) - f\left(\mathcal{N}(0,1)\right) \right\|_2 .$$

Convergence of return distributions for size portfolios

- Summary statistics of e_t:

Time horizon	Large firms	Small firms
1-day	0.052	0.080
2-day	0.033	0.063
4-day	0.025	0.047

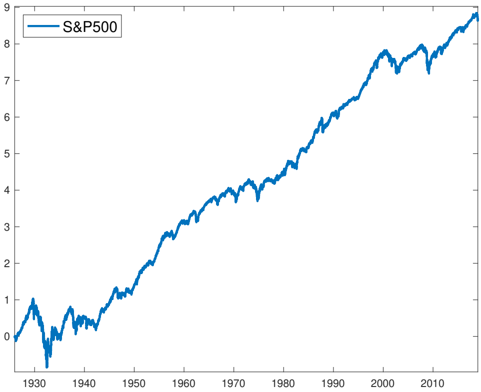
- Observation: the mean-reversion effect is stronger for large firms than for small firms.

The leverage effect in the S&P 500 data

- Reference: Ait-Sahalia, Yacine, Jianqing Fan, and Yingying Li. "The leverage effect puzzle: Disentangling sources of bias at high frequency." Journal of Financial Economics 109.1 (2013): 224-249.
- Data period: 2004 January to 2007 December.
- Frequency: one sample per minute.
- Robust estimation: $\rho = -0.77$.

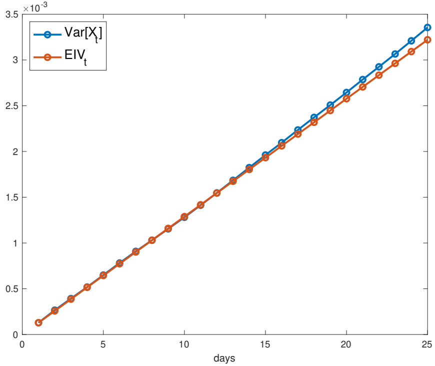
S&P 500 long historical data

- Period: 1926 January to 2018 December.
- Frequency: daily.
- Trajectory:



The interaction effect in the S&P 500 data

- The interaction effect in the S&P 500:



- The interaction effect for S&P 500 is weak.
- How does ρ affect its marginal return distribution?

Experiment to test the impact of the leverage effect on the return distribution

- If there is no leverage effect, then

$$X_t \Big| \int_0^t V_s \, ds \sim \mathcal{N}\left(r - \frac{1}{2}\int_0^t V_s \, ds, \int_0^t V_s \, ds\right).$$

- Assume the marginal density function of $X_{t+\Delta t} - X_t$ without the leverage effect is $g(x)$
- If the impact of the leverage effect is weak, we are able to fit the marginal distribution of return by $g(x)$
- What density function $g(x)$ to use?

Generalized hyperbolic distribution

- The generalized hyperbolic (GH) distribution is very powerful in fitting the empirical distribution, see e.g., Bibby, Sørensen (2003)
- In GH, there is no leverage effect and $\int_0^t V_s \, ds$ follows the generalized inverse Gaussian (GIG) distribution
- The following diffusion process has a stationary distribution following the GIG,

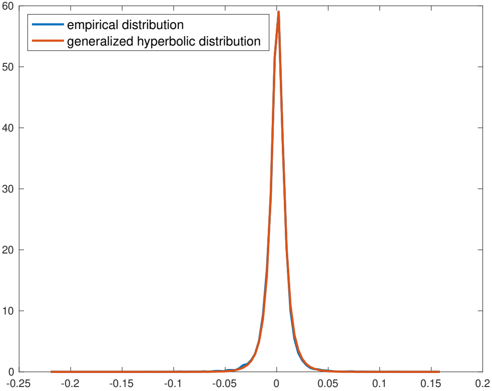
$$dV_t = \left(\beta_1 V_t^{2\alpha-1} - \beta_2 V_t^{2\alpha} + \beta_3 V_t^{2(\alpha-1)}\right) \, dt + \sigma V_t^{\alpha} dW_t$$

  where $\beta_1 = \frac{1}{2}\sigma^2(\lambda-1) + \sigma^2\alpha, \beta_2 = \frac{1}{4}(\sigma\gamma)^2, \beta_3 = \frac{1}{4}(\sigma\delta)^2$.
- Comparison with the CIR process:

$$dV_t = \kappa(\theta - V_t) \, dt + \sigma\sqrt{V_t} \, dW_t.$$

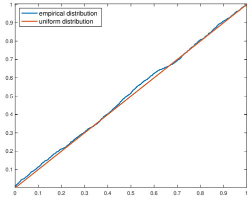
Fitness of the S&P 500 return

Figure: Fitness of the S&P 500 return data by the GH distribution at 1-day horizon



Kolmogorov–Smirnov test

- If the data is generated by the GH with the CDF $G(x)$, then $G(X_{t+\Delta t} - X_t)$ follows the uniform distribution.
- Using daily return with a 1-month distance
  $X_{\Delta t} - X_0, X_{23\Delta t} - X_{22\Delta t}, X_{45\Delta t} - X_{44\Delta t}, \dots$
  so that the data are very weakly dependent
- Fitness of the uniform distribution:



- K-S test: fails to reject the null hypothesis at 5% level
- The leverage effect only has little impact on the marginal return distribution of the S&P 500.

Convergence of the return distribution for S&P 500

- Compare the centralized and scaled return distribution with the standard Gaussian distribution

$$e_t = \left\| f\left(\frac{X_t - \mathbb{E}[X_t]}{\sqrt{\mathsf{Var}[X_t]}}\right) - f\left(\mathcal{N}(0,1)\right) \right\|_2 .$$

- Summary statistics of e_t:

Time horizon	S&P 500	Large firms	Small firms
1-day	0.054	0.052	0.080
2-day	0.033	0.033	0.063
4-day	0.021	0.025	0.047

- Observation: the mean-reversion effect is strong for the S&P 500!

Conclusion

- The leverage effect is important, but it **need not** to have a big impact on the return distribution. A strong leverage effect can be negated by a strong mean-reversion effect.
- When studying the distribution properties, one must consider the interaction effect, the mixture of the leverage effect and the mean-reversion effect.
- The annualized marginal variance of the return will grow over time until it converges for the strong interaction effect, but will barely change for the weak interaction effect.
- The interaction effect is stronger for small firms than large firms.
- Due to the weak interaction effect for the S&P 500, the strong leverage effect has little impact on the return distribution of the S&P 500.

Thank You!

Contact:

Dangxing Chen
dangxing@berkeley.edu