Rough Volatility: Evidence from two different volatility proxies

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The volatility process

- The most classical way to model the price S_t of a security is to use continuous semi-martingale dynamics of the form

$$d \log S_t = \mu_t + \sigma_t dZ_t,$$

with μ_t the drift process and W_t a Brownian motion.
- The coefficient σ_t is referred to as the volatility process.
$\rightarrow$ It is the key ingredient in models for pricing and hedging.
- We face two concerns :
    - Choosing the right estimator given the available data.
    - Considering the volatility dynamic that explains certain stylized facts and empirical behaviors.

Volatility models

The three main classes to model the volatility process are :

- Deterministic volatility :
  - **Pros :** Simplicity ; allows for closed formula in option pricing ; we can take advantage of the robustness of the Black-Scholes formula.
  - **Cons :** Not realistic at all.
- Local volatility [Dupire, 1994, Derman and Kani, 1994] :
  - **Pros :** Perfectly reproduce a given implied volatility surface.
  - **Cons :** The dynamic of such implied volatility surface is unrealistic.
- Stochastic volatility [Hull and White, 1987, Heston, 1993]...
  - **Pros :** A suitable dynamic of the volatility surface.
  - **Cons :** A less accurate static fits.

Solutions

- Local-stochastic volatility models.
- Introducing more volatility factors : double mean-reverting, Bergomi.
- **Rough volatility.**

Empirical Evidence from High Frequency Data I

- In stochastic volatility models, the smoothness of the sample path of the volatility is the same as that of a Brownian motion, i.e., α-Holder continuous with *alpha* $<$ H $=$ 1/2.
- Are stochastic volatility models supported by empirical data ?
- Can we estimate H from data ?
- The first challenging problem to perform this analysis is how to define a proxy for the volatility process starting from security prices, since cannot directly observe volatility.

Empirical Evidence from High Frequency Data II

- A possible choice is a statistical analysis of price time series, see for instance the pioneering paper by [Comte and Renault, 1998].
- The availability of good high-frequency data for security prices is required for an accurate reconstruction of the daily realized volatility.
- The original work by [Gatheral et al., 2014] and later the paper of [Bennedsen et al., 2016] made an extensive analysis starting from high-frequency price data, obtaining a value of $H < 1/2$.

Motivating the rough volatility model

- Empirically, the term structure of ATM skew is found to be proportional to $1/T^{\alpha}$ where $0 < \alpha < 1/2$ over a very wide range of expirations.
- Stochastic volatility models exhibit a constant ATM volatility skew for short dates and proportional to $1/T$ for long ones.
- In conventional models, the smoothness of the sample path of the volatility is the same as that of a Brownian motion, i.e. $1/2 - \varepsilon$ Holder continuous, $\forall \varepsilon > 0$.
- Are these model consistent with other volatility proxies ? Is the behavior the same when using option prices to deduce H ?

Measure of the regularity of the log-volatility

- We wish to verify the scaling of the volatility when taking other proxies.
- The quantity of interest is the scaling measure $m(q,\Delta)$ defined by :

$$m(q,\Delta) := \frac{1}{N} \sum_{k=0}^{\lfloor (N-1)/\Delta \rfloor} |\log(\hat{\sigma}_{t_{k+\Delta}}) - \log(\hat{\sigma}_{t_k})|^q$$

- $m(q,\Delta)$ is the discrete equivalent of $\mathbb{E}[|\log(\hat{\sigma}_\Delta) - \log(\hat{\sigma}_0)|^q]$ which is expected to verify

$$\mathbb{E}[|\log(\hat{\sigma}_\Delta) - \log(\hat{\sigma}_0)|^q] = K_q \Delta^{\zeta_q}, \tag{1}$$

- If the process is monofractal, we expect $\zeta_q = H \times q$, where H is the Hurst exponent.

Reminder

[Gatheral et al., 2014] have found H to be around 0.14.

Volatility proxies

- The ideal proxy is high frequency realized volatility !
  - **Pros :** Allows to estimate the daily volatility with enough historical data.
  - **Cons :** High frequency data is usually expensive, not available for all class of assets,
- Range-based volatility based on high, low, open and close asset price data.
  - **Pros :** Can be computed for a large class of assets (almost all quoted prices) ; not affected by microstructure noise ; easy to compute.
  - **Cons :** Constructed upon lognormal distribution of the underlying asset and continuous trading ; true ranges are unobserved.
- Option-based estimators, i.e. short time-to-maturity implied volatility or asymptotic formulas based on it.
  - **Pros :** More meaningful to practitioners ; translates the agents risk aversion in the options market.
  - **Cons :** Short-term options are not available for all dates, are noisy and need to be corrected.

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Range-based volatility I

- We carry out the analysis using as proxy Garman-Klass and Parkinson estimators for "exotic" indexes and stocks.

Reminder

Garman-Klass and Parkison volatility proxies are given by :

$$\sigma_{GK}^2 = \frac{1}{2}\left(\log\frac{H_t}{L_t}\right)^2 - (2\log 2 - 1)\left(\log\frac{C_t}{O_t}\right)^2,$$

and

$$\sigma_P^2 = \frac{1}{4\log 2}\left(\log\frac{H_t}{L_t}\right)^2,$$

where H_t, L_t, O_t, C_t are respectively high, low, open and close price data.

Range-based volatility II

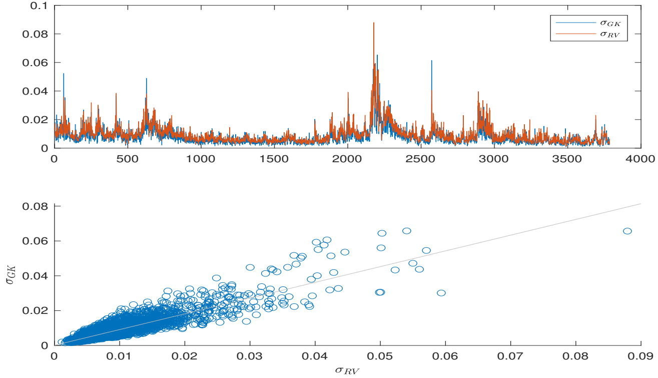


Figure – Top : Garman-Klass and Parkison volatility for S&P 500 in time. Bottom : Time series of Garman-Klass vs. realized volatility from high-frequency price data.

Scaling of $m(q, \Delta)$ and ζ_q

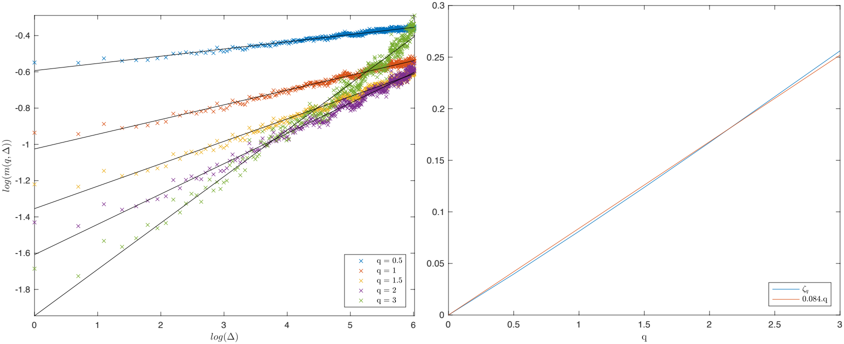


Figure – $\log m(q, \Delta)$ as a function of $\log \Delta$ (left), ζ_q (blue) and $0.075 \times q$ (red) (right), S&P 100 (Garman Klass volatility).

Results using other assets

Ticker	H for the whole period	H (first half)	H (second half)
SP100	0.0841	0.0897	0.0714
IBEX35	0.072	0.0753	0.071
HSI	0.0516	0.0605	0.0394
MEXBOL	0.0627	0.0737	0.0463
FTSE100	0.0751	0.0708	0.0728
ASX200	0.0489	0.0476	0.0415
TOTAL	0.0774	0.0835	0.0687
XIN9I	0.0674	0.0649	0.069
SHSZ300	0.0689	0.0718	0.0636
BCOM	0.014	0.0099	0.0238
INDU	0.0804	0.0838	0.067
USDEUR	0.0353	0.0393	0.0321
IBOV	0.0685	0.0724	0.0609
MICROSOFT	0.06	0.0717	0.0401
GOOGLE	0.0656	0.0724	0.0542
SP400	0.0715	0.0753	0.0592

Table – Estimates of H on the whole period and over two different time intervals for different indexes and stocks (Garman Klass volatility)

Results using the Parkinson estimator

Ticker	H for the whole period	H (first half)	H (second half)
SP100	0.0822	0.0888	0.0737
IBEX35	0.0644	0.0682	0.0648
HSI	0.0452	0.0555	0.0336
MEXBOL	0.0638	0.0738	0.0489
FTSE100	0.0774	0.0823	0.0669
ASX200	0.0513	0.0511	0.0422
TOTAL	0.0738	0.0856	0.0608
XIN9I	0.0595	0.0592	0.0593
SHSZ300	0.0591	0.0668	0.0499
BCOM	0.0127	0.00623	0.0251
INDU	0.08	0.08	0.0707
USDEUR	0.0265	0.0276	0.0261
IBOV	0.0694	0.0746	0.0603
MICROSOFT	0.0584	0.0685	0.0414
GOOGLE	0.0603	0.063	0.0542
SP400	0.0757	0.0822	0.0623

Table – Estimates of H on the whole period and over two different time intervals for different indexes and stocks (Parkinson volatility)

Distribution of the log-volatility increments

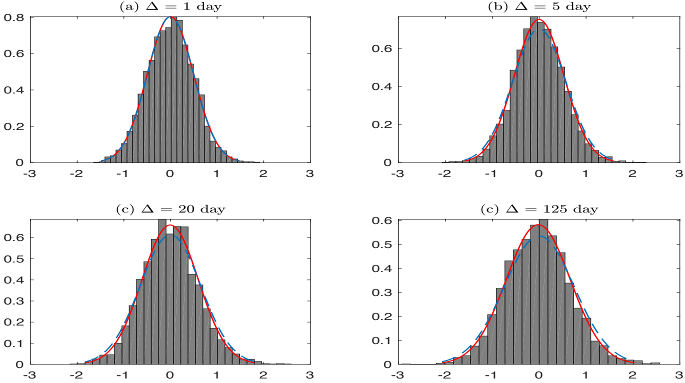


Figure – The distribution of log-volatility increments is close to Gaussian.

Rough volatility as a model for RBV I

- The scaling and distribution of the log-volatility suggest using fractional Brownian motion (fBm) to model the volatility.

Properties of a fBm with Hurst exponent H

- Self-similarity : $W^H_{at} \sim a^H W^H_t$
- Stationary increments : $W^H_t - W^H_s \sim W^H_{t-s}$
- Gaussian process with $\mathbb{E}[W^H_1] = 0$ and $\mathbb{E}[(W^H_1)^2] = 1$

An fBm W^H with Hurst parameter $H \in (0,1)$ is the only Gaussian process to satisfy with mean zero and autocovariance function

$$\mathbb{E}[W^H_t W^H_s] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}$$

Rough volatility as a model for RBV II

- A simple model would be :

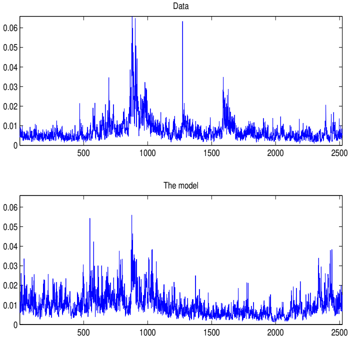
$$\sigma_t = \sigma_0 e^{\nu W_t^H}$$

- To account for stationarity we rely on the mean-reverting frational Ornstein-Uhlenbeck :

$$d\log(\sigma_t) = \alpha(m - \log(\sigma_t))dt + \nu dW_t^H, \quad H < 1/2.$$

- [Comte et al., 2012] presented a model with the same form but with $H > 1/2$ (we will call it fractional stochastic volatility (FSV)).

Model validation



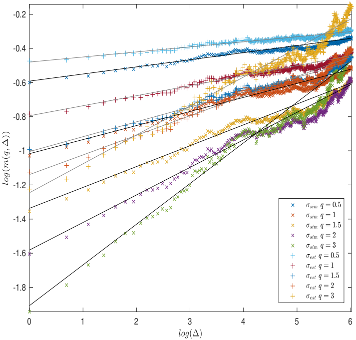


Figure – Garman Klass volatility of S&P 100 (top) and simulated paths (bottom) (left). The scaling of the model and the RBV through simulated range data (right).

FSV *vs.* RFSV - I

- The RFSV model seems to exhibit the desired properties found in the data.
- Why do smooth volatility models or with long memory seem to
- We conduct a simulation analysis to compare FSV and RFSV models.

	FSV	RFSV
H	0.7	0.08
α	0.25	5×10^{-4}
ν	0.25	0.45
m	-4.5	-5
X_0	-4.5	-5
$\mathbb{E}[\log(\sigma)]$	-4.6	-4.7
$\mathbb{V}ar[\log(\sigma)]$	0.21	0.33

FSV *vs.* RFSV - II

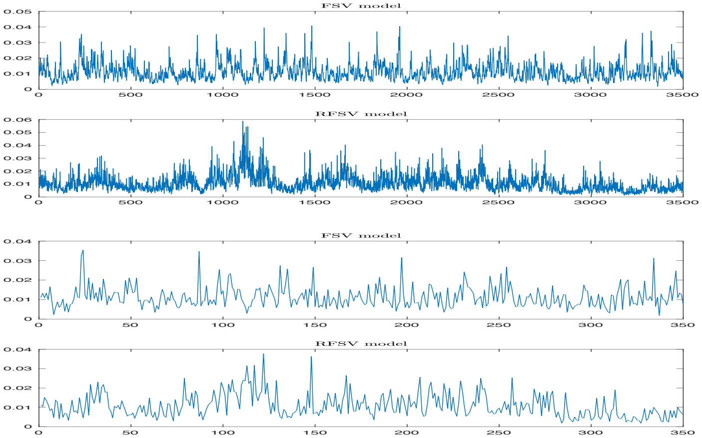


Figure – Plot of one path of the FSV and RFSV models for $H = 0.7$ and $H = 0.08$ observed daily (top two), and every 10 days (bottom two).

FSV *vs.* RFSV - III

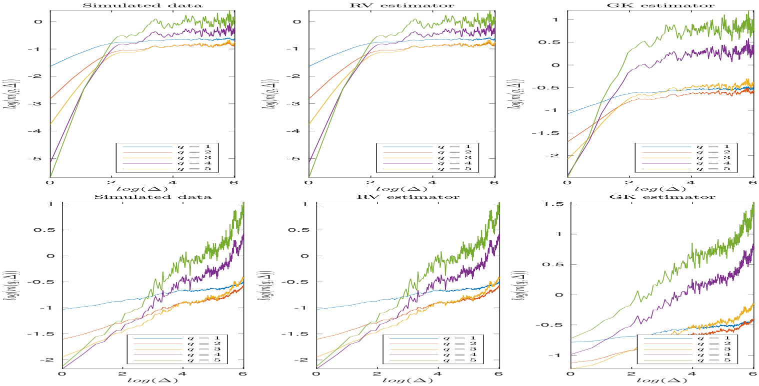


Figure – scaling of $\log(m(q, \Delta))$ for FSV model (top) and RFSV model (bottom)

The effect of the lag range of RV

Assume we have n observations for each day i for the price process $P_i^{j=1,\dots,n}$, and let $\Delta_j^n X$ the log price increments of day i, and $e^{\nu W_i^H}$:

$$\Delta_j^n X = \log(P_i^j) - \log(P_i^{j-1}).$$

Using the CLT for RV :

$$(\sigma_i^{RV})^2 = \sum_{j=1}^{n} (\Delta_j^n X)^2 \simeq e^{2\nu W_i^H} (1 + \sqrt{\frac{2}{n}}\xi)$$

$$\log(\sigma_i^{RV}) \simeq \nu W_i^H + \sqrt{\frac{1}{2n}}\xi \text{ where } \xi \sim \mathcal{N}(0,1).$$

Thus :

$$m(2,\Delta) \simeq \nu^2 \Delta^{2H} + \frac{1}{n}$$

RB volatility forecast using RFSV

- The forecast formula for the RFSV model stems from :

$$\mathbb{E}[W^H_{t+\Delta} \mid \mathcal{F}_t] = \frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{W^H_s}{(t-s+\Delta)(t-s)^{H+1/2}}ds$$

- RFSV forecast formula is :

$$\mathbb{E}[\log\sigma^2_{t+\Delta} \mid \mathcal{F}_t] = \frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{\log\sigma^2_s}{(t-s+\Delta)(t-s)^{H+1/2}}ds$$

- We compare RFSV forecast to :
  - AR(p) : $\widehat{\log(\sigma^2_{t+\Delta})} = K^{\Delta}_0 + \sum_{i=0}^{p} C^{\Delta}_i \log(\sigma^2_{t-i})$
  - HAR : $\widehat{\log(\sigma^2_{t+\Delta})} =$ $K^{\Delta}_0 + C^{\Delta}_0\log(\sigma^2_t) + C^{\Delta}_5\frac{1}{5}\sum_{i=0}^{p} C^{\Delta}_i \log(\sigma^2_{t-i}) + C^{\Delta}_{20}\frac{1}{20}\sum_{i=0}^{20}\log(\sigma^2_{t-i})$

Forecast results for $\log(\sigma^2_{t+\Delta})$ I

	AR(5)	AR(10)	HAR(3)	RFSV
SP100 $\Delta = 1$	0.451	0.446	0.443	0.466
SP100 $\Delta = 5$	0.644	0.635	0.546	0.557
SP100 $\Delta = 21$	0.897	0.894	0.734	0.718
IBEX35 $\Delta = 1$	0.594	0.594	0.582	0.622
IBEX35 $\Delta = 5$	0.843	0.824	0.728	0.728
IBEX35 $\Delta = 21$	1.18	1.17	0.943	0.908
HSI $\Delta = 1$	0.529	0.523	0.513	0.52
HSI $\Delta = 5$	0.647	0.633	0.575	0.577
HSI $\Delta = 21$	0.805	0.801	0.665	0.671
MEXBOL $\Delta = 1$	0.572	0.567	0.553	0.589
MEXBOL $\Delta = 5$	0.731	0.709	0.648	0.645
MEXBOL $\Delta = 21$	0.922	0.917	0.757	0.764
FTSE100 $\Delta = 1$	0.474	0.465	0.463	0.476
FTSE100 $\Delta = 5$	0.627	0.614	0.545	0.545
FTSE100 $\Delta = 21$	0.859	0.855	0.699	0.688
ASX200 $\Delta = 1$	0.536	0.524	0.524	0.527
ASX200 $\Delta = 5$	0.658	0.652	0.577	0.573
ASX200 $\Delta = 21$	0.806	0.793	0.707	0.688
TOTAL $\Delta = 1$	0.540	0.534	0.527	0.558
TOTAL $\Delta = 5$	0.720	0.704	0.640	0.636
TOTAL $\Delta = 21$	1.008	1.015	0.809	0.789
XIN9I $\Delta = 1$	0.587	0.58	0.568	0.582
XIN9I $\Delta = 5$	0.712	0.695	0.637	0.641
XIN9I $\Delta = 21$	0.913	0.918	0.762	0.758

Forecast results for $\log(\sigma^2_{t+\Delta})$ II

SHSZ300 $\Delta = 1$	0.574	0.568	0.56	0.572
SHSZ300 $\Delta = 5$	0.707	0.695	0.634	0.634
SHSZ300 $\Delta = 21$	0.896	0.904	0.772	0.753
BCOM $\Delta = 1$	0.846	0.838	0.805	0.83
BCOM $\Delta = 5$	0.876	0.854	0.821	0.825
BCOM $\Delta = 21$	0.956	0.937	0.874	0.854
INDU $\Delta = 1$	0.451	0.446	0.444	0.458
INDU $\Delta = 5$	0.617	0.612	0.532	0.541
INDU $\Delta = 21$	0.858	0.857	0.716	0.699
USDEUR $\Delta = 1$	0.530	0.514	0.507	0.521
USDEUR $\Delta = 5$	0.611	0.581	0.532	0.544
USDEUR $\Delta = 21$	0.755	0.728	0.618	0.638
IBOV $\Delta = 1$	0.602	0.595	0.587	0.617
IBOV $\Delta = 5$	0.779	0.754	0.68	0.691
IBOV $\Delta = 21$	1.010	1.008	0.843	0.836
MICROSOFT $\Delta = 1$	0.579	0.576	0.566	0.603
MICROSOFT $\Delta = 5$	0.749	0.737	0.668	0.673
MICROSOFT $\Delta = 21$	0.936	0.931	0.807	0.79
GOOGLE $\Delta = 1$	0.500	0.497	0.492	0.529
GOOGLE $\Delta = 5$	0.683	0.672	0.581	0.595
GOOGLE $\Delta = 21$	0.864	0.861	0.729	0.722
SnP400 $\Delta = 1$	0.454	0.451	0.445	0.464
SnP400 $\Delta = 5$	0.616	0.601	0.525	0.538
SnP400 $\Delta = 21$	0.816	0.81	0.668	0.67

Conclusions from this analysis

The previous analysis allowed to :

- Find that the scaling of less conventional volatility proxies on more "exotic" assets is monofractal with H of 0.1 and can be as low as 0.02.
- Verify that the RFSV succeeds to be .
- Confirm that mean-reverting model with Brownian of fractional Brownian diffusion with $H > 1/2$ are misleading and fail to satisfy the properties of the data.

Remaining question

What if the volatility estimation comes from option price data ?

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Rough volatility :
Evidence from option prices

Option Prices and Volumes

- Our dataset consists in daily close bid/ask prices and volumes of European call and put options, written on the S&P 500 index.
  - → The observations of option prices range from September 4, 2001 to January 31, 2012.
  - → We discard options with prices less than 2.5 cents of dollars, or with zero trading volume, to remove noise from data.
  - → We name prices of bid/ask call and put options as : c^a, c^b, p^a, p^b.
  - → We name call and put option volumes as : V^c, V^p.
- On each observation date t, the set of time-to-maturities $T - t$ may vary since options on S&P 500 index always expire on the third Friday of the month.

Calculating Implied Volatily I

- Market implied volatility $v_t(T,K)$ can be computted by inverting the Black-Scholes formula :

$$c_t(T,K) = P_t(T)(F_t(T)\Phi(d_+(v;T,K) - K\Phi(d_-(v;T,K))),$$
$$d_\mp(v;T,K) := \frac{1}{v\sqrt{T-t}} \log\frac{F_t(T)}{K} \pm \frac{1}{2}v\sqrt{T-t},$$

  where $P_t(T)$ and $F_t(T)$ are the zero-coupon bond and the forward price maturing at T.
- In the above equation, the unknown quantities are v, P, and F.
- In the following, we simply indicate with K_i the i-th strike quoted on maturity T, with c_i call prices on T at K_i strike level, and so on.

Calculating Implied Volatily II

- The values of the forward price $F(\tau)$ and zero coupon bond $D(\tau)$ are taken as solutions of the minimization problem :

$$\arg\min_{D,F} \left\{ \sum_i w_i \Big(\frac{1}{2}(C_i^{\mathrm{a}} - P_i^{\mathrm{b}}) + \frac{1}{2}(C_i^{\mathrm{b}} - P_i^{\mathrm{a}}) - D(\tau)(F(\tau) - K_i) \Big) \right\},$$

  where $C_i^{\mathrm{a,b}}$ and $P_i^{\mathrm{a,b}}$ are respectively the call and put market prices
- The weights w_i are given by :

$$w_i = \frac{\sqrt{\min\{V_i^C, V_i^P\}}}{\frac{1}{2}(C_i^{\mathrm{a}} - C_i^{\mathrm{b}}) + \frac{1}{2}(P_i^{\mathrm{a}} - P_i^{\mathrm{b}})},$$

  with V_i^C and V_i^P the trading volumes of call and put options at strike K_i.
- Finally, our implied volatility is taken as that of a call whose price would be the midprice between the bid and ask.

Calculating Implied Volatily III

- Once the implied volatility $v_t(T, K)$ are computed from the dataset we proceed to identify proxies for the spot volatility.
- Our first choice is the implied volatility with the shortest time-to-maturity and the strike closest to the spot price.
  - $\rightarrow$ Indeed, in most models the ATM implied volatility converges to the spot one as maturity goes to zero, see [Mühle-Karbe and Nutz, 2011].
- Since the options of the dataset are quoted on the third Friday of the month, the shortest time-to-maturity can vary between one month and one day.
  - $\rightarrow$ Alternatively, we could resort to some interpolation scheme to complete the surface for all time-to-maturities, and we can select a fixed time-to-maturity across observation dates.

Scaling of $m(q, \Delta)$ and ζ_q

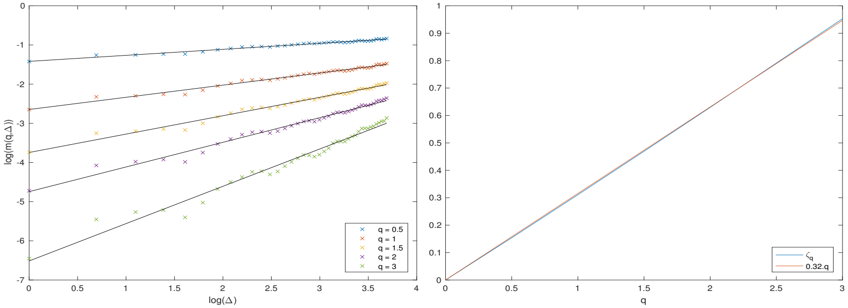


Figure – Scaling of $\log m(q, \Delta)$ (left), and ζ_q (blue) and $0.32 \times q$ (red) (right).

The value 0.32 is three times larger than H observed on RV of RBV. Maybe we can try asymptotic estimators to improve the result.

Refining the estimator : Including Smile Information

- We now study the robustness of the previous results by working with another spot-volatility proxy.
- We wish to include in our proxy not only one implied volatility quote, but also information on the the smile on different observation dates.
- A possible solution is introducing a stochastic volatility model calibrated on cross-sectional data for each t.
  - $\rightarrow$ Then, we can study the time series of the initial value of the volatility process, our spot volatility.
- This approach may suffer of instabilities due to calibration errors, since we are assuming that all the model parameters can vary on different observation dates.

Refining the estimator : Including Smile Information

- Alternatively, we can follow a different strategy
  - $\rightarrow$ First, we assume that the stochastic volatility model parameters, but the spot volatility, are the same on all observation dates.
  - $\rightarrow$ Then, we calibrate the model jointly across observation dates against call/put options near ATM and for short maturities.
- We restrict the dataset since the role of a stochastic volatility model is to catch the level/skew dynamics and not the shape of the smile.
- In doing so we refer to the paper of [Medvedev and Scaillet, 2007], where the authors derive an expansion formula for option implied volatility.
- In the practice we can use a hybrid method by partitioning the observation dates in smaller subsets and repeating the procedure in each subset.

Refining the estimation : The Medvedev-Scaillet estimator

- [Medvedev and Scaillet, 2007] consider a general two-factor model

$$\begin{aligned} dS_t =& (r - \mu(\sigma_t))S_t\,dt + \sigma_t S_t\,dZ_t + S_t\,dJ_t, \\ d\sigma_t =& a(\sigma_t)\,dt + \beta\sigma_t^{\phi}\left(\rho\,dZ_t + \sqrt{1-\rho^2}\,dW_t\right), \end{aligned}$$

  where Z_t and W_t are two independent standard Brownian motions, and J_t is an independent Poisson jump process whose intensity is given by $\lambda_t = \lambda_0\sigma_t^{\psi}$.
- Many classical stochastic volatility models can be obtained within this framework (Heston, Bates, ...)
- The authors derive an asymptotic expansion of the model for implied volatility near the ATM level for short time-to-maturities.

The Medvedev-Scaillet estimator

- The spot volatility is given by :

$$\sigma = \hat{\sigma} - I_1(0,\hat{\sigma})\sqrt{\hat{\tau}} + \left(I_1(0,\hat{\sigma})\frac{\partial I_1(0,\hat{\sigma})}{\partial \sigma} - I_2(0,\hat{\sigma}) + \frac{1}{2}\rho b(\hat{\sigma})\frac{\partial \mu(\hat{\sigma})}{\partial \sigma} \right)\hat{\tau} + O(\hat{\tau}\sqrt{\hat{\tau}}),$$

  where the functions I_1 and I_2 do not depend on the volatility drift a nor on the initial value of the volatility process σ (spot volatility), $\hat{\tau} = \hat{T} - t$ and $\hat{T}$ is the shortest available maturity.

$\rightarrow$ Notice that this proxy converges to the simple proxy of the previous section when $\hat{T} \rightarrow t$.

Calibration Procedure - I

- To employ the expansion formula we need first to filter the dataset to extract near-ATM short maturity options.
  - $\longrightarrow$ We select options with time-to-maturity ranging from 15 to 60 days and forward moneyness between 97% and 103%.
- Then, we select a model within the Medvedev and Scaillet (2007) framework, and we calibrate model parameters, but spot volatility, to option data using the first proxy.
  - $\longrightarrow$ We choose the Heston model, and the most complex model specification allowed by the framework.
- Finally, we use the second expansion to derive the spot volatility.
  - $\longrightarrow$ We repeat the analysis splitting the dataset into subsets of one year to asses the robustness of the procedure.
  - $\longrightarrow$ In each subset we use the corresponding parameter estimate to calculate the spot volatility.

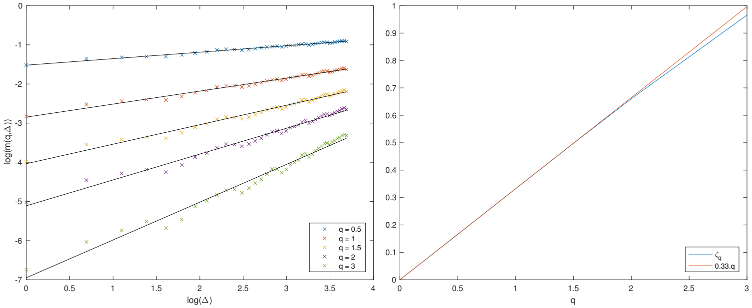
Calibration Procedure - II

Table – Parameters calibrated on quoted S&P500 option prices, from September 5, 2001 to January 31, 2012.

Parameter	Heston		General case
$\beta\rho$	-0.18	(0.00)	-3.27 (0.08)
ρ	-0.48	(0.00)	-0.39 (0.00)
ϕ	0		1.79 (0.02)
$\lambda_0\mathbb{E}(\Delta J)$	0		-0.6924 (0.03)
$\mathbb{E}(\Delta J)$	$--$	$--$	-0.17 (0.00)
ψ	$--$	$--$	1.11 (0.01)

Scaling of $m(q,\Delta)$ and ζ_q

Figure – Scaling of $\log m(q,\Delta)$ (left), and ζ_q (blue) and $0.33 \times q$ (red) (right).



The value 0.33 is the same as before and is smooth compared to results for other proxies.

Numerical Simulation of Option Prices - I

- Using option-based proxies, we deduce that volatility is rough but the value 0.3 is quite large.
- One may wonder why we obtain such value : mechanical effect from a conventional model ? a smoothing effect due to the implied volatility estimator ?
- We verify this by considering a toy rough volatility pricing model :

$$dS_t = \sigma_t S_t dZ_t, \quad d\log(\sigma_t) = \eta dW_t^H$$

where Z is a Bm, and W^H an fBm with $H = 0.04$ uncorrelated from the Z.

Numerical Simulation of Option Prices - II

We assume past volatility and price observed at $t_j \leq t_i$ and compute the price at t_i of an option with expiration date $t_k = t_i + \tau$:

- Generate M paths of the volatility on $[t_{i+1}, t_k]$:
  - We generate a vector $X = (X_{t_1}, ..., X_{t_n})$ of independent standard Gaussian.
  - We use Cholesky decomposition of the W^H covariance to obtain $(W^H_{t_1}, W^H_{t_n}) = LX$:

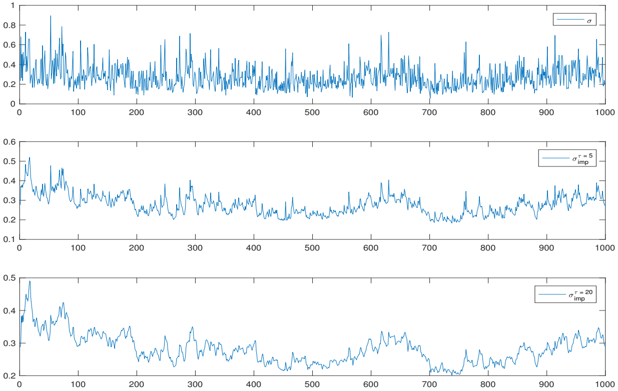
$$W^H_{t_j} = \sum_{p=1}^{i} l_{jp} X_p + \sum_{p=i+1}^{j} l_{jp} X_p.$$

- We generate M trajectories of the spot volatility denoted by σ^m.
- The price at t_i of an ATM option with maturity τ is :

$$\frac{1}{M} \sum_{m=1}^{M} C_{BS} \left(S_{t_i}, \tau, \sqrt{\frac{1}{\tau} \sum_{p=i+1}^{k} (\sigma^m_{t_p})^2} \right),$$

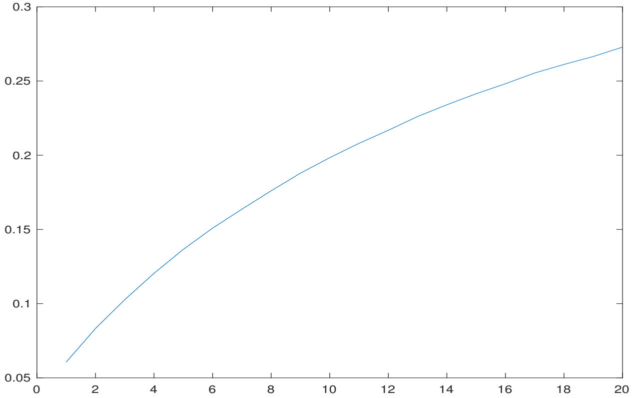
Numerical Simulations : Results I

Figure – Sample paths of spot volatility and implied volatility for $\tau = 5$ and $\tau = 20$.



Numerical Simulations : Results II

Figure – Estimated values of the Hurst parameter using implied volatility as a function of time to maturity.



Analytical illustration of the upward bias I

- We look at the analytical approximation of the scaling $\hat{m}^\tau(2,\Delta) = \mathbb{E}[(\hat{v}^\tau(\Delta) - \hat{v}^\tau(0))^2]$.
- We use the Mandelbrot and Van Ness representation of an fBm :

$$W_t^H = c_H\Big(\int_0^t (t-s)^{H-1/2} dW_s + \int_{-\infty}^0 \big((t-s)^{H-1/2} - (-s)^{H-1/2}\big) dW_s\Big)$$

.

- We deduce :

$$\hat{m}^\tau(2,\Delta) = A\Delta^{2H}\big(f_1(\theta) + f_2(\theta)\big),$$

with

$$A = \frac{c_H^2 \nu^2}{(H+1/2)^2}, \quad \theta = \tau/\Delta,$$

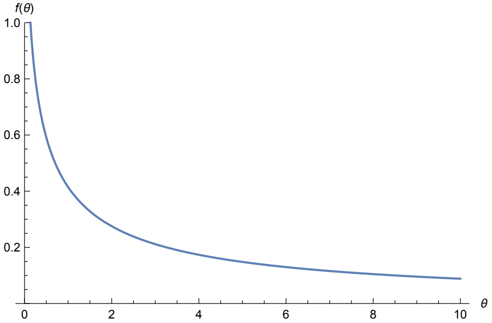
$$f_1(\theta) = \frac{1}{\theta^2}\int_{-\infty}^0 \big((1+\theta-s)^{H+1/2} - (1-s)^{H+1/2} - (\theta-s)^{H+1/2} + (-s)^{H+1/2}\big)^2 ds,$$

$$f_2(\theta) = \frac{1}{\theta^2}\int_0^1 \big((1+\theta-s)^{H+1/2} - (1-s)^{H+1/2}\big)^2 ds.$$

Analytical illustration of the upward bias II

→ When θ is small $\hat{m}^{\tau}(2,\Delta) \sim \nu^2 \Delta^{2H}$. Otherwise we should add the multiplicative factor $f(\theta) = \frac{c_H^2}{(H+1/2)^2}\left(f_1(\theta) + f_2(\theta)\right)$

Figure – The function f for $H = 0.04$.



Some Resources on Rough Volatility

- 1939 days since Volatility is Rough and many topics have been covered and other challenges are yet to be explored :
  - Smile, skew and term structure calibration (a recent paper on the joint S&P 500/VIX smile calibration).
  - Foundations of rough volatility from microscopic price formation.
  - Portfolio selection under the non-Markovian nature of rough volatility.
- A rough volatility repository with more than 100 papers on the topic : `https://sites.google.com/site/roughvol/home`
- A python Jupyter Notebook to familiarize with the topic : `https://tpq.io/p/rough_volatility_with_python.html#cite_note-BayerFriz`

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