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Abstract

We model asset prices in the most general sensible form as special semimartingales. This approach allows us to also include jumps in the asset price process. We show that the existence of an equivalent martingale measure, which is essentially equivalent to no-arbitrage, implies that the asset prices cannot exhibit predictable jumps. Hence, in arbitrage-free markets the occurrence and the size of any jump of the asset price cannot be known before it happens. In practical applications it is basically not possible to distinguish between predictable and unpredictable discontinuities in the price process. The empirical literature has typically assumed as an identification condition that there are no predictable jumps. Our result shows that that identification condition follows from the existence of an equivalent martingale measure, and hence essentially comes for free in arbitrage-free markets.

1 Introduction

Semimartingales are the most general processes such that a stochastic integral can be defined and hence are also the most general stochastic process for asset prices. Special semimartingales are semimartingales that have finite conditional means, which is necessary to sensibly define instantaneous returns. Ansel and Stricker (1991) show that a suitable formulation of absence of arbitrage implies that security gains must be special semimartingales. The class of special semimartingales includes many process, e.g. Itô processes, jump processes and mixed jump-Itô processes.

Back (1990) and Schweizer (1992) model asset prices as special semimartingales. They derive a formula for the local risk premium of an asset which is proportional to its covariance with the state price density process. In both papers, predictable jumps are implicitly included. Predictable jumps are discontinuities in the path of a stochastic process, whose realizations are known just before they happen.

Empirically, it is not possible to distinguish a predictable jump from a non-predictable jump. However, these two jumps have different properties which can have a huge effect on econometric estimators. Hence, the econometrics literature for models with discontinuities generally excludes predictable jumps from the asset price processes. For example Barndorff-Nielsen and Shephard (2004b) have shown that the realized power variation and its extension

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the realized bipower variation can be used to separately estimate the integrated volatility of the continuous and the jump component of a certain class of stochastic processes. However, this estimation strategy works only if we exclude predictable jumps. This identification assumption is unnecessary as it is essentially a consequence of the absence of arbitrage.

The idea of the proof is based on two facts. First, a local martingale does not have any predictable jumps. Hence, in particular we need only to show that the predictable finite variation part of the asset price process cannot have predictable jumps. Second, the existence of an equivalent martingale measure puts restrictions on the predictable finite variation part. We obtain a CAPM like representation, where the predictable finite variation part is proportional to the "covariance" between the state price density and the local martingale part of the asset price. As these both processes are also local martingales, they cannot have predictable jumps, which in turn implies that the predictable finite variation part cannot have any jumps at all.

2 The Model

We have essentially the same model as in Back (1990) and Schweizer (1992). We refer to those papers for the underlying motivation and to Kallenberg (1997) for the probabilistic concepts. Assume a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, satisfying the usual conditions, is given. The discounted gains process is denoted by $X(t)$. We could start by modeling an asset price process, dividend process and discount-rate process individually, but we do not lose generality by directly starting with $X(t)$. Under the real-world measure $\mathbb{P}$ the discounted gains process $X(t)$ is assumed to be a special semimartingale

$$X(t) = X_0 + A(t) + M^{\mathbb{P}}(t)$$

where $A(t)$ is the predictable finite variation part and $M^{\mathbb{P}}(t)$ is a local martingale. The predictable σ-field is the σ-field generated by the left-continuous, adapted processes. We say that the stochastic process $A(t)$ is predictable, if it is measurable with respect to the predictable σ-field. In particular, if $A(t)$ has a predictable discontinuity, it is known right before it happens. The decomposition of a special semimartingale into a predictable finite variation part and a local martingale is unique. In a general semimartingale, $A(t)$ is not assumed to be unique. It is the predictability of A(t) that implies the uniqueness of the decomposition of a semimartingale.

For example consider

$$X(t) = a(t) + B(t) + \sum_{i=1}^{N(t)} Y_i$$

where $B(t)$ is a Brownian motion, $\sum_{i=1}^{N(t)} Y_i$ a compound Poisson process independent of B and $a(t)$ is a predictable finite variation process. The Poisson process $N(t)$ has intensity λ and the jump sizes Y_i are i.i.d. with $E[Y] = \kappa < \infty$. As $\sum_{i=1}^{N(t)} Y_i$ is of finite variation, $X(t)$ is a semimartingale with $B(t)$ being the local martingale part. The unique special semimartingale representation takes the form:

$$X(t) = \underbrace{B(t) + \left(\sum_{i=1}^{N} (t) Y_i - t\lambda\kappa \right)}_{M^{\mathbb{P}}(t)} + \underbrace{t\lambda\kappa + a(t)}_{A(t)}$$

The compensated jump process is now part of the local martingale, while the compensator of the jump process plus $a(t)$ form the predictable finite variation process. The only assumption that we have made about $a(t)$ is that it is a predictable finite variation process. In particular, $a(t)$ could be a discontinuous process, i.e. it could have predictable jumps. The main contribution of this paper is to show that the predictable process $a(t)$ has to be continuous in arbitrage-free markets.

Following Back's (1990) heuristic, the predictable finite variation part corresponds to the the conditional mean:

$$E_t \left[dX(t) \right] = E_t \left[dA(t) \right] + E_t \left[dM^{\mathbb{P}} \right] = dA(t)$$

as the differential of the predictable finite variation part is known just before t. Of course, this is just an heuristic as any rigorous statement would involve stochastic integrals.

As it is well-known the existence of an equivalent martingale measure is essentially equivalent to the absence of arbitrage opportunities. "Essentially" means that this statement depends on the precise definition of arbitrage opportunities; see Kreps (1981) and Stricker (1990) for a discussion. An equivalent martingale measure $\mathbb{Q}$ for X is a probability measure that is equivalent to $\mathbb{P}$ (i.e. $\mathbb{P}$ and $\mathbb{Q}$ have the same null sets) and has the property that X is a martingale with respect to $\mathbb{Q}$. The equivalence implies the existence of the Radon-Nikodym derivative $Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$, which defines a strictly positive martingale Z with $Z_0 = 1$:

$$Z_t = E^{\mathbb{P}} \left[Z_T | \mathfrak{F}_t \right] = \frac{d\mathbb{Q}}{d\mathbb{P}} | \mathfrak{F}_t$$

The martingale property of X under $\mathbb{Q}$ is equivalent to the statement that XZ is a $\mathbb{P}-$martingale. A more general concept is a martingale density (introduced by Schweizer (1992)):

Definition 1. *A local $\mathbb{P}$-martingale Z with $Z_0 = 1$ is called a martingale density for X if the process XZ is a local $\mathbb{P}$-martingale. Z is called a strict martingale density, if, in addition, Z is strictly positive.*

All the results that we can derive for a martingale density will of course hold for an equivalent martingale measure.[1]

A key concept in working with semimartingales are the predictable covariation process $\langle ., . \rangle$ and the covariation process $[., .]$. The conditional covariation process can be interpreted as a conditional covariance. If M_1 and N_2 are two local martingales such that the product MN is a special semimartingale, then the conditional covariation process $\langle M, N \rangle$ is the predictable finite variation part in the canonical decomposition of MN. If $X_1 = A_1 + M_1$ and $X_2 =$

[1] The martingale density assumption is weaker than the assumption about an equivalent martingale measure, as XZ does not need to be a real martingale. Hence for a martingale density, $\mathbb{Q}$ is in general only a subprobability, i.e. $\mathbb{Q}(\Omega) \leq 1$.

$A_2 + M_2$ are special semimartingales, and if X_1X_2 is a special semimartingale, then $\langle X_1, X_2\rangle$ is defined as

$$\langle X_1, X_2\rangle(t) = \langle M_1, M_2\rangle(t) + \sum_{0\leq s\leq t} \Delta X_1(s)\Delta X_2(s)$$

The jumps are denoted by $\Delta X(s) = X(s) - X(s-) \neq 0$. The jumps in the above representation are predictable and we will show in the following that the existence of an equivalent martingale measure implies that such jumps cannot occur. The conditional quadratic covariation should not be confused with the quadratic covariation process $[.,.]$. $\langle .,.\rangle$ is the $\mathbb{P}$-compensator of $[.,.]$. If M_1 and M_2 are semimartingales the quadratic covariation process is defined by

$$[M_1, M_2] = M_1M_2 - \int M_1(t-)dM_2(t) - \int M_2(t-)dM_1(t)$$

We use the following result from Kallenberg (1997):

Proposition 1. *A local martingale is predictable iff it is a.s. continuous.*

We conclude, that a predictable local martingale cannot have any jumps:

Corollary 1. *A local martingale does not have predictable jumps.*

As we will refer several times to Yoerup's lemma (Dellacherie and Meyer (1982), VII.36), we state it here for convenience:

Lemma 1. *Let M be a local martingale and A a predictable process of finite variation. Then the quadratic variation process $[M, A]$ is a local martingale.*

We can now state our main theorem:

Theorem 1. *Let Z be a strict martingale density for X. If XZ is a special semimartingale, then X cannot have predictable jumps.*

Proof. By Yoerup's lemma, $[Z, A]$ is a local $\mathbb{P}$-martingale. Hence, ZA is a special semimartingale:

$$d(ZA) = Z_-dA + A_-dZ + d[Z, A]$$

As ZX is a special semimartingale, $ZM^{\mathbb{P}}$ is one as well and thus $\langle Z, M^{\mathbb{P}}\rangle$ exists. Next, we apply the product rule to XZ:

$$\begin{aligned} d(XZ) &= X_-dZ + Z_-dX + [Z, X] \\ &= X_-dZ + Z_-dM^{\mathbb{P}} + Z_-dA + d[Z, A] + d[Z, M^{\mathbb{P}}] - d\langle Z, M^{\mathbb{P}}\rangle + d\langle Z, M^{\mathbb{P}}\rangle \\ &= \text{local } \mathbb{P}\text{-martingale } + Z_-dA + d\langle Z, M^{\mathbb{P}}\rangle \end{aligned}$$

In the last line we have used Yoerup's lemma again. But as XZ is a local martingale by assumption, the two last terms, which are predictable and of finite variation, must vanish. Hence, we conclude

$$dA = -\frac{1}{Z_-}d\langle Z, M^{\mathbb{P}}\rangle.$$

Hence, for all predictable jumps Δ in $A(t)$ one must have

$$\Delta A(t) = \Delta \left(-\frac{1}{Z_(t-)} d\langle Z(t), M^{\mathbb{P}(t)} \rangle \right) = \frac{\Delta Z(t)}{Z(t-)} \Delta M^{\mathbb{P}}(t)$$

As Z and $M^{\mathbb{P}}$ are local $\mathbb{P}$-martingales, they cannot have any predictable jumps, and thus neither can A. In conclusion, X cannot have any predictable jumps. □

Corollary 2. *Assume that $X = X_0 + A + M^{\mathbb{P}}$ is a special semimartingale and that there exists an equivalent martingale measure $\mathbb{Q}$ for X with respect to $\mathbb{P}$, which is defined by the Radon-Nikodym derivative Z. Assume that both X and Z are locally square-integrable. Then X cannot have any predictable jumps.*

Proof. The local square-integrability ensures that $\langle M^{\mathbb{P}}, Z\rangle$ is well-defined. By definition $dM^{\mathbb{P}}Z - d\langle M^{\mathbb{P}}, Z\rangle$ is a local martingale. Hence, $M^{\mathbb{P}}Z$ is a special semimartingale with decomposition $(dM^{\mathbb{P}}Z - d\langle M^{\mathbb{P}}, Z\rangle) + d\langle M^{\mathbb{P}}, Z\rangle$. As AZ is a special semimartingale, we conclude that XZ is a special semimartingale. If Z defines an equivalent martingale measure, it is also a strict martingale density and hence we can apply Theorem 1. □

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