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Abstract

Contingent convertible bonds (CCBs) are new debt instruments that automatically convert to equity when the issuing firm or bank reaches a specified level of financial distress. This paper presents a formal model of CCBs with finite maturity and where the firm's value process is driven by a jump diffusion process. We are able to derive closed-form solutions for the value of the CCBs. In this paper we can completely characterize two different types of CCBs: In the first case the number of shares granted at conversion is fixed a priori. In the second specification the number of shares granted at conversion is chosen a posteriori such that the value of the shares equals a specified value. Incorporating jumps into the dynamics of the firm's value process is important for two reasons. First it can solve the predictability problem of the conversion and default event, i.e. including jumps into the firm's value process creates non-zero credit spreads for short maturities. Second, the evaluation of CCBs depends on the capital structure. Without jumps the evaluation of contingent convertible bonds can be independent of the amount of straight debt. In a model with jumps the valuation of straight debt and contingent convertible debt is interlinked as jumps could be large enough to trigger conversion and default simultaneously. Furthermore, it is observed that short-term debt has very different features than long-term debt. Our model can capture the effect of the maturity on the debt contracts.

In order to apply CCBs in practice it is desirable to base the conversion on observable market prices that can constantly adjust to new information in contrast to accounting triggers. We can show how to use credit spreads and the risk premium of credit default swaps to construct the conversion trigger and to evaluate the contracts under this specification.

CCBs are intended to avoid bank bailouts of the type that occurred during the subprime mortgage crisis when banks were in trouble to recapitalize themselves and regulators feared the consequences of default contagion. Hence, the second focus of this paper is to analyze whether CCBs can be used as a regulation instrument. It is crucial to require that the parameters of the CCBs are chosen such that they satisfy a no-early-default condition. In this case a regulation that combines a restriction on the maximal leverage ratio and the requirement of issuing a certain fraction of CCBs as part of the whole debt, can efficiently lower the default probability without reducing the total value of the firm. However, if this condition is violated, CCBs can increase the default risk of a bank.

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1 Introduction

This paper presents a formal model for a new regulatory hybrid security for financial firms, so-called contingent convertible bonds. This instrument has the features of normal debt in normal times, but converts to equity when the issuing firms are under financial stress. Although these bonds could in principle be used by any firm, the focus here is to analyze their potential as a regulatory instrument for banks.

As the ongoing financial crisis has illustrated, banks play an important role in the economy. When they are healthy, banks channel savings into productive assets. But when they are distressed, this role is compromised and banks lend less with adverse effects on investment, output and employment. In this situation governments often intervene, but as we could see during the past crisis, the measures are costly to taxpayers and may be limited in effectiveness. There are several reasons why banks may inadequately recapitalize on their own in the first place.

First, there is the so-called debt overhang problem. If a bank suffers substantial losses, the managers, who should act in the interest of the shareholders, may prefer not to issue new equity. If a distressed bank issues new equity, the bank's bondholders profit from this as the new capital increases the likelihood that they will get repaid. On the other hand, existing shareholders bear costs as their claims on the firm are diluted. In this sense issuing new equity creates a transfer from existing shareholders to bondholders. Hence, in order to satisfy capital requirements shareholders may prefer the bank to sell risky assets or to reduce new lending instead of issuing new equity. If during a financial crisis other banks are in trouble, too, they will also cut lending and thus the economy as a whole suffers.

Second, there is the moral hazard problem of a government bailout. A government bailout represents an implicit guarantee for the bondholders that their debt will be repaid. If bondholders believe that the government will not allow a bank to fail, the bondholders may be more willing to lend money to a bank that pursues more risky strategies and have less incentives to control the bank. This problem is particularly severe for the most important financial institutes, that are "too big to fail".

Furthermore, bankruptcy reorganization for banks is different than default restructuring for other firms. Banking business relies on confidence. If a bank is in trouble, there is the danger of a bank run as clients and short-term creditors may withdraw their capital. As in the case of Lehman Brothers, distress for a financial firm often leads to partial or complete liquidation in contrast to a restructuring according to Chapter 11 which can help a "normal" company to return to economic viability.

In summary, the debt overhang problem can make banks reduce lending or sell assets instead of recapitalizing themselves and maintaining their lending capacity. If restructuring takes place, it is usually ineffective and disruptive and can affect other institutions. The possibility of a government bailout can increase the riskiness of the strategies of a bank. For this reason the discussion in the aftermath of the financial crisis has focussed on a resolution mechanism that can allow quick and less disruptive recapitalization of distressed banks, but does not shift the costs of risky activities to the government. A possible solution is contingent convertible bonds.

Contingent convertible bonds are instruments that convert into equity if the bank is financially distressed. The bank would issue these bonds before a crisis and if a certain trigger is reached, conversion takes place automatically. The automatic conversion of debt into equity would transform an undercapitalized bank into a well capitalized bank at no cost to the taxpayer.

The key issues for specifying CCBs are: When does conversion take place and how many shares are given to the bondholders at conversion? The automatic conversion should be triggered by the same mechanism that triggers default. From our understanding bankruptcy is caused if the value of the firm's assets is below a default barrier. Hence, conversion should take place if the value of the firm's assets reaches another barrier, namely the conversion barrier. In view of the application as a regulation instrument it is sensible to require this conversion barrier to be higher than the default barrier. In addition to the trigger, the rate at which the debt converts into equity has to be specified. There are basically two different approaches. In the first approach each dollar of debt converts into a fixed quantity of equity shares. In this case the total value of the portfolio of shares granted at conversion depends on the stock price at conversion. In the second approach the conversion is specified in terms of the market value of equity. In this case the number of shares granted at conversion depends on the stock price at the time of conversion. This paper considers both types of converting debt into equity.

CBBs have been traded only very recently. In 2009 Lloyd's bank issued the first £7 billion CCBs. The first and only time so far that contingent convertible bonds were used as a regulatory instrument was in Switzerland, when the Credit Suisse Group AG issued $2 billion of these new bonds on February 14th, 2011. The coupon payments of the contingent convertible bonds were substantially higher than for normal debt: 7.875% vs. 4% on average.

The idea of contingent convertible bonds has been a very vivid area of research in the last 3 years. However, the literature on formal models of contingent convertible bonds is still very limited. Qualitative discussions can be found in Flannery (2009a+b), Squam Lake Working Group on Financial Regulation (2009), McDonald (2010) and Calomiris and Herring (2011). The first structural formal model is presented in Albul, Jaffee and Tchistyi (2010). Their model is based on Leland's (1994b) structural credit risk model with optimal default barrier. The firm's value process follows a geometric Brownian motion and the bonds are of infinite maturity. Conversion is triggered when the firm's value process reaches an exogenous conversion barrier and the conversion value is expressed in terms of the market value of equity. Their paper provides many very interesting insights concerning regulation and capital structure decisions. Our paper is closely related to Albul, Jaffee and Tchistyi's (2010) work, but our model is based on Hilberink and Rogers' (2002) and Chen and Kou (2009)'s jump process framework. As in Albul, Jaffee and Tchistyi we work with a structural credit risk model, in which the optimal default barrier is chosen endogenously by the shareholders by trading off tax benefits and bankruptcy costs. However, the bonds in our model have finite maturity and are issued such that we obtain a stationary debt structure. The firm's value in our model follows a specific jump diffusion process, namely a Kou process. This particular choice of process allows us to obtain a non-zero credit spread limit for a maturity approaching zero. The valuation of CCBs becomes considerable more challenging under a jump diffusion process. A jump that triggers conversion can be sufficiently large to trigger bankruptcy as well. Hence, the conversion value of CCBs explicitly depends on the features of the straight debt. In Albul, Jaffee and Tchistyi the valuation of straight debt and contingent convertible debt could be separated. In our framework we can show that the valuation of the two different debt instruments is interlinked. Furthermore, we do not only treat the conversion barrier as exogenously given, but we also consider the case of it being chosen optimally by the shareholders or the firm.

This is the first model where the conversion value is also specified as a fixed number of shares. In order to price the conversion value that depends on a fixed number of shares, we have to model the stock price process as well. As shares are essentially claims on a fraction of the equity, the stock price process will depend on the capital structure and has to be modeled endogenously. As we explicitly model the stock price process, we can completely characterize the dilution costs to the old shareholders at conversion.

Specifying the conversion trigger in terms of the firm's value process is conceptionally appealing and allows us to derive analytical solutions for all prices. However, the firm's value process is unobservable and it would be desirable to base the conversion event on observable prices. Some literature, e.g. De Spiegeleer and Schoutens (2011), propose to trigger conversion when the stock price process first crosses a barrier level. As we will show the stock price is in general not a sufficient statistic for the firm's value process. Therefore, conversion based on the stock price cannot be incorporated into our modeling framework. Moreover, it is possible that more than one stock price and CCB price are consistent with our equilibrium conditions if conversion is based on the stock price. These shortcomings are not appealing. However, we can show that the unobservability of the firm's value process can be circumvented by using credit spreads or the risk premiums of credit default swaps (CDS). Credit spreads and CDS risk premiums have the same advantages as stock prices as they constantly adjust to new information in contrast to accounting triggers. We will prove that they are a sufficient statistic for the firm's value process. Thus, defining the conversion event in terms of credit spreads or CDS risk premiums is equivalent to using the firm's value process. In summary, our evaluation formulas can be applied in practice with a trigger event based on observable market prices.

We extend our model into several directions. First, we introduce a general approach which allows all parameters of the firm's value process to change after the conversion. In particular we can model the special case, that the drift of the firm's value process increases after conversion as interest payments decrease. Second, we introduce exogenous noise trading into the stock price process and can show that under certain conditions our pricing formulas are not affected. Third, we show that contracts, for which the number of shares granted at conversion is fixed a priori, are more robust against manipulation by the contingent convertible bondholders than contracts with a fixed value at conversion. It is possible to design a contract that is robust against manipulation by the equity holders and contingent convertible bondholders.

In the last part of the paper we analyze whether contingent convertible bonds can be used as a regulation instrument. We can show that under a technical assumption, the no-early-default condition, a regulation that combines a restriction on the maximal leverage ratio and the requirement of issuing a certain fraction of contingent convertible bonds as part of the whole debt, can efficiently lower the default probability without reducing the total value of the firm. However, if the no-early-default condition is violated, a regulation based on contingent convertible debt can actually increase the risk. Therefore, it is crucial to make sure that this condition is satisfied. If the contingent convertible bonds are issued in a smaller amount than the straight debt, have a long maturity and the conversion barrier is sufficiently high, the no-early-default condition is generally satisfied.

The paper is organized as follows. In Section 2 we present the formal model of Chen and Kou. In Section 3 we add the contingent convertible bonds to the model. In Section 4 we derive closed-form prices for CCBs. Section 5 discusses the choice of the default and conversion barrier. In Section 6 we consider the case where conversion is based on observable market prices instead of the unobservable firm's value process. In Section 7 we present some numerical simulations. Section 8 discusses the optimal design of CCBs. In Section 9 we consider extensions to our model. Section 10 focusses on contingent convertible bonds as a regulation instrument. Section 11 concludes. Most of the proofs and the special case of a pure diffusion process are collected in the Appendix.

2 Model for Normal Debt

In this section we review Chen and Kou's (2009) model, in which the firm's value process follows a particular jump-diffusion process. It is based on Leland's (1994a) diffusion and Hilberink and Rogers' (2002) jump diffusion structural credit risk models with optimal default barrier. As the model presented here applies to any firm and as we will only consider the special case of banks in the section about regulation, we use the generic terminology "firm" instead of "bank".

The value of the firm's assets at time t is denoted by V_t and under the risk-neutral measure $\mathbb{P}$ [1] it evolves as

$$dV_t = V_t(dZ_t + (r - \delta)dt) \tag{2.1}$$

where Z is some martingale, r is the constant riskless interest rate and δ represents the proportional rate at which a part of the assets is disbursed to investors.[2]

We will follow Chen and Kou (2009) and specify Z as a jump diffusion process with double exponentially distributed jumps. Note that as the firm has bondholders and shareholders, δ cannot be seen as a dividend rate. First the coupons and principal repayments have to be paid before the residual can be paid out as dividends. By assuming a constant riskless interest rate we neglect the interest rate risk.

The firm is partly financed by debt which has two features: Its time structure and its riskiness. In order to obtain a stationary debt structure, debt is constantly retired and reissued. Assume that at every point in time the firm issues new debt in the amount of p_D, i.e. in the time interval $(t, t+dt)$ new bonds with face value $p_D dt$ are issued. The debt has the maturity profile φ, where φ can be any non-negative function with $\int_0^\infty \varphi(s)ds = 1$ and can be interpreted as a density function. We will choose the maturity profile as $\varphi(t) = me^{-mt}$, i.e. the maturity of a specific bond is chosen randomly according to an exponentially distributed random variable. Of all the debt issued in $(t, t+dt)$ the debt with face value $p_D\varphi(s)dtds$ will mature in the time interval $(t+s, t+s+ds)$. If we also consider all the debt that was issued before $t = 0$ the face value of debt maturing in $(s, s+ds)$ is

$$\int_{-\infty}^0 p_D\varphi(s-x)dxds = \int_s^\infty p_D\varphi(y)dyds = p_D\Psi(s)ds \quad , \quad \Psi(s) \equiv \int_s^\infty \varphi(y)dy.$$

For our maturity profile this equals

$$\int_{-\infty}^0 p_D me^{-m(s-x)}dxds = p_D ds.$$

[1] As we consider only one martingale measure in this section, E will denote the expected value with respect to $\mathbb{P}$.

[2] We use the drift $r - \delta$ because we work under the risk-neutral measure $\mathbb{P}$. Under the real world probability measure the drift would be $\mu - \delta$, where μ is some constant.

The face value of all the newly issued debt is $p_D ds$. Hence, the face value of all the debt maturing in $(s, s+ds)$ is equal to the face value of the newly issued debt. Thus, the face value of debt stays constant and at every point in time equals

$$P_D = p_D \int_0^\infty \Psi(s)ds = \frac{p_D}{m}.$$

Note, that for our maturity profile $\varphi(t) = me^{-mt}$ the parameter m is a measure of maturity. As m increases a higher fraction of the debt matures earlier. If default never occurs, the average maturity of debt is:

$$\int_0^\infty t\varphi(t)dt = \int_0^\infty t(me^{-mt})dt = \frac{1}{m}.$$

For $m = 0$ only consol bonds are issued as in Leland (1994b). The choice of the exponential maturity is needed in our model to express the debt in terms of some Laplace transform. This enables us to derive an explicit solution for double exponential jump diffusion processes.

We assume that all debt is of equal seniority and is paid by coupons at the fixed rate $c_D dt$ for the time interval $(t, t+dt)$ until either maturity or default occurs. The first time the value of the firm falls to some level V_B or lower and thus default happens is denoted by τ. The default barrier V_B will be chosen optimally by the shareholders. In the case of default, we assume that a fraction α of the value of the firm's assets is lost. The value of a bond issued at time 0 with face value 1 and maturity t is therefore

$$d_D(V, V_B, t) \;=\; E\left[\int_0^{t\wedge\tau} c_D e^{-rs} ds\right] + E\left[e^{-rt}\mathbb{1}_{\{t<\tau\}}\right] + \frac{1}{P_D}(1-\alpha)E\left[V(\tau)e^{-r\tau}\mathbb{1}_{\{\tau\le t\}}\right].$$

The first term represents the net present value of all coupons up to the minimum of t and τ. The second term can be interpreted as the net present value of the firm's repayment when default does not happen. The last term is the net present value of the assets if bankruptcy occurs. We assume that the face value of all debt is P_D and thus a bondholder with a bond with face value 1 gets the fraction $1/P_D$ of the value $(1-\alpha)V(\tau)$ that remains after bankruptcy. Note that if V were continuous, $V(\tau)$ would simply be V_B, but for a process with jumps this need not be the case.

The total value of all debt outstanding given our assumptions about the maturity profile was derived by Chen and Kou (2009) and is given in the next proposition.

Proposition 1. *The total value of all outstanding debt for the maturity profile $\varphi(t) = me^{-mt}$ is*

$$\begin{aligned} D(V, V_B) &= \int_0^\infty p_D \Psi(t) d_D(V, V_B, t)dt \\ &= \frac{c_D P_D + mP_D}{m+r} E\left[1 - e^{-(m+r)\tau}\right] + (1-\alpha)E\left[V(\tau)e^{-(m+r)\tau}\mathbb{1}_{\{\tau<\infty\}}\right]. \end{aligned}$$

The main problem now is to compute the two expectations. For V following a geometric Brownian motion an explicit solution is easily available. Note that the expectation $E\left[V(\tau)e^{-(r+m)\tau}\mathbb{1}_{\{\tau<\infty\}}\right]$ is bounded from above by V_B and below by 0. In the two extreme cases for the maturity rate m, Lebesgue's dominated convergence theorem yields the following corollary:

Corollary 1. *The value of the debt for $m \to \infty$ and $m \to 0$ is given by*

$$\lim_{m\to 0} D(V,V_B) = \frac{c_D P_D}{r} E\left[1-e^{-r\tau}\right] + (1-\alpha)E\left[V(\tau)e^{-r\tau}\mathbb{1}_{\{\tau<\infty\}}\right]$$
$$\lim_{m\to\infty} D(V,V_B) = P_D.$$

The limit for $m \to 0$ corresponds to the case of consol bonds as in Leland's (1994b) paper.

The total coupon rate of all the debt equals $C_D = c_D P_D$. As by the choice of our maturity profile, P_D is constant over time and c_D is assumed to be fixed, we get a stationary debt structure: A unit of bonds issued one year ago will look exactly the same as a unit of bonds issued today. If the value of the firm's assets does not change they will also have the same price.

According to Modigliani and Miller the total value of the firm is expressed as the sum of the asset value plus tax benefits minus bankruptcy costs. We assume that there is a proportional corporate tax rate $\bar{c}$ and coupon payments can be offset against tax. Thus, for the total coupon rate $C_D = c_D P_D$ the firm receives an additional income stream of $\bar{c}C_D dt$.

Definition 1. *The tax benefits associated to the debt are denoted by TB_D and the bankruptcy costs by BC. The total value of the firm is defined as*

$$G_{debt}(V,V_B) = V + TB_D(V,V_B) - BC(V,V_B).$$

Proposition 2. *The total value of the firm equals*

$$G_{debt}(V,V_B) = V + \frac{\bar{c}C_D}{r} E\left[1-e^{-r\tau}\right] - \alpha E\left[V(\tau)e^{-r\tau}\mathbb{1}_{\{\tau<\infty\}}\right]. \tag{2.2}$$

Proof. See Appendix 12.1 □

The value of the firm consists of the value of its assets plus the net present value of the tax rebates minus the net present value of the losses on default. Now, we can express the value of the firm's equity as

$$EQ_{debt}(V,V_B) = G_{debt}(V,V_B) - D(V,V_B).$$

Optimal capital structure and optimal endogenous default are two interlinked problems. The optimal debt level P_D and the optimal bankruptcy trigger V_B have to be chosen simultaneously. When a firm chooses P_D in order to maximize the total value of the firm at time 0, the decision depends on V_B. Vice versa, the optimal default trigger V_B is a function of the amount of debt P_D. Leland (1994a+b) and Leland and Toft (1996) have shown how to choose P_D and V_B according to a two-stage optimization problem. In the first stage, for a fixed P_D, equity holders choose the optimal default barrier by maximizing the equity value subject to the limited liability constraint. In a second stage, the firm determines the amount of debt P_D that maximizes the total value of the firm. More precisely, the first stage problem is

$$\max_{V_B} EQ_{debt}(V,V_B) \qquad \text{such that } EQ_{debt}(V',V_B) > 0 \text{ for all } V' > V_B$$

The "smooth pasting" condition as derived in Leland and Toft (1996) delivers an optimality criterion:

$$\frac{\partial EQ_{debt}}{\partial V}(V, V_B)|_{V=V_B} = 0.$$

In the case of two-sided jumps Chen and Kou prove that the solution to the smooth pasting condition indeed maximizes the equity, respecting the constraint that the value of equity must remain non-negative at all times. The optimal default barrier will be denoted by V_B^* and is clearly a function of P_D. The second stage optimization is formulated as

$$\max_{P_D} G_{debt}(V, V_B^*(P_D)).$$

In our model bankruptcy occurs at an endogenously determined asset value V_B. For all asset values larger than V_B equity has a positive value. Note, that this does not mean that bankruptcy occurs when debt service payments exceed the cash flow δV. At any point in time the firm issues bonds which are worth D, but has to make a debt service of P_D and after-tax coupon payment of$(1-\bar{c})C_D$. Hence, $(\delta V - (1-\bar{c})C_D - P_D + D)$ is the payout rate to the shareholders. As V falls, the cash flow δV declines and the price of the debt $D(V, VB)$ will fall as well, which can result in a negative payout rate to the shareholders. As long as the equity value is positive, new stock can be issued to meet debt service requirements. Hence, bankruptcy can only occur when the equity value becomes zero.

The initial total coupon rate C_D will be chosen, such that debt sells at par, i.e. the price of the debt equals its face value. Therefore, if the value of the firm's assets does not change, the firm can use the newly issued debt to repay the face value of the old debt. As we will see later the optimal value of V_B depends on the coupon rate C_D. Hence, for a fixed amount of debt P_D, the initial coupon payment C_D is computed by solving the following two equations simultaneously. First, debt has to sell at par:

$$P_D = D(V_0, V_B, P_D, C_D). \tag{2.3}$$

Second, the optimality criterion is to maximize the equity value:

$$\left(\frac{\partial EQ_{debt}(V_0, V_B, P_D, C_D)}{\partial V_0}\right)_{V_0=V_B} = 0. \tag{2.4}$$

3 A Model for Contingent Convertible Debt

3.1 Modeling Conversion

The special property of contingent convertible bonds is that the debt automatically converts to equity if the firm or bank reaches a specified level of financial distress. We will model this by introducing a barrier V_C. The first time the value of the firm falls to or below this level, the convertible bond fully converts into equity. Hence, the conversion time is defined as

$$\tau_C = \inf(t \in (0, \infty) : V(t) \leq V_C).$$

The challenge of modeling convertible debt lies in the specification of the conversion value. First, we present a model where the conversion value is based on a fixed number of shares. In this case the stock price processes has to be modeled as well. We label them fixed share contingent convertible bonds (FSCCB). Second, we consider a model where the conversion value is based on the market value of equity, i.e. the number of equity shares depends on the stock price at conversion. Flannery (2002) labeled this kind of contingent convertible bonds as "reverse convertible debentures" (RCD) and we will adopt this name:

1. **FSCCB:** Conversion value in terms of a fixed number of shares.

   Here, the number of shares granted in exchange for a contingent convertible bond are fixed in the contract. Therefore, the conversion value depends on the stock market price at the time of conversion. We will model the stock price endogenously as a fraction of the equity. There are basically two ways to choose the fixed number of shares granted at conversion. Either this number is fixed at time zero without any reference to other prices; in this case we obtain a unique price for the FSCCBs. The alternative is to express this number in terms of the stock price $S(t)$ at time $t = 0$. The number of shares granted at conversion for a single CCB will then equal $\frac{\ell}{S(0)}$. The coefficient ℓ is a contract term. The value of the corresponding shares at time τ_C is $S(\tau_C)$. Hence, at the time of conversion bondholders receive equity valued at its market price $\ell \frac{S(\tau_C)}{S(0)}$. However, the stock price at time 0 depends on the features of the CCBs, while the price of the CCBs also depends on the stock price $S(0)$. Hence, the stock price $S(0)$ and the price of CCBs have to be determined in an equilibrium. We will show that for this case there exist in general two equilibrium prices for FSCCBs.

2. **RCD:** Conversion value in terms of the market value of equity.

   Bondholders receive equity valued at its market price in the amount of ℓ at the time of conversion τ_C. The coefficient ℓ is a contract term that determines the fraction of the conversion value to the face value of the convertible bond at the time of conversion. In the following we will assume that the value of equity for $V_t = V_C$ is sufficient to pay the conversion value. This makes sense as the bondholders would only agree on a contract, where it is known a priori that it is possible to fulfill the contractual obligations. There are basically two ways of how the payments at conversion can be specified. In a model without jumps both approaches coincide. Suppose that conversion is triggered by a jump in V_t that crosses the conversion barrier V_C. In the first approach, we assume that the number of shares granted to the contingent convertible bondholders is determined as if the firm's value process first touches V_C, conversion takes place at this time and then the firm's value process jumps to the value V_{τ_C}. This means the number of shares granted at conversion for a single bond with face value 1 is $n' = \frac{\ell}{S(V_C)}$, where $S(V_C)$ denotes the stock price given the firm's value process is equal to V_C. The value of the payment is then $n' \cdot S(\tau_C)$. We can think of $S(V_C)$ as a hypothetical stock price and show that it is known at time zero when the contract is written. Hence, the number n' is a constant and this kind of RCD contract is actually a FSCCB contract. We will label it RCD1.

   In the second approach the number of shares granted is determined based on the stock price $S(\tau_C) = S(V_{\tau_C})$, i.e. $n' = \frac{\ell}{S(V_{\tau_C})}$. In this case conversion takes place after the firm's value process has crossed the conversion barrier. If the crossing happens by a jump, $S(V_{\tau_C})$ is smaller than $S(V_C)$. This implies that the number n' is a random variable, which makes the contract different from the above FSCCBs. We use the name RCD2 for this second specification. In this paper we have solved the model for both cases. The first approach has the advantage that RCDs and FSCCBs can be incorporated into the same framework. Hence, in the main part of the text we focus on this contract specification. In the Appendix we present the detailed solution to the second approach.

In analogy to the normal debt case we denote by P_C the total value of the convertible debt. The fixed coupon paid by a unit contingent convertible debt is c_C and the total amount of the coupon payments equals C_C. The maturity profile of the contingent convertible and the normal debt is the same, but it is straightforward to relax this assumption. In summary we have the following equations for a bond d_D with face value 1 and a contingent convertible bond d_C with face value 1:

$$\tau = \inf(t \in (0,\infty) : V_t \leq V_B) \qquad \tau_C = \inf(t \in (0,\infty) : V_t \leq V_C)$$

$$P_D = p_D \int_0^\infty \Psi(s)ds = \frac{p_D}{m} \qquad P_C = p_c \int_0^\infty \Psi(s)ds = \frac{p_C}{m}$$

$$C_D = c_D P_D \qquad C_C = c_C P_C.$$

Implicitly, we make the following assumption:

Assumption 1. *The conversion level is always equal or larger than the bankruptcy level:*

$$V_C \geq V_B.$$

If contingent convertible debt is to be used as a regulation instrument, it is sensible to make the even stronger assumption that $V_C > V_B$. In the case where $V_C \leq V_B$ the contingent convertible debt degenerates to straight debt without any recovery payment.

The pricing structure of contingent convertible bonds is similar to that of straight debt bonds: The price consists of the net present value of the coupon payments until conversion, the net present value of the firm's repayment if conversion does not occur and finally the conversion value if conversion happens before maturity. The two different types of CCBs considered in this paper distinguish themselves only in the conversion value. The value of a single contingent convertible bond with face value 1 and maturity t equals

$$d_C(V, V_B, V_C, t) = E\left[\int_0^{t\wedge\tau_C} c_C e^{-rs} ds\right] + E\left[e^{-rt}\mathbb{1}_{\{t<\tau_C\}}\right] + conv(V, V_B, V_C, t)$$

where $conv(V, V_B, V_C, t)$ is the conversion value of the respective bond. Following the same argument as for straight debt we obtain the following proposition:

Proposition 3. *The total value of all outstanding convertible debt equals*

$$\begin{aligned} CB(V, V_B, V_C) &= \int_0^\infty p_C \Psi(t) d_C(V, V_B, V_C, t)dt \\ &= \left(\frac{c_C P_C + m P_C}{m+r}\right) E\left[1 - e^{-(m+r)\tau_C}\right] + CONV(V, V_B, V_C) \end{aligned}$$

where $CONV$ is the total conversion value

$$CONV(V, V_B, V_C) = \int_0^\infty p_C \cdot \Psi(t) \cdot conv(V, V_B, V_C, t)dt$$

Proof. See Appendix 12.2 □

We have decided to use the terminology "conversion value" instead of "conversion payment" for $CONV$. The reason is that $CONV$ is not an actual cash payment like the coupon payments or the repayment of the principal value if conversion does not take place. The conversion value represents a redistribution of the equity value among shareholders in the event of conversion. This perspective is important when evaluating the value of the equity. A more appropriate but less practicable name for $CONV$ would be "the total value of the shares granted to contingent convertible bondholders at conversion".

3.2 Consistency and Equilibrium Requirements

The total tax benefits are the sum of the tax benefits of the straight debt and the tax benefits of the contingent convertible debt:

$$\begin{aligned} TB(V_t, V_B, V_C) &= TB_D(V_t, V_B, V_C) + TB_C(V_t, V_B, V_C) \\ &= \frac{\bar{c}C_D}{r} E\left[1 - e^{-r\tau}\right] + \frac{\bar{c}C_C}{r} E\left[1 - e^{-r\tau_C}\right] \end{aligned}$$

where $C_D = c_D P$ and $C_C = c_C P_C$ denote the total values of the coupon payments and $\bar{c}$ is the tax rate. As before coupon payments are tax deductible. The bankruptcy cost are

$$BC(V_t, V_B) = \alpha E\left[V(\tau)e^{-r\tau}\mathbb{1}_{\{\tau<\infty\}}\right].$$

The total value of the firm equals

$$G = V_t + TB - BC.$$

The value of the equity consists of the total value of the firm minus the payments which the equity holders have to make to the bondholders. The payments to the holders of straight debt have a different structure than the payments to contingent convertible bondholders. The value of a contingent convertible bond could be split into two parts: First, the value of the coupon payments and the repayment of the principal value if conversion does not take place. These are actual cash payments. Second, the value of the shares granted at conversion. The conversion shares given to the contingent convertible bondholders are not a cash payment but represent a redistribution among shareholders. The value of the conversion shares depends on the value of the equity as a whole. For the old shareholders it represents an actual cost, but it does not change the value of the equity as a whole. The total equity will be only affected by the value of actual cash payments. Hence, the conversion value for the various contingent convertible bonds $CONV(V, V_B, V_C)$ does not directly enter the valuation formula for the total equity. The term $CB - CONV$ equals the principal value paid in the case of no conversion and the value of the coupon payments of the contingent convertible bonds. We define the total equity as

$$\begin{aligned} &EQ(V_t, V_B, V_C) \\ =&V_t + TB(V_t, V_B, V_C) - D(V_t, V_B) - (CB(V_t, V_B, V_C) - CONV(V_t, V_B, V_C)) - BC(V_t, V_B). \end{aligned}$$

Note that at any time t the following budget equation has to hold:

$$V_t + TB = EQ + D + CB - CONV + BC.$$

In Section 3.3.2 we introduce dilution costs $DC(V_t, V_B, V_C)$. These are the costs that the old shareholders have to bear because the claim on equity will be distributed among more shareholders after the conversion. At the time of conversion there is only one value transfer: The contingent convertible bondholders receive the conversion value $CONV$ and the old equity holders suffer from a loss in value equal to the dilution costs DC. As there is no other value created or destroyed the budget equation requires the dilution costs to equal the conversion value.

Lemma 1. *The dilution costs DC coincide with the conversion value $CONV$.*

After having specified the concrete form of the dilution costs and the conversion value, we will verify in Section 3.3.2 that the above lemma holds in our model. In this sense our model is consistent. Denote by

$$EQ_{old}(V_t, V_B, V_C) = EQ(V_t, V_B, V_C) - DC(V_t, V_B, V_C)$$

the value of the equity for the old shareholders.[3] By Lemma 1 this can be written as

$$EQ_{old}(V_t, V_B, V_C) = V_t + TB(V_t, V_B, V_C) - BC(V_t, V_B) - D(V_t, V_B) - CB(V_t, V_B, V_C)$$

In a model without contingent convertible debt the equity holders choose the conversion barrier V_B such that it maximizes the value of the equity subject to the constraint that the value of the equity is strictly positive for a firm's value process larger than V_B. In a model including contingent convertible debt the default decision before conversion is made by the old shareholders. However, if default does not happen before conversion, the contingent convertible bondholders become equity holders as well. The optimal default barrier for the old and new equity holders after conversion can be different than for the old shareholders before conversion. The fact, that the old shareholders cannot commit to their optimal choice before conversion, will be called the commitment problem. Hence, for a given amount of debt P_D and P_C the old shareholders will choose a default barrier that maximizes the value of their equity subject to the limited liability constraint and the commitment problem:

$$\max_{V_B} EQ_{old}(V_t, V_B) \qquad \text{s.t. } EQ_{old}(V', V_B) > 0 \text{ for all } V' > V_B \text{ and s.t. the commitment problem.}$$

In Section 5 we will formulate the problem formally and derive a general solution to it. For a given level of debt P_D and P_C, the coupon values will be determined at time $t = 0$ such that all the debt sells at par:

$$P = D(V_0, V_B, P_D, P_C, C_D, C_C) \tag{3.1}$$

$$P_C = CB(V_0, V_B, V_C, P_D, P_C, C_D, C_C). \tag{3.2}$$

As the variables V_B, C_D and C_C are determined endogenously, the remaining choice variables are $V_C, P_D, P_C, m, \ell, \bar{c}, V_0$ and r. In the following we will usually suppress the dependence of the functions on all the parameters and use a short-hand notation where we only implicitly write the dependence on the variables of interest.

[3]The labeling "total equity" for EQ and "equity for the old shareholders" for EQ_{old} is our own notation. In other papers, e.g. Albul, Jaffee, Tchistyi (2010), the equity for the old shareholders is just called equity.

3.3 Modeling the Stock Price Process

3.3.1 Stock Price Process without CCBs

A single stock is a claim on a fixed portion of the equity of a firm. Hence, we can define the stock price process as the value of the equity for the old shareholders divided by the number of shares.

Definition 2. *If the capital structure of a firm includes only straight debt, but no contingent convertible debt, the stock price is defined as*

$$S_t = S(V_t) = \frac{EQ_{debt}(V_t)}{n}$$

where n is the number of shares $n = EQ(V_0)_{debt}/S(0)$.

3.3.2 Dilution Costs

Shareholders do not only profit from the additional tax benefits from issuing contingent convertible bonds, but also face the risk of dilution. This creates a tradeoff. In more detail, at the time of conversion all the cash payments of the contingent convertible bonds, i.e. the coupon payments and the repayment of the face value, fall to zero. Hence, the total value of the equity of a firm at conversion is the same as the total value of the equity of an identical firm that did not issue any CCBs. At the time of conversion holders of contingent convertible bonds become equity holders. As this implies that new shares are issued, the value of the shares of the old shareholders decreases: "The size of the cake stays the same, but is divided among more people." Here we want to model the dilution costs for the old shareholders. Denote by n the number of shares of the old shareholders and by n' the number of shares of all the new shareholders after conversion. Hence the costs of dilution DC are defined as

$$DC(V_t, V_B, V_C) = E\left[\frac{n'}{n+n'} EQ(V_{\tau_C}) e^{-(r+m)(\tau_C - t)} \mathbb{1}_{\{\tau_C < \infty\}} \mathbb{1}_{\{\tau_C < \tau\}} | \mathfrak{F}_t \right].$$

DC corresponds to the value of the shares of the new shareholders. It is calculated as the present value of their fraction of the total equity weighted by the maturity profile. Hence, the value of the equity for the old shareholders equals

$$EQ_{old}(V_t, V_B, V_C) = EQ(V_t, V_B, V_C) - DC(V_t, V_B, V_C).$$

The main difference between our two different contingent convertible bonds is the number of shares granted to the bondholders.

3.3.3 Stock Price Process with CCBs

If the capital structure of a firm includes debt and CCBs we define the stock price in the following way:

Definition 3. *If Assumption 1 is satisfied, the endogenous stock price process is defined as*

$$S_t = S(V_t) = \begin{cases} \frac{EQ(V_t) - DC(V_t)}{n} & \text{if } t < \tau_C \\ \frac{EQ(V_t)}{n+n'} & \text{if } \tau_C \le t < \tau \\ 0 & \text{if } t \ge \tau \end{cases}$$

where n is the number of "old" shares

$$n = \frac{EQ(V_0) - DC(V_0)}{S_0}$$

and n' is the number of "new" shares issued at conversion.[4]

Note that $EQ_{old}(V_t) = EQ(V_t) - DC(V_t)$ is just the value of the equity for the old shareholders.

3.4 Pricing CCBs

3.4.1 Pricing FSCCBs

The old shareholders own a number of shares that is equal to the value of equity to them divided by the price of the stock at time $t = 0$:

$$n = \frac{EQ(V_0) - DC(V_0)}{S_0}.$$

The number n is fixed at time zero. Note, that the stock price S_0 has to be determined endogenously and will depend on the features of the CCBs. Assume first, that n' is fixed and does not depend on S_0. The contingent convertible bondholders receive a fixed number of shares at conversion if and only if conversion and bankruptcy do not happen at the same time. This condition is captured by $\tau_C < \tau$ or equivalently $V_{\tau_C} > V_B$.

Proposition 4. *If the value of the shares, that holders of a single contingent convertible bond with face value 1 receive at conversion, is $n'S(\tau)/P_C$, then the value of the individual bond satisfies*

$$d_C(V, V_B, V_C, t) = E\left[\int_0^{t\wedge\tau_C} c_C e^{-rs} ds\right] + E\left[e^{-rt}\mathbb{1}_{\{t<\tau_C\}}\right] + \frac{n'}{P_C} E\left[S(\tau_C)e^{-r\tau_C}\mathbb{1}_{\{\tau_C\leq t\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right].$$

Under the assumption of an exponential maturity profile $\varphi(t) = me^{-mt}$ the total value of the convertible debt CB is given by:

$$CB(V, V_B, V_C) = \left(\frac{c_C P_C + mP_C}{m + r}\right) E\left[1 - e^{-(m+r)\tau_C}\right] + n'E\left[S(\tau_C)e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right].$$

Proof. See Appendix 12.2 □

In Lemma 1 we have already noted that $CONV(V_0) = DC(V_0)$. Hence, instead of evaluating the conversion value, we will focus on the dilution costs, which are equal to

$$DC(V_t) = \frac{n'}{n + n'} E\left[EQ(V_{\tau_C})e^{-(r+m)(\tau_C - t)}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}|\mathfrak{F}_t\right] \tag{3.3}$$

We will evaluate this expression analytically in Section 4.3.1.

The value of the equity $EQ(V_{\tau_C})$ is independent of the contingent convertible bonds as conversion has already taken place. We confirm that the consistency result in Lemma 1 is satisfied for our choice of the stock price process.

[4]The definition of the stock price implicitly assumes that no new shares can be issued before conversion, i.e. the number of old shares n at time $t = 0$ is the same as the number of old shares n at conversion $t = \tau_C$. This restrictive assumption is only needed for the evaluation of FSCCBs. For the evaluation of RCD1 and RCD2 contracts and all the other results in this paper, this assumption is not necessary and can be relaxed.

Corollary 2. *The conversion value equals the dilution costs:*

$$CONV(V_t) = DC(V_t)$$

Proof. Plugging in the definition of $S(t)$ yields

$$\begin{aligned} CONV(V_0) =& n' E\left[S(\tau_C)e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] \\ =& \frac{n'}{n+n'}E\left[EQ(V_{\tau_C})e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\tau\}}\mathbb{1}_{\{\tau_C<\infty\}}\right] = DC(V_0) \end{aligned}$$

□

Assume that the number n' of total shares granted at conversion is expressed in terms of the stock price. If a single FSCCBs gives a bondholder ℓ/S_0 shares, then the total number of shares equals

$$n' = \frac{P_C \ell}{S_0} = \frac{n\ell P_C}{EQ(V_0) - DC(V_0)}.$$

This is an intuitive way of specifying the contract as it just says how much of the face value of debt are convertible bondholders going to get in terms of the current stock price if conversion takes place. However, introducing CCBs into the capital structure changes the stock price. As soon as the agents in the economy anticipate that contingent convertible capital will be issued they will discount the current stock price by the dilution costs. As the dilution costs and the stock price at time 0 are interlinked variables, specifying n' in terms of S_0 will in general lead to two possible equilibrium prices for FSCCBs. Therefore, it is undesirable to specify the conversion value of FSCCBs in terms of the stock price S_0, and one should avoid such a contract design.

Proposition 5. *If $n' = \frac{P_C \ell}{S_0}$, then there exist two different combinations of prices for $\{S_0, DC(0)\}$, which satisfy the consistency and equilibrium conditions for FSCCBs. The dilution costs $DC(V_0)$ at time $t = 0$ for a contingent convertible bond with such a fixed number of shares equal:*

$$DC(V_0) = \frac{EQ(V_0) + n'S_0}{2} \pm \sqrt{\left(\frac{EQ(V_0) + n'S_0}{2}\right)^2 - E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] n'S_0}.$$

Proof. See Appendix 12.2 □

In order to rule out multiple equilibrium prices, the FSCCB contract has to be specified such that n and n' are fixed a priori and chosen independently of S_0. After the equilibrium stock price is realized, we can determine a posteriori for which contract parameter ℓ the condition $n' = \frac{\ell P_C}{S_0}$ is satisfied. In our numerical analysis we will refer to n' as a fraction of the face value of debt in terms of the current market value of equity, because it is easier to interpret. However, this will not be the contract definition but an a posteriori consequence of the contract specification.

3.4.2 Pricing RCDs

The conversion value of FSCCBs depends on the stock market price at conversion. RCDs are designed to avoid this uncertainty and the conversion value is defined as a fraction ℓ of the face value of convertible debt. There are basically two ways of how to conceptualize the conversion value for RCDs. In the first case, which we will label RCD1, we introduce the hypothetical stock price $S(V_C)$, which would be realized if the firm's value process only touches but does not jump over the conversion barrier V_C. The total number of shares granted to the bondholders equals $n' = \frac{\ell P_C}{S(V_C)}$. Given certain assumptions about the dynamics of the firm's value process the value $S(V_C)$ is uniquely defined and known a priori at time 0. Hence, RCD1s are actually a specific version of FSCCB contracts. Although the stock price appears in the definition of n' we will show that there exists a unique equilibrium price. RCD1 have the advantage that they are relatively easy to implement.

In the second approach, labeled as RCD2, we try to take into consideration the fact that if conversion is triggered by a jump in the firm's value process, the stock price at conversion $S(V_{\tau_C}) = S_{\tau_C}$ is lower than the hypothetical stock price $S(V_C)$. In particular, it can be so low that the value of the equity is not sufficient to make the promised payment. If the value of the equity after conversion is sufficiently large, the bondholder get $n' = \frac{\ell P_C}{S_{\tau_C}}$ shares, otherwise they take possession of the whole firm. Note, that n' is a random variable in this case, which makes RCD2 different from FSCCB contracts. Because of the possibility of complete dilution at conversion the stock price process has to be modeled differently for RCD2s than for the other contracts. Most of the derivations for RCD2s are explained in Appendix 12.3.

RCD1

In the case of RCD1s we make the assumption that the value of the equity at conversion is sufficient to make the promised payment. Relaxing this assumption is straightforward as the agents would just price in the additional risk. However, it seems sensible that the agents would usually only agree on a contract where it is known a priori that the contractual obligations can be fulfilled.

Assumption 2. *The parameters of the RCD1 contract are chosen such that*

$$EQ(V_C) \geq \ell P_C,$$

i.e. the equity value at conversion is sufficient to give shares to the bondholders with a value equal to the promised payment.

Proposition 6. *If the value of the shares given to holders of contingent convertible bonds at conversion is ℓ, the values of the individual bonds of RCD1 under Assumption 2 satisfy*

$$d_C(V, V_B, V_C, t) = E\left[\int_0^{t\wedge\tau_C} c_C e^{-rs} ds\right] + E\left[e^{-rt}\mathbb{1}_{\{t<\tau_C\}}\right] + \ell E\left[\frac{S(\tau_C)}{S(V_C)} e^{-r\tau_C}\mathbb{1}_{\{\tau_C\leq t\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right].$$

The total value of the convertible debt RCD1 for an exponential maturity profile $\varphi(t) = me^{-mt}$ under Assumption 2 is given by:

$$CB(V, V_B, V_C) = \left(\frac{c_C P_C + mP_C}{m+r}\right) E\left[1 - e^{-(m+r)\tau_C}\right] + \ell P_C E\left[\frac{EQ(V_{\tau_C})}{EQ(V_C)} e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right].$$

Proof. See Appendix 12.2 □

Note, that the value of the equity after conversion is independent of any features of the contingent convertible debt. Hence, we obtain a unique equilibrium price for RCD1 contracts although the number of new shares n' is defined with respect to some stock price.

RCD2

For RCD2s we do not require Assumption 2. The conversion value for RCD2s requires us to distinguish several cases. If $\tau_C < \tau$, i.e. the downward movement of V_{τ_C} is not sufficient to trigger bankruptcy, the contingent convertible bondholders receive a payment. If on the one hand the value of the equity is sufficiently large, they get a number of stocks such that the value of the total payment equals ℓP_C. If on the other hand the value of the equity is insufficient to make the promised payment to the contingent convertible bond holders, they take possession of the whole equity and the old shareholders are completely diluted out. We assume that the face value of all contingent convertible debt is P_C and thus a bondholder with a bond with face value 1 gets a fraction $1/P_C$ of the value of the equity $EQ(V_{\tau_C})$ after conversion in this case.

Proposition 7. *If the payment to holders of contingent convertible bonds at conversion is ℓ, the values of the individual bonds of RCD2 satisfy*

$$
\begin{aligned}
d_C(V,V_B,V_C,t) =& E\left[\int_0^{t\wedge\tau_C} c_C e^{-rs} ds\right] + E\left[e^{-rt}\mathbb{1}_{\{t<\tau_C\}}\right] + \ell E\left[e^{-r\tau_C}\mathbb{1}_{\{\tau_C\le t\}}\mathbb{1}_{\{\ell P_C\le EQ(V_{\tau_C})\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] \\
&+ \frac{1}{P_C}E\left[e^{-r\tau_C}EQ(V_{\tau_C})\mathbb{1}_{\{\tau_C\le t\}}\mathbb{1}_{\{\tau_C<\tau\}}\mathbb{1}_{\{\ell P_C>EQ(V_{\tau_C})\}}\right].
\end{aligned}
$$

For an exponential maturity profile $\varphi(t) = me^{-mt}$ the total value of the convertible debt RCD2 is given by:

$$
\begin{aligned}
CB(V,V_B,V_C) =& \left(\frac{c_C P_C + mP_C}{m+r}\right) E\left[1-e^{-(m+r)\tau_C}\right] + \ell P_C E\left[e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\tau\}}\mathbb{1}_{\{\ell P_C\le EQ(V_{\tau_C})\}}\right] \\
&+ E\left[EQ(V_{\tau_C})e^{-(m+r)\tau_C}\mathbb{1}_{\{V(\tau_C)>V_B\}}\mathbb{1}_{\{\ell P_C>EQ(V_{\tau_C})\}}\mathbb{1}_{\{\tau_C<\infty\}}\right].
\end{aligned}
$$

Proof. See Appendix 12.3 □

A detailed treatment of RCD2s is provided in Appendix 12.3. In Appendix 12.11 we show that the two contracts RCD1 and RCD2 are identical if no jumps are included in the firm's value process. The idea is that without jumps the firm's value process has to touch the conversion barrier at the time of conversion, i.e. $V_{\tau_C} = V_C$. As a consequence $EQ(V_{\tau_C}) = EQ(V_C)$ and thus $S(V_{\tau_C}) = S(V_C)$. Under Assumption 2 the number of shares granted at conversion is $\frac{\ell P_C}{S(V_{\tau_C})}$ for RCD2s and $\frac{\ell P_C}{S(V_C)}$ for RCD1s. As both numbers coincide, the two contracts are the same.

In Appendix 12.4 we compare RCD and FSCCB contracts in terms of the number of shares n' granted at conversion for different conversion parameters ℓ.

4 Evaluating the Model

4.1 Dynamics of the Firm's Value Process

Now we make some explicit assumptions about the martingale in (2.1) and assume that it is a Lévy process. Thus V can be expressed as

$$V_t = V_0 \exp(X_t) \tag{4.1}$$

where X is a Lévy process. The key tool for our analysis will be the Laplace exponent of X. The moment generating function of a Lévy process is of the form

$$E[\exp(zX_t)] = \exp(t\psi(z)) \tag{4.2}$$

for some function ψ being analytic in the interior of its domain of definition. The Lévy-Khintchine representation theorem characterizes a Lévy process, identifying a drift term, a Brownian motion component and a jump component:

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zy\mathbb{1}_{\{|y|<1\}})\nu(dy)$$

where b is the drift, σ corresponds to the Brownian motion component and ν is the Lévy measure identifying the jumps. The roots of $\psi(z) = \lambda$ will be of importance in the following. We can now define the first passage times in terms of X: τ is the first hitting time defined as $\tau = \tau_x = \inf(t \geq 0 : X(t) \leq x)$ with $x = \log(V_B/V)$ and $\tau_C = \tau_{x_C} = \inf(t \geq 0 : X(t) \leq x_C)$ with $x_C = \log(V_C/V)$.

For a general Lévy process it is very difficult to characterize the distribution of the first passage times. Following Chen and Kou (2009) we propose a two-sided jump model for the evolution of the firm's assets with a double exponential jump diffusion process. The main advantage of the double exponential distribution is that it leads to an analytical solution for various Laplace transforms of the first passage times. Due to the conditional memoryless property of the exponential distribution we can also analytically evaluate the Laplace transform of the conversion time for jumps that are too small to trigger default. We assume that under the risk-neutral measure the value of the firm's assets V follows

$$dV_t = V_t \left((r-\delta)dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t}(Z_i - 1)\right) - \lambda\xi dt \right)$$

where N is a Poisson process with constant intensity rate λ. Z_i are i.i.d. random variables and the $Y_i = \log(Z_i)$ possess a double exponential density:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y}\mathbb{1}_{\{y\geq 0\}} + q\eta_2 e^{\eta_2 y}\mathbb{1}_{\{y<0\}}$$

where η_1, η_2, p and q are positive numbers and $p+q=1$. The parameters p and η_1 correspond to the upward jumps and q and η_2 to the downward jumps respectively. The mean percentage jump size ξ is given by

$$\xi = E[Z-1] = E[e^Y - 1] = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1.$$

The sources of randomness N, W and Y are assumed to be independent. In order to ensure that $\xi < \infty$, we assume that $\eta_1 > 1$. This condition implies that the average upward jump cannot exceed 100%, which is reasonable in reality. Applying Itô's lemma for jump diffusions yields

$$V_t = V_0 \exp(X_t) = V_0 \exp\left(bt + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right).$$

where $b = r - \delta - \frac{1}{2}\sigma^2 - \lambda\xi$ and for $z \in (-\eta_2, \eta_1)$ the Lévy-Khintchine formula is given by

$$\begin{aligned}\psi(z) &= bz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1)\lambda f_Y(y)dy \\ &= bz + \frac{1}{2}\sigma^2 z^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - z} + \frac{q\eta_2}{\eta_2 + z} - 1\right).\end{aligned}$$

Note that under the above risk-neutral measure V_t is a martingale after proper discounting, i.e.

$$V_t = E\left[e^{-(r-\delta)(T-t)} V_T | \mathfrak{F}_t\right]$$

where $\mathfrak{F}_t$ is the information up to time t. Kou and Wang (2003) prove the following results:

Lemma 2. *For any $\rho > 0$ it holds*

$$E\left[e^{-\rho\tau}\right] = \frac{\eta_2 - \beta_{3,\rho}}{\eta_2}\frac{\beta_{4,\rho}}{\beta_{4,\rho} - \beta_{3,\rho}} e^{x\beta_{3,\rho}} + \frac{\beta_{4,\rho} - \eta_2}{\eta_2}\frac{\beta_{3,\rho}}{\beta_{4,\rho} - \beta_{3,\rho}} e^{x\beta_{4,\rho}}$$

where τ denotes the first passage time of X_t to x and $-\beta_{3,\rho} > -\beta_{4,\rho}$ are the two negative roots of the equation $\psi(\beta) = \rho$.

Lemma 3. *For any $\rho > 0$ and $\theta > -\eta_2$ it holds*

$$E\left[e^{-\rho\tau + \theta X_\tau}\mathbb{1}_{\{\tau_x < \infty\}}\right] = e^\theta\left(\frac{\beta_{4,\rho} + \theta}{\eta_2 + \theta}\frac{\eta_2 - \beta_{3,\rho}}{\beta_{4,\rho} - \beta_{3,\rho}} e^{x\beta_{3,\rho}} + \frac{\beta_{3,\rho} + \theta}{\eta_2 + \theta}\frac{\beta_{4,\rho} - \eta_2}{\beta_{4,\rho} - \beta_{3,\rho}} e^{x\beta_{4,\rho}}\right)$$

where τ denotes the first passage time of X_t to x and $-\beta_{3,\rho} > -\beta_{4,\rho}$ are the two negative roots of the equation $\psi(\beta) = \rho$.

For the evaluation of contingent convertible bonds we need to consider the case where conversion occurs but the jumps are not large enough to trigger bankruptcy. For this reason we show the following proposition.

Proposition 8. *Assume that X_t follows a Kou process and τ denotes the first passage time to $x < 0$, i.e. $\tau = \inf(0 \leq t : X_t \leq x)$. It holds that for $y > 0$, $\theta > -\eta_2$ and $\rho > 0$:*

$$\begin{aligned}&E\left[e^{-\rho\tau + \theta X_\tau}\mathbb{1}_{\{\tau < \infty, -(X_\tau - x) < y\}}\right] \\ &= \left(\frac{\eta_2 - \beta_{3,\rho}}{\beta_{4,\rho} - \beta_{3,\rho}} e^{x\beta_{3,\rho}} + \frac{\beta_{4,\rho} - \eta_2}{\beta_{4,\rho} - \beta_{3,\rho}} e^{x\beta_{4,\rho}}\right) e^{\theta x} + e^{\theta x}\frac{\eta_2}{\theta + \eta_2}\frac{\left(1 - e^{-(\theta + \eta_2)y}\right)}{(1 - e^{-\eta_2 y})} \\ &\cdot\left(\frac{e^{x\beta_{3,\rho}}}{\beta_{4,\rho} - \beta_{3\rho}}\left(\frac{\eta_2 - \beta_{3,\rho}}{\eta_2}\beta_{4,\rho} - (\eta_2 - \beta_{3,\rho}) - e^{-\eta_2 y}\frac{(\eta_2 - \beta_{3,\rho})(\beta_{4,\rho} - \eta_2)}{\eta_2}\right)\right. \\ &\left. + \frac{e^{x\beta_{4,\rho}}}{\beta_{4,\rho} - \beta_{3,\rho}}\left(\frac{\beta_{4,\rho} - \eta_2}{\eta_2}\beta_{3,\rho} - (\beta_{4,\rho} - \eta_2) + e^{-\eta_2 y}\frac{\eta_2 - \beta_{3,\rho}}{\eta_2}(\beta_{4,\rho} - \eta_2)\right)\right)\end{aligned}$$

where $-\beta_{3,\rho} > -\beta_{4,\rho}$ are the two negative roots of the equation $\psi(\beta) = \rho$.

Proof. See Appendix 12.5 □

Definition 4. *The function J is defined as*

$$J(x,\theta,y,\rho) = E\left[e^{-\rho\tau+\theta X_\tau}\mathbb{1}_{\{\tau<\infty,-(X_\tau-x)<y\}}\right],$$

where $x<0$, $\theta>-\eta_2$, $y>0$, $\rho>0$ and $\tau=\inf(t\geq 0: X_t\leq x)$. The explicit form of $J(x,\theta,y,\rho)$ is given in Proposition 8.

Corollary 3. *Assume that X_t follows a Kou process and τ denotes the first passage time to $x<0$, i.e. $\tau=\inf(t\geq 0: X_t\leq x)$. It holds that for $y>0$ and $\rho>0$*

$$\begin{aligned} E\left[e^{-\rho\tau}\mathbb{1}_{\{-(X_\tau-x)<y\}}\right] =& \frac{e^{x\beta_{3,\rho}}}{\beta_{4,\rho}-\beta_{3\rho}}\left(\frac{\eta_2-\beta_{3,\rho}}{\eta_2}\beta_{4,\rho}-e^{-\eta_2 y}\frac{(\eta_2-\beta_{3,\rho})(\beta_{4,\rho}-\eta_2)}{\eta_2}\right) \\ &+\frac{e^{x\beta_{4,\rho}}}{\beta_{4,\rho}-\beta_{3,\rho}}\left(\frac{\beta_{4,\rho}-\eta_2}{\eta_2}\beta_{3,\rho}+e^{-\eta_2 y}\frac{\eta_2-\beta_{3,\rho}}{\eta_2}(\beta_{4,\rho}-\eta_2)\right). \end{aligned}$$

Definition 5. *The function G is defined as*

$$G(x,y,\rho) = E\left[e^{-\rho\tau}\mathbb{1}_{\{-(X_\tau-x)<y\}}\right],$$

where $x<0$, $y>0$, $\rho>0$ and $\tau=\inf(t\geq 0: X_t\leq x)$. The explicit form of $G(x,y,\rho)$ is given in Corollary 3.

4.2 Evaluation of the Debt

Now it is straightforward to derive an expression for the firm's debt D. The following three propositions are shown in Chen and Kou (2009).

Proposition 9. *The value of the firm's debt equals*

$$\begin{aligned} D =& \frac{C_D+mP_D}{r+m}\left(1-\frac{\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_B}{V}\right)^{\beta_{3,r+m}}-\frac{\beta_{3,r+m}}{\eta_2}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_B}{V}\right)^{\beta_{4,r+m}}\right) \\ &+(1-\alpha)V_B\left(\frac{\beta_{4,r+m}+1}{\eta_2+1}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_B}{V}\right)^{\beta_{3,r+m}}+\frac{\beta_{3,r+m}+1}{\eta_2+1}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_B}{V}\right)^{\beta_{4,r+m}}\right) \end{aligned}$$

where $-\beta_{3,\rho}>-\beta_{4,\rho}$ are the only two negative roots of $\psi(\beta)=\rho$ and the value is independent of the specification of the contingent convertible bonds.

Proposition 10. *If the capital structure does not include any contingent convertible debt the total value of the firm G_{debt} equals*

$$\begin{aligned} G_{debt} &= V+\frac{\bar{c}C}{r}\left(1-\frac{\beta_{4,r}}{\eta_2}\frac{\eta_2-\beta_{3,r}}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_B}{V}\right)^{\beta_{3,r}}-\frac{\beta_{3,r}}{\eta_2}\frac{\beta_{4,r}-\eta_2}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_B}{V}\right)^{\beta_{4,r}}\right) \\ &\quad -\alpha V_B\left(\frac{\beta_{4,r}+1}{\eta_2+1}\frac{\eta_2-\beta_{3,r}}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_B}{V}\right)^{\beta_{3,r}}+\frac{\beta_{3,r}+1}{\eta_2+1}\frac{\beta_{4,r}-\eta_2}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_B}{V}\right)^{\beta_{4,r}}\right). \end{aligned}$$

Recall that if the capital structure does not include any contingent convertible debt the value of the equity of the firm EQ_{debt} is the difference between the total value of the firm and the value of its debt:

$$EQ_{debt}(V, V_B) = G_{debt}(V, V_B) - D(V, V_B).$$

Proposition 11. *For all $V > V_B$ and $V_B > V_B^*$ the function $EQ_{debt}(V, V_B)$ is a strictly increasing function in V*

$$\frac{\partial EQ_{debt}(V, V_B)}{\partial V} > 0$$

where V_B^* is the optimal default barrier as defined in Section 5.

As we will show in Section 5 later, the smallest possible default barrier, that we need to take into consideration is V_B^*. Therefore, Proposition 11 is general enough for our purposes. In order to evaluate the conversion value, we need to calculate the value of the equity at the time of conversion $EQ(V_{\tau_C})$.

Lemma 4. *The value of the equity at conversion $EQ(V_{\tau_C})$ satisfies*

$$EQ(V_{\tau_C}) = EQ_{debt}(V_{\tau_C}) = \sum_i \alpha_i V_{\tau_C}^{\theta_i} = \sum_i V_0^{\theta_i} \alpha_i e^{X(\tau_C)\theta_i}$$

The coefficients α_i and θ_i are defined in Appendix 12.5.

Proof. See Appendix 12.5. □

Definition 6. *The function $T : (V_B, \infty) \to (0, \infty)$ is defined as*

$$T(V_t) = EQ_{debt}(V_t, V_B)$$

Corollary 4. *The condition $EQ(V_{\tau_C}) \geq \ell P_C$ is equivalent to $V_{\tau_C} \geq T^{-1}(\ell P_C)$.*

Proof. We have seen that $EQ(V_{\tau_C}) = EQ_{debt}(V_{\tau_C}) = T(V_{\tau_C})$. As by Proposition 11 $T(.)$ is strictly increasing, its inverse exists and is strictly increasing as well. □

4.3 Evaluation of the CCBs

4.3.1 Evaluation of FSCCBs

The only difficulty is to evaluate the dilution costs given by equation 3.3.

Lemma 5.

$$E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] = \sum_i \alpha_i V_0^{\theta_i} J\left(\log\left(\frac{V_C}{V_0}\right), \theta_i, \log\left(\frac{V_C}{V_B}\right), r+m\right)$$

where α_i and θ_i are as in Lemma 4.

Proof. The total equity value is the difference between the total value of the firm and the value of actual debt payments: In Lemma 4 we have shown that $EQ(V_{\tau_C})$ has a structure of the form

$$EQ(V_{\tau_C}) = \sum_i \alpha_i V_{\tau_C}^{\theta_i}.$$

Therefore, calculating $E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right]$ boils down to

$$\sum_i E\left[\alpha_i V_{\tau_C}^{\theta_i} e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] = \sum_i \alpha_i V_0^{\theta_i} E\left[e^{\theta_i X_{\tau_C}-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{-(X_{\tau_C}-x_C)<\log(V_C/V_B)\}}\right].$$

□

We conclude:

Theorem 1. *The price of FSCCBs for $t<\tau_C$ equals*

$$\begin{aligned} CB(V_t) =& \frac{C_C+mP_C}{r+m}\left(1-\frac{\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_C}{V_t}\right)^{\beta_{3,r+m}}-\frac{\beta_{3,r+m}}{\eta_2}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_C}{V_t}\right)^{\beta_{4,r+m}}\right) \\ &+\frac{n'}{n+n'}\left(\sum_i \alpha_i V_t^{\theta_i} J\left(\log\left(\frac{V_C}{V_t}\right),\theta_i,\log\left(\frac{V_C}{V_B}\right),r+m\right)\right). \end{aligned}$$

where α_i and θ_i are as in Lemma 4.

Proof. By Proposition 4, Lemma 1 and equation 3.3 the price of the FSCCB is given by

$$\begin{aligned} CB(V,V_B,V_C) = & \left(\frac{c_C P_C + mP_C}{m+r}\right)E\left[1-e^{-(m+r)\tau_C}\right] \\ &+\frac{n'}{n+n'}E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right]. \end{aligned}$$

The first summand corresponds to the value of the coupon payments and the repayment of the face value if conversion does not take place. The second summand is the value of the dilution costs. Applying Lemma 2, we can explicitly calculate the Laplace transformation of the conversion time appearing in the first summand. Lemma 5 gives us an explicit expression for the expectation in the second summand. □

4.3.2 Evaluation of RCDs

RCD1:

Proposition 12. *The price of RCD1s for $t<\tau_C$ equals*

$$\begin{aligned} CB(V_t) =& \frac{C_C+mP_C}{r+m}\left(1-\frac{\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_C}{V_t}\right)^{\beta_{3,r+m}}-\frac{\beta_{3,r+m}}{\eta_2}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_C}{V_t}\right)^{\beta_{4,r+m}}\right) \\ &+\frac{\ell P_C}{\sum_i \alpha_i V_C^{\theta_i}}\left(\sum_i \alpha_i V_t^{\theta_i} J\left(\log\left(\frac{V_C}{V_t}\right),\theta_i,\log\left(\frac{V_C}{V_B}\right),r+m\right)\right). \end{aligned}$$

where α_i and θ_i are as in Lemma 4.

Proof. By Proposition 6 the price of RCD1s is given by

$$CB(V, V_B, V_C) = \left(\frac{c_C P_C + m P_C}{m+r}\right) E\left[1 - e^{-(m+r)\tau_C}\right] + \ell P_C E\left[\frac{EQ(V_{\tau_C})}{EQ(V_C)} e^{-(m+r)\tau_C} \mathbb{1}_{\{\tau_C < \infty\}} \mathbb{1}_{\{V_{\tau_C} > V_B\}}\right].$$

The first summand is the value of the coupon payments and the repayment of the face value if conversion does not happen. The second term corresponds to the conversion value. Lemma 2 allows us to calculate the expectation in the first term. We combine Lemma 4 and Lemma 5 to derive an expression for the second summand. Note, that $EQ(V_C)$ is nonrandom and thus can be taken out of the expectation. □

RCD2:

Theorem 2. *The price of RCD2s for $t < \tau_C$ equals*

$$\begin{aligned}
CB =& \frac{C_C + mP_C}{r+m}\left(1 - \frac{\beta_{4,r+m}}{\eta_2}\frac{\eta_2 - \beta_{3,r+m}}{\beta_{4,r+m} - \beta_{3,r+m}}\left(\frac{V_C}{V_t}\right)^{\beta_{3,r+m}} - \frac{\beta_{3,r+m}}{\eta_2}\frac{\beta_{4,r+m} - \eta_2}{\beta_{4,r+m} - \beta_{3,r+m}}\left(\frac{V_C}{V_t}\right)^{\beta_{4,r+m}}\right) \\
&+ \ell P_C \cdot G\left(\log\left(\frac{V_C}{V_t}\right), \log\left(\frac{V_C}{\max(T^{-1}(\ell P_C), V_B)}\right), m+r\right) \mathbb{1}_{\{V_C > T^{-1}(\ell P_C)\}} \\
&+ \sum \alpha_i V_t^{\theta_i}\left(J\left(\log\left(\frac{V_C}{V_t}\right), \theta_i, \log\left(\frac{V_C}{V_B}\right), m+r\right)\right. \\
&\left. - J\left(\log\left(\frac{V_C}{V_t}\right), \theta_i, \log\left(\frac{V_C}{T^{-1}(\ell P_C)}\right), m+r\right) \mathbb{1}_{\{V_C > T^{-1}(\ell P_C)\}}\right) \mathbb{1}_{\{V_B < T^{-1}(\ell P_C)\}}
\end{aligned}$$

Proof. See Appendix 12.3. □

5 Optimal Default Barrier

Only Straight Debt without Contingent Convertible Debt

Choosing the optimal debt level P_D and the optimal bankruptcy trigger V_B are two entangled problems. When a firm chooses P in order to maximize the total value of the firm at time 0, the decision depends on V_B. Vice versa, the optimal default trigger V_B is a function of the amount of debt P_D. Following Leland (1994a+b) P_D and V_B are chosen according to a two-stage optimization problem. In the first stage, for a fixed P_D, equity holders choose the optimal default barrier by maximizing the equity value subject to the limited liability constraint. In a second stage, the firm determines the amount of debt P_D that maximizes the total value of the firm. In this section, we will focus on the first stage problem. The solution to the whole problem is presented in Section 10. First, we summarize how the first stage problem is solved in the case with only straight debt and in the next subsection we extend the analysis to a firm that issues contingent convertible debt and straight debt. The maximization problem is

$$\max_{V_B} EQ_{debt}(V_t, V_B) \qquad \text{such that } EQ_{debt}(V', V_B) > 0 \qquad \forall V' > V_B.$$

The case of only straight debt was already considered in Chen and Kou (2009):

Proposition 13. *The optimal default barrier without CCBs solves the smooth pasting condition*

$$\left(\frac{\partial(V + TB_D(V,V_B) + BC(V,V_B) - D(V,V_B))}{\partial V}\,|_{V=V_B}\right) = 0$$

and equals:

$$V_B^* = \frac{\frac{C_D+mP_D}{r+m}\beta_{3,r+m}\beta_{4,r+m} - \frac{\bar{c}C_D}{r}\beta_{3,r}\beta_{4,r}}{\alpha(\beta_{3,r}+1)(\beta_{4,r}+1) + (1-\alpha)(\beta_{3,r+m}+1)(\beta_{4,r+m}+1)}\frac{\eta_2+1}{\eta_2}.$$

In the following we want to outline the arguments presented in Chen and Kou (2009).

Proof. Denote by V_B^* the solution to

$$\frac{\partial EQ_{debt}(V,V_B)}{\partial V}\,|_{V=V_B} = 0.$$

By the formula for the equity without contingent convertible debt we can easily verify that

$$V_B^* = \frac{\frac{C_D+mP_D}{r+m}\beta_{3,r+m}\beta_{4,r+m} - \frac{\bar{c}C_D}{r}\beta_{3,r}\beta_{4,r}}{\alpha(\beta_{3,r}+1)(\beta_{4,r}+1) + (1-\alpha)(\beta_{3,r+m}+1)(\beta_{4,r+m}+1)}\frac{\eta_2+1}{\eta_2}.$$

Define $H(V,V_B)$ by

$$H(V,V_B) = \frac{\partial}{\partial V_B}EQ(V,V_B).$$

The proof consists of the following steps:

1. The optimal V_B satisfies $V_B \geq V_B^*$.
2. It holds that $\frac{\partial EQ_{debt}(V,V_B)}{\partial V} \geq 0$ for all $V \geq V_B \geq V_B^*$, i.e. the equity value is increasing in the firm's value.
3. It holds that $H(V,V_B) \leq 0$ for all $V \geq V_B \geq V_B^*$. Hence

   $$EQ_{debt}(V,y_1) \geq EQ(V,y_2) \qquad \text{for all } V_B^* \leq y_1 \leq y_2 \leq V,$$

   i.e. the firm will choose the lowest default barrier that satisfies the non-negativity constraint

First, by definition $EQ_{debt}(V_B^*,V_B^*) = 0$ and by step 2 $EQ_{debt}(V,V_B^*)$ is nondecreasing in V. Thus V_B^* satisfies the non-negativity constraint $EQ_{debt}(V',V_B^*) \geq 0$ for all $V' \geq V_B^*$. Second, any $V_B \in (V_B^*,V]$ cannot yield a higher equity value because of step 3. □

Case 1: Optimal default barrier V_B for exogenous conversion barrier V_C

The optimization problem for the old shareholders changes, when contingent convertible bonds are included in the capital structure. We have to consider two different “notions of equity” here: The value of the equity for the old shareholders equals

$$EQ_{old}(V, V_B, V_C) = V + TB_D + TB_C - BC - D - CCB.$$

The value of the debt excluding any features of the contingent convertible bonds is

$$EQ_{debt}(V, V_B) = V + TB_D - BC - D.$$

The old shareholders will choose the default barrier V_B such the value of their equity EQ_{old} is maximized subject to the constraint, that EQ_{old} has to be nonnegative. There are basically two different solutions to this optimization problem. Either the resulting default barrier V_B is larger than the conversion barrier V_C or smaller. After conversion, the contingent convertible bondholders will become equity holders and the optimization problem of the equity holders is the same as in the case of only straight debt. If the old shareholders decide on a default barrier V_B that is smaller than V_C, they will not be able to commit to it as after conversion V_B^* (see last subsection) will be chosen. However, it is possible that the value of the equity for the old shareholders $EQ_{old}(V, V_B^*, V_C)$ becomes negative for $V > V_C$. As the old shareholders anticipate this, they will choose a default barrier larger than V_C in this case.

More formally, the optimization problem is formulated as follows:

Definition 7. *If the capital structure includes straight debt and contingent convertible bonds, the old shareholders choose V_B to maximize*

$$\max_{V_B} EQ_{old}(V, V_B, V_C) \qquad \text{such that } EQ_{old}(V', V_B, V_C) > 0 \text{ for all } V' > V_B.$$

subject to the commitment problem that $V_B = V_B^*$ if $V_B < V_C$.

We need to clarify what happens in the case where the default and conversion barrier are crossed at the same time for $V_B < V_C$. Passing both barriers simultaneously can occur because a jump that crosses the conversion barrier is large enough to cross the default barrier as well. We treat the crossing as if it happened sequentially. First the conversion barrier is passed and the contingent convertible bondholders become equity holders. For the equity holders, consisting of the old and new shareholders, the optimal default barrier will be V_B^*, but not the former barrier V_B. Therefore the crossing of V_B will not trigger default. Default happens only, if the new barrier V_B^* is passed.

Next, we need to clarify how to treat the case $V_B \geq V_C$. If default happens before conversion, all the payments linked to the contingent convertible bonds are nil as well. Hence, we will assume that value of the payments of a contingent convertible bond are the same as if $V_B = V_C$:

$$CCB(V, V_B, V_C) = CCB(V, V_B, V_B) \qquad \text{for } V_B > V_C$$

How does the commitment problem affect the optimal choice of V_B in this case? If $V_B \geq V_C$ and a jump crosses both barriers at the same time, we treat this case as if the crossing had happened sequentially. First, the default barrier is passed and the firm defaults. Second, the conversion barrier is passed. But as default has already taken place, the value of the equity is zero and the contingent convertible bondholders cannot be compensated with stocks. In particular, the contingent convertible bondholders cannot change the default barrier to V_B^* as in the previous case. If the default barrier and conversion barrier coincide, i.e. $V_B = V_C$, we can either assume that default happens first or that conversion takes place first. We have decided, that first default should take place and after that we deal with the conversion. This is a purely technical convention, which does not affect any of our main results qualitatively, but simplifies the exposition.

We will now show, that there are only two possible solutions to the optimization problem.

Theorem 3. *There are only two possible solutions for the optimal default barrier. Either the optimal default barrier coincides with the optimal default barrier with only straight debt or it equals the maximum of the conversion barrier and* $V_B^{**}: V_B = V_B^*$ *or* $V_B = \max(V_C, V_B^{**})$, *where*

$$V_B^{**} = \frac{\frac{C_D + C_C + m(P_D + P_C)}{r+m}\beta_{3,r+m}\beta_{4,r+m} - \frac{\bar{c}(C_D + C_C)}{r}\beta_{3,r}\beta_{4,r}}{\alpha(\beta_{3,r}+1)(\beta_{4,r}+1) + (1-\alpha)(\beta_{3,r+m}+1)(\beta_{4,r+m}+1)}\frac{\eta_2 + 1}{\eta_2}.$$

V_B^{**} *equals the optimal default barrier of a firm with only straight debt with face value* $P_D + P_C$ *and coupon* $C_D + C_C$. *If*

$$EQ_{old}(V, V_B^*, V_C) \geq EQ_{debt}(V, V_B^{**}, P_D + P_C, C_D + C_C) \qquad \text{for all } V \geq V_C$$

for $V_B^{**} > V_C > V_B^*$ *or*

$$EQ_{old}(V, V_B^*, V_C) \geq 0 \qquad \text{for all } V \geq V_C.$$

for $V_B^* < V_B^{**} \leq V_C$ *then*

$$V_B = V_B^*$$

otherwise

$$V_B = \max(V_C, V_B^{**}).$$

Proof. See Appendix 12.6. □

The intuition behind the proof is the following. V_B^* can only be the optimal default barrier, when it is feasible, i.e. the equity value of the old shareholders is always positive before conversion $EQ_{old}(V, V_B^*, V_C) > 0$ for all $V > V_C$. Even, when it is feasible to choose V_B^*, it may be optimal for the old shareholders to default before conversion. We show that the equity value of the old shareholders for $V_B > V_C$ is the same as for a firm that issues only straight debt in the amount $P_D + P_C$ with coupon $C_D + C_C$: $EQ_{old}(V, V_B, V_C, P_D, P_C, C_D, C_C) = EQ_{debt}(V, V_B, P_D + P_C, C_D + C_C)$. The optimal default barrier for this amount of straight debt is V_B^{**}. If $V_B^{**} > V_C$ and $EQ_{debt}(V, V_B^{**}, P_D + P_C, C_D + C_C) > EQ_{old}(V, V_B^*, V_C)$, then the old shareholders will prefer to default before conversion. However, if $V_B^{**} < V_C$, the old shareholder can get at most $EQ_{debt}(V, V_C, P_D + P_C, C_D + C_C)$ if they decide to default before conversion, i.e. the optimal default barrier before conversion is V_C itself. However, we can show that in this case the old shareholders will always prefer to default after conversion, i.e. take $V_B = V_B^*$ if it is feasible.

We are particularly interested in the case, where the default and conversion barrier are not the same. Hence, we assume

Assumption 3. *The default barrier $V_B^* < V_C$ satisfies the no-early-default condition: For $V_B^{**} \leq V_C$*

$$EQ_{old}(V, V_B^*, V_C) \geq 0 \qquad \text{for all } V \geq V_C$$

and for $V_B^{**} > V_C$

$$EQ_{old}(V, V_B^*, V_C) \geq EQ_{debt}(V, V_B^{**}, P_D + P_C, C_D + C_C) \qquad \text{for all } V \geq V_C.$$

The no-early-default condition is composed of two statements: First, V_B^* is a feasible default barrier, i.e. it satisfies the limited liability constraint. Second, it is never profitable for the old shareholders to default before conversion, i.e. the value of the equity for the old shareholders for V_B^* is always larger than the corresponding value for the optimal default barrier larger than the conversion barrier. Assumption 3 is easily testable for a given amount of debt.

Proposition 14. *If Assumption 3 is satisfied, then the firm chooses the same default barrier as in the case without contingent convertible capital:*

$$V_B = V_B^*.$$

Lemma 6. *For a fixed amount of debt P_D and P_C and fixed coupon values C_D and C_C the default barrier V_B^{**} is always larger than V_B^*.*

Case 2: V_B and V_C chosen optimally

In the previous subsection we assumed that V_C is exogenously given. In this subsection we treat V_C as a choice variable. From a decision theoretical point of view, V_B and V_C are different. V_B is not agreed on explicitly, when the debt is issued. By the very nature of debt, default happens when the cash payments cannot be made anymore. As long as the equity value is positive ($EQ_{old} > 0$), shareholders can (and will) always issue more equity to avoid default. When $EQ_{old} < 0$, the firm defaults. Hence, V_B is agreed on only implicitly as all agents anticipate the shareholders' actions. If the parameters change, e.g. more debt is issued, the default barrier V_B changes as well, as it is not a contract term. In contrast, V_C is specified in the contract a priori and cannot be changed a posteriori.

We will analyze two cases: In the first case the shareholders choose V_C to maximize the value of their equity and in the second case V_C is determined to maximize the total value of the firm. The main result of this section is that contingent convertible debt can degenerate to straight debt without recovery payment. In this case the conversion barrier will coincide with the default barrier V_B^{**}. If conversion takes place before default, the optimal conversion barrier of the shareholders is strictly higher than the optimal conversion barrier for the firm as a whole.

First, we consider the two-dimensional optimization problem of the shareholders. We will simplify it to a two-stage optimization problem. The first stage is to choose V_B optimally for a given V_C subject to the commitment problem, i.e. we have the same problem as in the previous subsection. The second stage is to choose a V_C, that maximizes the equity value for the old shareholders. However, the solution may not be unique. We adopt the convention that the smallest possible conversion barrier is chosen by the old shareholders if the solution is not unique. This is motivated by the fact, that the shareholders have also additional costs from dilution which are not explicitly modeled here, e.g. less control over the company. Hence, a lower conversion barrier should be preferred. In summary, the second stage optimization problem is

$$V_C = \inf\{\arg\max_{V_C} EQ_{old}(V, V_B(V_C), V_C) \text{ s.t. } V \geq V_C \geq V_B(V_C)\}$$

Define $\bar{V}_C$ as the smallest conversion barrier, such that the limited liability constraint for V_B^* is satisfied:

$$\bar{V}_C = \inf\{V_C \geq V_B^* : EQ_{old}(V, V_B^*, V_C) \geq 0 \text{ for all } V \geq V_C\}.$$

This infimum exists as the above set is non-empty (e.g. $V_C = V$ is included in the set). Next, we define $V_C^* \geq \bar{V}_C$ as the smallest conversion barrier that maximizes the value of the equity of the old shareholders for the default barrier V_B^*:

$$V_C^* = \inf\{\arg\max_{V_C : V \geq V_C \geq \bar{V}_C} EQ_{old}(V, V_B^*, V_C)\}.$$

Note, that $EQ_{old}(V, V_B^*, V_C)$ is a continuous function in V_C and the set over which we are maximizing is compact. Hence, a maximum exists. The infimum of the nonempty set is well-defined and unique.

Proposition 15. *Assume that the shareholders choose $\{V_B, V_C\}$ according to the two stage optimization problem in order to maximize EQ_{old}. If*

$$EQ_{old}(V, V_B^*, V_C^*) \geq EQ_{debt}(V, V_B^{**}, V_B^{**}) \text{ for all } V > V_B^{**}$$

then the optimal solution is

$$V_B = V_B^* \text{ and } V_C = V_C^*,$$

otherwise

$$V_B = V_B^{**} \text{ and } V_C = V_B^{**}$$

Proof. See Appendix 12.6. □

Now we change the optimization problem and consider the case where V_C is chosen to maximize the total value of the firm. This affects only the second stage, while the first stage remains unaffected:

$$V_C = \inf\{\arg\max_{V_C} G(V, V_B, V_C) \text{ s.t. } V \geq V_C \geq V_B(V_C)\}$$

Proposition 16. *Assume that V_C is chosen to maximize the total value of the firm in a second stage. If $G(V, V_B^*, \bar{V}_C) > G(V, V_B^{**}, V_B^{**})$ for all $V > V_B^{**}$, then the optimal solution is*

$$V_B = V_B^* \text{ and } V_C = \bar{V}_C,$$

otherwise

$$V_B = V_B^{**} \text{ and } V_C = V_B^{**}$$

Proof. See Appendix 12.6. □

Note, that in the case where $V_B = V_B^*$, the optimal conversion barrier for the old equity holders is higher than the conversion barrier that is optimal for the firm as a whole. The reason is that the firm and the old shareholders face different tradeoffs. The total value of the firm is strictly increasing in a lower conversion barrier as a late conversion means more tax benefits. The old shareholders also profit from a late conversion as it implies more tax benefits and a lower conversion value. However, there is also a cost to the old shareholders if conversion takes place later, as the coupon and face value payments for the contingent convertible bonds increase. Therefore, it is in general not optimal for the old shareholders to choose the lowest possible conversion barrier.

The key result of this section is the following: If the conversion barrier is chosen endogenously by the firm, the contingent convertible bonds could degenerate to straight debt without recovery payment. In this case the optimal default barrier V_B^{**} will be larger than V_B^* for the same face value of debt and the same coupon payments. As we will discuss in Section 10, a higher default barrier implies a higher default risk. Thus, a regulator prefers a lower default barrier. Therefore, this section gives a strong argument for V_C being fixed exogenously by the regulator such that the contingent convertible debt does not degenerate to debt without any conversion payment. Hence, in the following we focus on an exogenously given V_C which satisfies the no-early-default condition from Assumption 3.

6 Conversion Triggered by Observable Market Prices

6.1 Stock Price as a Sufficient Condition for Conversion

The firm's value process is in general not observable. Our model so far has specified the event of conversion in terms of the firm's value process. In this subsection we want to analyze whether conversion could also be specified in terms of the observable stock price.

The stock price process S_t can be expressed as a function of V_t. Recall that in the case of a firm that has not issued any contingent convertible bonds, but only straight debt, the relationship is very simple:

$$S_t = \frac{EQ(V_t)}{n}.$$

As $\partial EQ(V_t)/\partial V_t > 0$ (see Proposition 11), the stock price is a strictly increasing function in the value of the firm's assets. If the firm also issues contingent convertible bonds, the situation becomes more complicated. In Section 3.3.3 we have defined the stock price $S_t = S(V_t)$ as a function of V_t:

$$S_t = S(V_t) = \begin{cases} \frac{EQ_{old}(V_t)}{n} = \frac{EQ(V_t)-DC(V_t)}{n} & \text{if } t < \tau_C \\ \frac{EQ(V_t)}{n+n'} & \text{if } \tau_C \leq t < \tau \end{cases}$$

We have shown that the dilution costs equal the conversion value for all specifications of CCBs. Hence, it follows for all types of CCBs that

$$EQ(V_{\tau_C}) - DC(V_{\tau_C}) = \frac{n}{n+n'} EQ(\tau_C).$$

Hence, the stock price is a continuous function in V_t, even when we include contingent convertible bonds. Given our closed form solutions for the conversion value and the dilution costs we can give an explicit formula for the stock price as a function of V_t. However, if we include contingent convertible debt, $EQ_{old}(V_t)$ and therefore also $S(V_t)$ are not necessarily strictly increasing functions in V_t any more. First, we consider the special case, where $S(.)$ is still a strictly increasing function.

Assumption 4. *Assume that the mapping between S_t and V_t is strictly increasing for $V_t > V_B$. This implies that the mapping $S(.)$ is invertible for $V > V_B$ and its inverse $S(.)^{-1}$ is strictly increasing as well.*

Proposition 17. *Under Assumption 4 the stock price is a sufficient statistic for conversion, i.e.*

$$\tau_C = \inf\{t \geq 0 : V_t \leq V_C\} = \inf\{t \geq 0 : S_t \leq S_C\} \qquad \text{with probability 1}$$

with $S_C = S(V_C)$.

Proof. Obviously, $V_t \leq V_C$ implies $S_t \leq S_C$. As $S(.)$ is invertible with a strictly increasing inverse, $S_t \leq S_C$ also implies $V_t \leq V_C$. □

Corollary 5. *If Assumption 4 is satisfied, we can base the conversion event on the stock price and obtain a unique pricing equilibrium. The prices for the different CCBs, which are presented in Section 4.3, are still valid.*

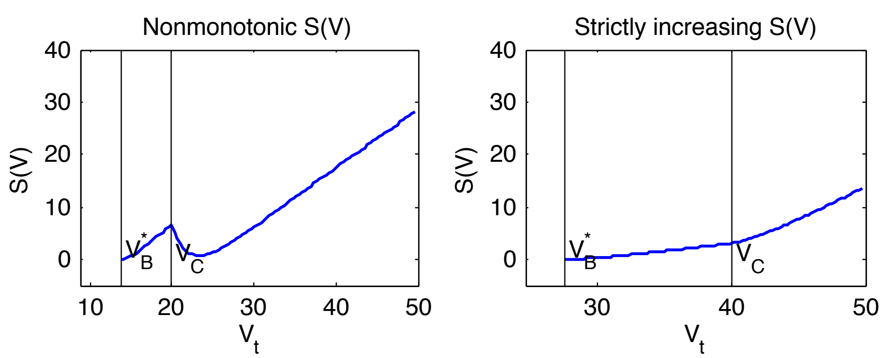

Figure 1: Stock price as a function of the firm's value process.

The situation changes if Assumption 4 is violated.[5] In Figure 1 we plot the stock price as a function of the firm's value process for two different RCD1 contracts. In the right plot

[5]Sundaresan and Wang (2010) present a structural model for CoCo Bonds, in which conversion is based on the stock price process. They claim that, in order to obtain a unique equilibrium, mandatory conversion must not result in any value transfer between equity and CoCo holders (Theorem 1). However, their claim is wrong. A pricing equilibrium can exist and be unique if the trigger price and conversion ratio are chosen independently. The only crucial assumption is that the mapping between the stock price and the firm's assets is strictly monotonic. Sundaresan and Wang also illustrate their argument in a two period model. However, they are just stating, that we need a monotonic relationship between the stock price and the firm's value to obtain a unique pricing equilibrium. In a two period model this condition coincides with their stronger condition that there is no value transfer between shareholders and CoCo holders.

Assumption 4 is satisfied. However, as we can see from the left plot, there exists parameter values such that $S^{-1}(S_t) = \{V_t^L, V_t^M, V_t^H\}$ contains three elements, i.e. the stock price S_t is the equilibrium price for the firm's values $V_t^L < V_t^M < V_t^H$. Particularly problematic is that $V_t^L < V_C < V_t^H$, i.e. the event of conversion and not conversion are consistent under a certain stock price. If conversion is based on the stock price and the stock price is a nonmonotonic function of the firm's value process, then the prices cannot be evaluated within our modeling framework.

More formally, define a conversion triggering stock price S_C. Next, we define the conversion time based on observables as

$$\tilde{\tau}_C = \inf\{t \geq 0 : \tilde{S}_t \leq S_C\}$$

where $\tilde{S}_t$ is the stock price process. This can be used to define a new CCB contract $\widetilde{CB}$, which is identical to the former one except for the conversion time. Hence, the equity $\widetilde{EQ}$ and stock price function $\tilde{S}(V)$ under this new conversion time change as well. Note, that as the CCB price now explicitly depends on the stock price process (through $\tilde{\tau}$), an equilibrium stock price has to be the solution to the equation $\tilde{S}_t = \tilde{S}(V_t)$. In more detail, an equilibrium stock price is a function $\tilde{S}(.) : (V_B, \infty) \to \mathbb{R}_0^+$, that solves the following equation for all $V_t \in (V_B, \infty)$:

$$\begin{aligned}
n \cdot \tilde{S}(V_t) =& EQ_{debt} + \widetilde{TB}_C(V_T) - \widetilde{CB} \qquad \Leftrightarrow \\
n \cdot \tilde{S}(V_t) =& EQ_{debt} + \frac{\bar{c}C_C}{r} E\left[1 - e^{-r\tilde{\tau}_C}\right] - \left(\frac{c_C P_C + mP_C}{m+r}\right) E\left[1 - e^{-(m+r)\tilde{\tau}_C}\right] \\
& - E\left[n' \tilde{S}(V_{\tilde{\tau}_C}) e^{-(m+r)\tilde{\tau}_C} \mathbb{1}_{\{\tilde{\tau}_C < \tau < \infty\}}\right]
\end{aligned}$$

This is a fix point problem. However, the fix point in this case is a function. In order to prove existence and uniqueness of a solution we need to apply fix point theorems for infinitely dimensional Banach spaces. We can show, that if we restrict the set of possible solutions to functions $\tilde{S}(V_t)$ for which the conversion time can be expressed in the form $\inf\{t \geq 0 : V_t \leq V_C\}$ for some V_C, then it is possible that no equilibrium exists. If we simplify the model to discrete time, we can show that multiple solutions can exist. It seems to be a nontrivial problem to make a general statement about the existence and uniqueness of a solution. For this reason we will propose different observable prices as conversion triggers in the next subsection, which do not suffer from this shortcoming.

6.2 Conversion based on credit spreads and credit default swaps

In this paper we have developed a consistent and complete model for CCBs where the event of conversion is based on the unobserved firm's value. For practical purposes we need to specify conversion in terms of an observable variable. As we have seen in the last subsection, defining the event of conversion in terms of the stock price will only lead under Assumption 4 to the same pricing formulas as a model where conversion is based on the firm's value process. The reason is that the stock price implicitly depends on the features of the CCBs. As a consequence it is possible that a particular stock price is consistent with two different firm's values. Hence, the stock price is in general not a sufficient statistic for the firm's value process. Our goal is it to find an observable variable that could fully reveal the firm's value process. We propose the credit spread and the risk premium of a portfolio of credit default swaps (CDS). As we will show, both, the credit spread and the risk premium for CDSs, are not affected by the features of CCBs and this will allow us to use them as a sufficient statistic for the firm's value process.

First, we prove that the credit spread fully reveals the stock price. The credit spread is defined as the risk premium between a risky and an identical risk-free bond.

Definition 8. *The price of a unit default-free coupon bond with face value 1, maturity t and coupon c is denoted by*

$$b(c,t) = \int_0^t ce^{-rs}ds + e^{-rt}.$$

Lemma 7. *The aggregated total value of default-free bonds for a firm that issues p default-free unit bonds with maturity profile $\varphi(t) = me^{-mt}$ equals*

$$B(C) = \frac{C + Pm}{m + r}$$

with $C = Pc$ and $P = p/m$.

Proof.

$$\begin{aligned} B(C) &= \int_0^{-\infty} p\Psi(t)b(c,t)dt = \int_0^{-\infty} pc\left(\frac{1-e^{-rt}}{r}\right)e^{-mt}dt + \int_0^{\infty} pe^{-rt}e^{-mt}dt \\ &= pc\left(\frac{1}{mr} - \frac{1}{r(m+r)}\right) + \frac{p}{m+r} = pc\frac{1}{m(m+r)} + \frac{Pm}{m+r}. \end{aligned}$$

□

The credit spread π is defined as the difference in the coupon payments of a risky and an identical risk-free unit bond, that trade at the same price:

$$b(c,t) = d_D(V, V_B, c+\pi, t)$$

We will first look at an aggregated credit spread Π, i.e.

$$B(C) = D(V, V_B, C + \Pi)$$

where we assume that the portfolio of risky and risk-free bonds have the same face value P_D and maturity profile.

Lemma 8. *The aggregated credit spread Π equals*

$$\Pi = C_D - \tilde{C}$$

where C_D is the total coupon value of the risky debt and

$$\tilde{C} = D(V, V_B, C_D)(m + r) - P_D \cdot m$$

Proof. The result follows from

$$D(V, V_B, C_D) = B(\tilde{C}) = \frac{\tilde{C} + P_D m}{m + r}.$$

□

Recall that $C_D = P_D \cdot c_D$. We conclude: The credit spread of a single bond equals

$$\pi = c_D - D(V, V_B, C_D)\frac{m+r}{P_D} + m$$

Economically, it only makes sense to consider positive spreads. A necessary condition is that the recovery payment in the case of default has a lower present value than the repayment of the face value.

Assumption 5. *In the following we assume that the value of the total straight debt, if it was risk-free, is larger than total value of the risky debt:*

$$B(C_D) > D(V, V_B).$$

Assumption 5 has an important implication:

Lemma 9. *Assumption 5 implies that the value of the total straight debt, if it was risk-free, is larger than the value of the largest possible recovery payment:*

$$B(C_D) > D(V, V_B) \qquad \Rightarrow \qquad \frac{C_D + mP_D}{r+m} > (1-\alpha)V_B.$$

Proof. See Appendix 12.7. □

If we want to make a statement about the relationship between the credit spread π and the firm's value process, we need to analyze the dependency of $D(V, V_B)$ on V.

Lemma 10. *If the condition $\frac{C_D+mP_D}{r+m} \geq \frac{\eta_2}{\eta_2+1}\frac{\beta_{3,r+m}+1}{\beta_{3,r+m}}(1-\alpha)V_B$ is satisfied, then the value of the straight debt is an increasing function in the firm's value:*

$$\frac{\partial D(V)}{\partial V} > 0 \qquad \textit{for all } V \geq V_B.$$

Proof. See Appendix 12.7. □

Note that $\frac{\eta_2}{\eta_2+1}\frac{\beta_{3,r+m}+1}{\beta_{3,r+m}} > 1$. This means, that the condition in the above lemma is stronger than Assumption 5. However, as the next lemma shows, it will always be satisfied in our case.

Lemma 11. *If the default barrier is chosen optimally as $V_B = V_B^*$ (i.e. V_B is chosen optimally by the shareholders as in the case with only straight debt), then the condition*

$$\frac{C_D + mP_D}{r+m} \geq \frac{\eta_2}{\eta_2+1}\frac{\beta_{3,r+m}+1}{\beta_{3,r+m}}(1-\alpha)V_B$$

is always satisfied.

Proof. See Appendix 12.7. □

Now we can completely characterize the relationship between π and V.

Corollary 6. *If the default barrier is chosen as $V_B = V_B^*$, the credit spread $\pi(V)$ as a function of the firm's value process is strictly decreasing:*

$$\frac{\partial \pi}{\partial V} < 0 \qquad \text{for all } V \geq V_B.$$

Our goal was it to express the conversion trigger in terms of an observable process. The next theorem shows that the credit spread is a suitable candidate.

Theorem 4. *Assume that $V_B = V_B^*$. There exists a unique value π_C such that $\pi(V_C) = \pi_C$. If the conversion time is defined as*

$$\tau_C^* = \inf(t \in [0, \infty) : \pi(V_t) \geq \pi_C)$$

we get exactly the same evaluation formulas for CCBs as in the case where the conversion time is defined as

$$\tau_C = \inf(t \in [0, \infty) : V_t \leq V_C).$$

Proof. As $\pi(V)$ is strictly decreasing on $[V_B, \infty)$ and $V_C > V_B$, existence and uniqueness of π_C follow. Furthermore, the strict monotonicity implies that with probability 1

$$\{t \in [0, \infty) : V_t \leq V_C\} = \{t \in [0, \infty) : \pi(V_t) \geq \pi(V_C)\} = \{t \in [0, \infty) : \pi(V_t) \geq \pi_C\}$$

holds. □

Another sufficient statistic for the firm's value process is the risk premium of CDSs. A credit default swap is an agreement that the seller of the CDS will compensate the buyer in the event of a loan default. The buyer of the CDS makes a series of payments to the seller and, in exchange, receives a payoff if the loan defaults. We define the CDS fee as $\tilde{\pi}$. For a unit straight debt bond with face value 1 and maturity t the CDS risk premium has to satisfy

$$\int_0^t \tilde{\pi} e^{-rs} ds = b(t, c_D) - d_D(V, V_B, t).$$

This means that a CDS together with a defaultable bond has the same value as an otherwise identical default-free bond.

In the following analysis we construct an index, which is a strictly monotonic function in the firm's value process. Our index is a portfolio of CDS contracts such that the whole debt is "insured". For this purpose, we have to make the weak assumption that a CDS for a risky bond with every possible maturity is issued. It is important to note that this portfolio, that fully insures the aggregated debt, does not need to actually exist. As long as we observe market prices for CDS contracts for every possible maturity, we can calculate the price of our artificial index. The price of the portfolio is a weighted average of the CDS prices, where the different maturities have the same weights as in the debt portfolio.

Proposition 18. *The risk premium for credit default swaps on the aggregated debt satisfies the following equation:*

$$\tilde{\pi} \frac{P_D}{m + r} = \left(B(C_D) - D(V, V_B)\right).$$

Proof. Aggregation yields:

$$\int_0^\infty p_D \int_0^t \tilde{\pi} e^{-rs} ds \Psi(t) dt = \int_0^\infty p_D (b(t, c_D) - d_D(V, V_B, t)) \Psi(t) dt.$$

We only need to show the statement for the LHS.

$$\int_0^\infty p_D \frac{\tilde{\pi}}{r} \left(1 - e^{-rt}\right) e^{-mt} dt = \tilde{\pi} p_D \frac{1}{m(m+r)} = \tilde{\pi} \frac{P_D}{m+r}.$$

□

This allows us to completely characterize the relationship between $\tilde{\pi}$ and V:

Proposition 19. *If the default barrier is chosen as $V_B = V_B^*$, the CDS risk premium $\tilde{\pi}(V)$ as a function of the firm's value process is strictly decreasing:*

$$\frac{\partial \tilde{\pi}}{\partial V} < 0 \qquad \textit{for all } V \geq V_B.$$

By the same argument as in Theorem 4 we conclude that conversion can be based on the CDS risk premium.

One of the arguments of critics of CoCo-Bonds was that the conversion event cannot be based on the observable stock price process. Indeed, there exist parameter values, for which our evaluation formulas based on the firm's value process and a model where conversion is triggered by movements in the stock price, differ. However, we have shown that the unobservability of the firm's value process can be circumvented by using credit spreads or the CDS risk premium. Credit spreads have the same advantages as stock prices as they constantly adjust to new information in contrast to accounting triggers. As credit spreads are not affected by the features of CCBs, they are a sufficient statistic for the firm's value process. Thus, defining the conversion event in terms of credit spreads is equivalent to using the firm's value process. The same holds for CDS risk premiums.[6]

7 Numerical Examples

In this subsection we will calculate several scenarios numerically and discuss their interpretations. For the computations the values of the following parameters are fixed:

$$V_0 = 100, r = 7.5\%, \delta = 7\%, \alpha = 50\%, \bar{c} = 35\%.$$

[6] A special case are firms that are "too big to fail" (TBTF). As debt of these firms is implicitly protected by a government guarantee, the credit spread should be zero, i.e. the debt should be considered to be risk free. However, even the big banks, that enjoyed this government guarantee, had to pay a risk premium on their debt during the past crisis. This implies that the debt of TBTF firms is not completely insured. Either there is some uncertainty about the government bail-out taking place or debt holders fear a hair cut after a bail-out. In either case, the credit spread as a sufficient statistic for the firm's value process works. Assume for example that the loss after default for unprotected debt is $\alpha = 0.5$. If the probability of a bail-out is 80%, then the expected bankruptcy loss of the debt is $\tilde{\alpha} = 0.8 \cdot 0 + 0.2 \cdot 0.5 = 0.1$. If on the other hand a bailout is certain, but a hair cut of 10% is to be expected, the expected bankruptcy loss of the debt is $\tilde{\alpha} = 0.1$. Either way, we could apply the methods of this section, where we replace α with $\tilde{\alpha}$ in the formula for the debt.

These parameter values are similar to those used by Leland (1994a), Leland and Toft (1996) and Hilberink and Rogers (2002) and are chosen to be consistent with the U.S. environment.

We display the spread of normal debt and contingent convertible bonds as a function of log-maturity. By letting $\log(m^{-1})$ vary between -4 to 10 we receive mean maturity profiles from about a week to 1000 years. For example a log-maturity $\log(m^{-1})$ of 1 corresponds to an average maturity of $\frac{1}{m} = 2.7$ years. The spread here is defined as an aggregate spread, i.e.

$$spread = \frac{C_D}{P_D} - r$$

for traditional debt and respectively

$$spread_C = \frac{C_C}{P_C} - r$$

for contingent convertible debt.

We consider three different firms. The parameters of the firm's value process are chosen such that the amount of "uncertainty" for all three firms is the same, i.e. the quadratic variation is kept constant: $\langle \log\left(\frac{V_t}{V_0}\right)\rangle = \sigma^2 + 2\lambda\left(\frac{p}{\eta_1^2} + \frac{1-p}{\eta_2^2}\right) = 0.25$

1. "No jumps": $\sigma = 0.25$
2. "Infrequent large jumps": $\sigma = 0.15, \eta_1 = 2, \eta_2 = 2, p = 0.5, \lambda = 0.2$.
   On average every five years the firm's value jumps. With 50% probability the firm losses one third of its value, while with 50 % probability it gains one third.
3. "Frequent moderate jumps": $\sigma = 0.15, \eta_1 = 10, \eta_2 = 10, p = 0.25, \lambda = 0.5$.
   On average every two years the firm's value jumps. With 75% probability the firm losses 1/11 of its value, while with 25% probability it gains 1/11.

We analyze the aggregate credit spreads and the dilution costs for different choices of the conversion value parameter ℓ, the conversion barrier V_C and the amount of straight debt and contingent convertible debt. For the straight debt we consider three different levels of debt $P_D = 10, 30$ and 40 and for the contingent convertible debt we vary $P_C = 10$ and $P_C = 40$. The value of the firm's assets is hold constant, which means that we swap debt respectively CCBs for equity. For Figure 2 to 10 we assume that for every maturity the coupon values are determined such that the debt sells at par. This can be interpreted as the case of a firm that creates a new capital structure. In Figure 11 to 16 we fix the coupon payments such that the debt sells at par at time 0 for $V_0 = 100$. Here we think of a firm that has set up its capital structure at time $t = 0$ and we follow the dynamics of the value of its assets over time.

7.1 Comparing RCD and FSCCB contracts

In Figure 2 we plot the spread and the default and conversion barrier for a firm without jumps. The conversion ratio ℓ of the RCD1 contract is set to 1, i.e. the Coco bondholders receive equity at conversion which has the same market value as the face value of the CCBs. The most striking result is that the spread of the CCBs is completely independent of the capital structure and equal to zero. This result illustrates that a model without jumps produces unrealistic results.

The humped-shaped form for normal debt was already found by Leland and Toft. A higher curve corresponds to a higher leverage in terms of normal debt. This makes sense as for a larger leverage the default barrier is higher which in turn implies a higher default risk. Therefore the spread as a risk premium is also higher. Firms with low levels of debt have a small spread, which increases with maturity. These firms are far away from the bankruptcy level V_B and thus the credit spread as a measurement of risk is low. As maturity is growing the firm has more time to approach the critical level V_B and thus the spread increases. As the leverage increases the spread curve becomes more humped. Why is the spread of the highly levered firm falling for a certain level of maturity. This can be explained by the argument that if a firm has survived for a long period of time it is very likely that its value has gone up. Thus conditioning on survival for a long time the firm's value has to be on average far away from V_B and hence the lower spread indicates the decrease in riskiness. Note, that the credit spreads for normal debt are equal to zero for short maturity.

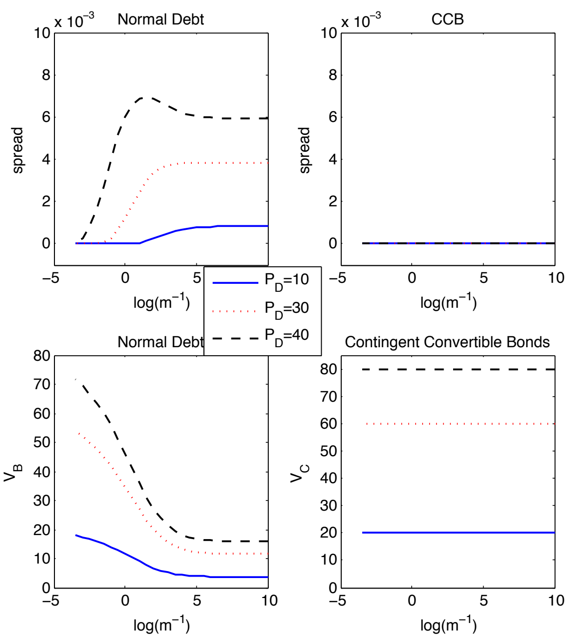

Figure 2: "No jumps": RCD1 spread as a function of maturity for different levels of debt and different conversion barriers. The parameters are $\ell = 1, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 60, 80$.

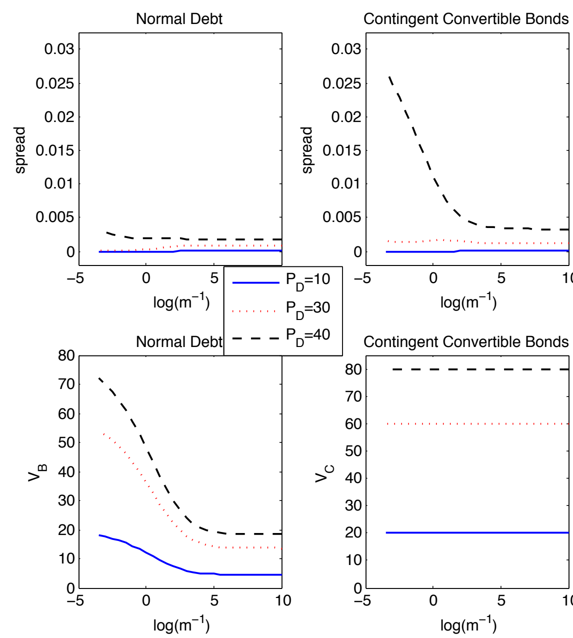

Figure 3: "Frequent, moderate jumps": RCD1 spread as a function of maturity for different levels of debt and different conversion barriers. The parameters are $\ell = 1, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 60, 80$.

In Figure 3 and 4 we consider the two firms with jumps. In both cases we observe substantial credit spreads which depend strongly on the amount of straight debt. Figure 2 has shown, that if no jumps are included in the firm's value process, if ℓ is equal to 1 and if the equity value at conversion is sufficiently high, then RCDs are risk-free. The contingent convertible debt holders will always receive their full payments, but the time when this happens may be random. In the case of RCD1 contracts with jumps the expected value of the equity at the conversion time τ_C is in general lower than the value of the equity for $V_t = V_C$. Hence, the conversion value is lower than the face value of the CCBs, which results in the positive credit spreads. The larger the jumps, the smaller is the expected value of the equity at conversion and therefore the higher the spread. The curves of contingent convertible bonds have a similar shape as the curves for the straight debt. Note that the limiting credit spreads are nonzero for both bonds. The spreads for the contingent convertible bonds are higher than for straight debt in Figures 3 and 4. This is mainly due to the fact that conversion happens substantially earlier than default for most maturities. For short maturities, where the default and conversion barrier are relatively close, conversion is most likely to occur by a jump. If the conversion barrier is crossed by a jump the equity value after conversion is lower than if it is passed by a continuous movement, which results in a higher credit spread for the contingent convertible bonds.

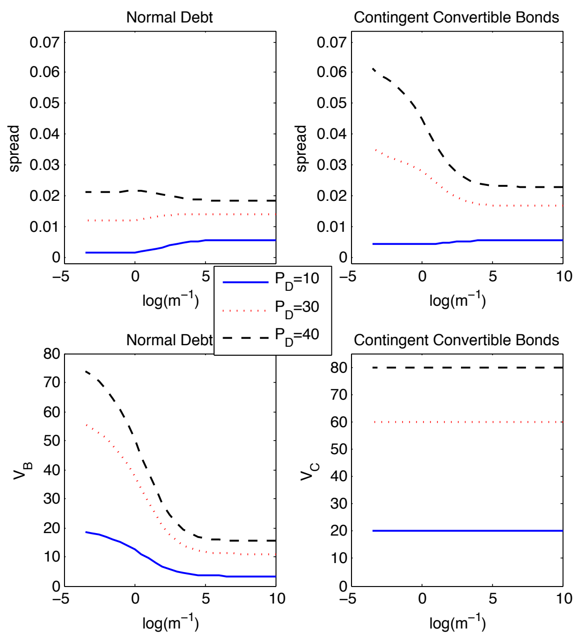

Figure 4: "Infrequent, large jumps": RCD1 spread as a function of maturity for different levels of debt and different conversion barriers. The parameters are $\ell = 1, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 60, 80$.

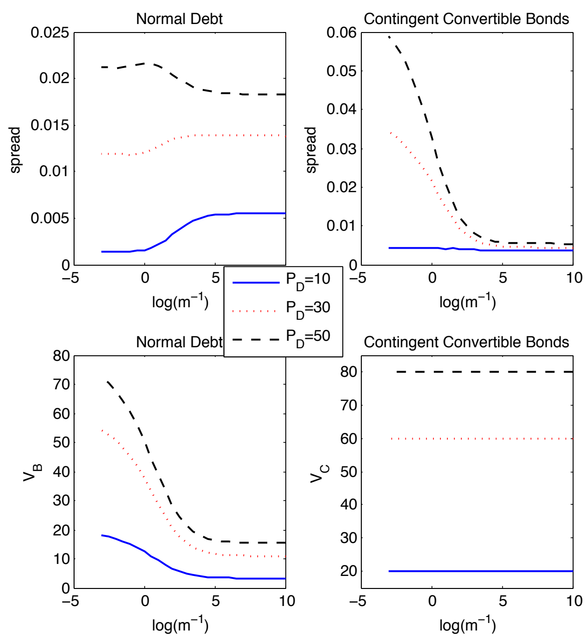

Figure 5: "Infrequent, large jumps": RCD2 spread as a function of maturity for different levels of debt and different conversion barriers. The parameters are $\ell = 1, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 60, 80$.

In Figure 5 we consider a firm with infrequent, moderate jumps for RCD2 contracts. As we can see the magnitude of the spreads is very similar to the corresponding RCD1 contract, however the spreads for long maturities for RCD2 contracts are lower than for RCD1 contracts. If the value of the equity at conversion is sufficient to make the promised payment, than the face value of RCD2s equals the conversion value for $\ell = 1$. For long maturities the default barrier is relatively low, which makes it more likely that the equity value at conversion is high. Hence, for long maturities RCD2s are almost risk-free.

In Figure 6 we plot the spread and the default and conversion barrier for FSCCBs. The conversion barrier is $\ell = 1$ which means that for a contingent convertible bond with face value 1, the debt holders will get $\frac{1}{S_0}$ shares at conversion. Without loss of generality we can normalize $S_0 = 1$ and hence think of ℓ as the number of shares granted at conversion. Of course, one share at time $t = 0$ has a substantially higher value than a share at time $t = \tau_C$. Hence, we expect the conversion value to be relatively low. This is exactly, what we observe: The spreads for FSCCBs are substantially higher than for RCDs, due to the lower conversion value. In Figure 7 we increase ℓ to 1.5. The higher conversion value dramatically lowers the spread. As we have discussed before, the specification of the number of shares at conversion in terms of the stock price S_0 at time 0 is problematic as it leads to multiple equilibria. Here, we have focussed on the equilibrium with the lower conversion value.

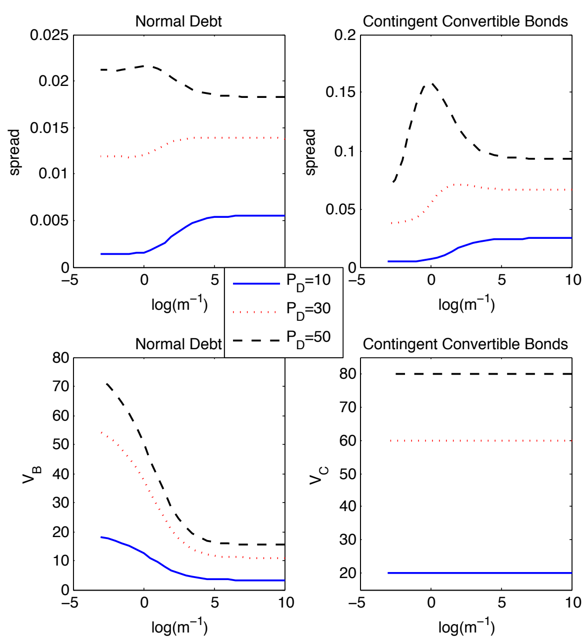

Figure 6: "Infrequent, large jumps": FSCCB spread as a function of maturity for different levels of debt and different conversion barriers. The parameters are $\ell = 1, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 60, 80$.

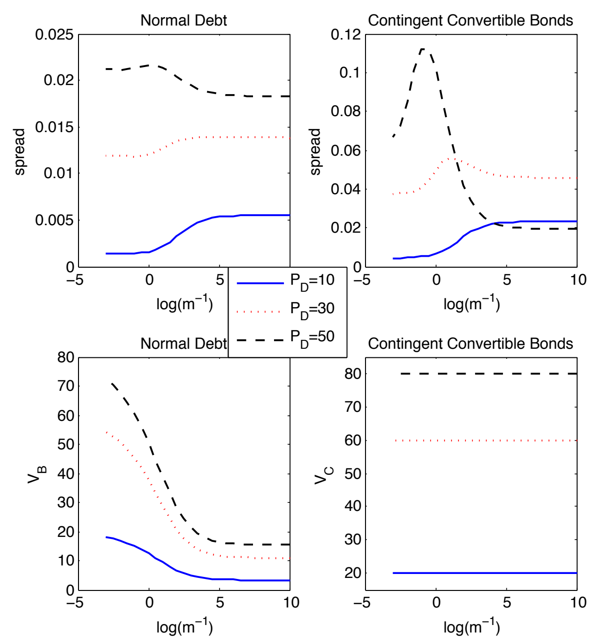

Figure 7: "Infrequent, large jumps": FSCCB spread as a function of maturity for different levels of debt and different conversion barriers. The parameters are $\ell = 1.5, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 60, 80$.

We already know, that a RCD1 is a special version of a FSCCB contract. How do we have to choose the parameter ℓ_{RCD1} such that the contract coincides with a given FSCCB contract with parameter ℓ_{FSCCB}? Simple calculations show that

$$\frac{\ell_{RCD1}}{\ell_{FSCCB}} = \frac{S(V_C)}{S(0)}.$$

In Figure 8 we plot the corresponding ratio. For long-term maturities with $P_D = 30$ a FSCCB contract that promises three times the face value of debt in terms of the current market value of equity is equivalent to an RCD1 contract that pays the exact face value of debt in terms of shares with value $S(V_C)$.

In Figures 4 and 5 we have seen that the spreads for RCD1 and RCD2 contracts are very similar. However, the number of shares granted at conversion differs. The main difference between the two contracts is that the conversion value of RCD1s is based on $EQ(V_C)$, while the conversion value of RCD2s is primarily based on $EQ(V_{\tau_C})$. If these two values coincide, there is no difference between the two RCD contracts. In Figure 9 we plot the ratio

$$E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right]/EQ(V_C)E\left[e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right].$$

For long-term contracts the ratio takes values from 0.8 to 0.9, i.e. if all other parameters of the contract stay the same, the number of shares granted under RCD2s should be 10% to 20% higher than under RCD1s. Figure 10 plots the ratio of the dilution costs for RCD2s and RCD1s. We observe that the dilution costs for RCD2s are actually around 5% to 12 % higher than for RCD1s in the long run, i.e. less than the expected 10% to 20%. This can be explained by the argument that for a low value of the firm's assets V_{τ_C} at conversion it is likely that the old shareholders lose (almost) any claim on the company and in this case the dilution costs for RCD1 and RCD2 contracts are the (almost) same.

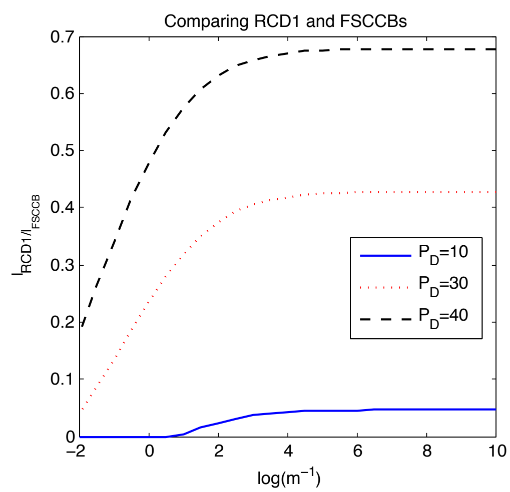

Figure 8: "Infrequent, large jumps": Comparing RCD1s with FSCCBs. We plot ℓ_{RCD1}/ℓ_{FSCCB} such that the two contracts are equal. The parameters are $P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 60, 80$.

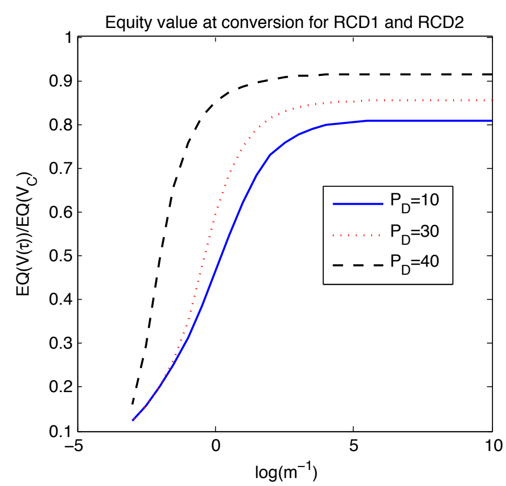

Figure 9: "Infrequent, large jumps": Comparison of the total equity value at conversion for RCD1 and RCD2 for different constant conversion barriers. The parameters are $\ell = 1$, $P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 60, 80$.

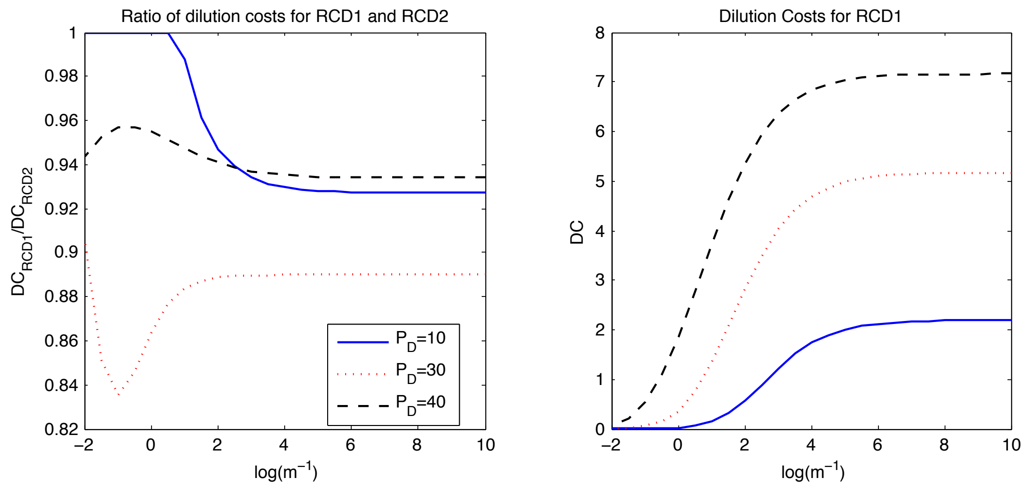

Figure 10: "Infrequent, large jumps": Ratio of dilution costs between RCD1 and RCD2. The parameters are $P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 60, 80$.

7.2 Assumption 3

If Assumption 3 is satisfied, the optimal default barrier is $V_B^* < V_C$ and we can apply the solution formulas developed in this paper. If the assumption is violated, the contingent convertible debt degenerates to straight debt without any recovery payment. From the perspective of a regulator, the contingent convertible bonds become unattractive. Hence, it is important to know under which parameter constellations Assumption 3 holds. It requires that V_B^* satisfies the limited liability constraint and that no default barrier larger than V_C yields a higher value for the old shareholders. We define a "critical" equity value:

$$EQ_{critical}(V) = \max(0, EQ_{debt}(V, \max(V_B^{**}, V_C), P_D + P_C, C_D + C_C)).$$

As long as $EQ_{old}(V, V_B^*, V_C) > EQ_{critical}(V)$ for $V > V_C$, Assumption 3 is satisfied. In Figure 11 we see that for an average maturity of 1 year and an amount of contingent convertible debt (RCD1) that does not exceed the straight debt, the assumption holds. However, if the amount of contingent convertible debt is large as in Figure 12, the assumption is violated. Having very short maturity debt can also create problems. In Figure 13 we set the average maturity to 1/10 year and the equity value of the old shareholders crosses the critical barrier. On the other hand, long term debt seems to elevate the chances of satisfying the condition. In Figure 14 we combine a large amount of contingent convertible debt ($P_C = 40$) with a long maturity (10 years) and the assumption holds.

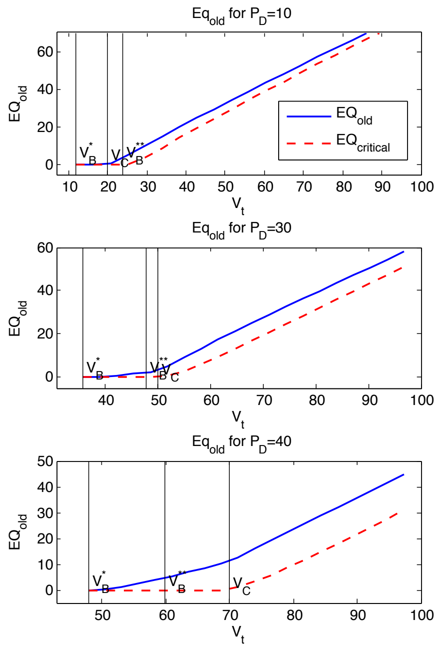

Figure 11: Testing Assumption 3 for RCD1s. The parameters are $\ell = 1, m = 1, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 50, 70$.

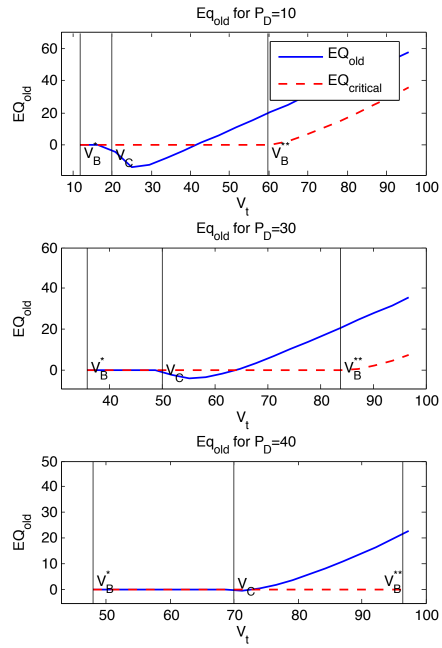

Figure 12: Testing Assumption 3 for RCD1s. The parameters are $\ell = 1, m = 1, P_C = 40, P_D = 10, 30, 40$ and $V_C = 20, 50, 70$.

Why can $EQ_{old}(V, V_B^*, V_C)$ be negative and why does it eventually become positive again? The equity value for the old shareholders can only be negative if the cash payments related to the contingent convertible debt (i.e. coupon payments C_C and face value P_C) are very high. If conversion takes place, the old shareholders are freed from all the cash payments of the convertible debt. The total value of the equity that remains after conversion has by definition a non-negative value. Thus, independently of how small the share of the old equity holders is after conversion, it will have a non-negative value. Therefore, if the firm's value process falls sufficiently low and the cash payments for the contingent convertible bonds are high, the equity value EQ_{old} can become negative. However, the prospect of conversion will eventually lead to a positive price, if V_t falls further.

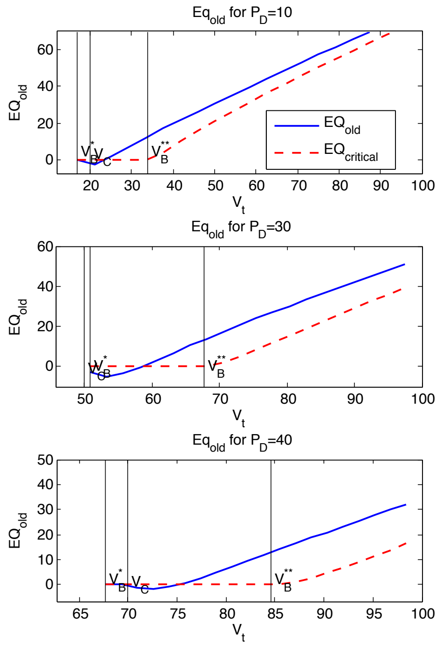

Figure 13: Testing Assumption 3 for RCD1s. The parameters are $\ell = 1, m = 10, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 50, 70$.

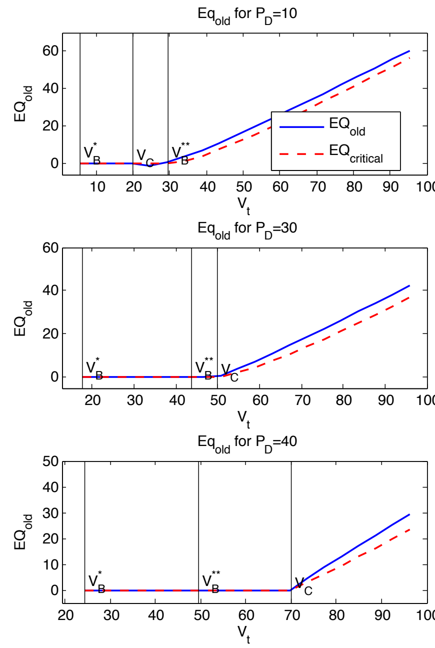

Figure 14: Testing Assumption 3 for RCD1s. The parameters are $\ell = 1, m = 0.1, P_C = 40, P_D = 10, 30, 40$ and $V_C = 20, 50, 70$.

In the case of RCD2s and FSCCBs the findings are similar as depicted in Figures 15 and 16. The key results from these simulations are that as long as the conversion ratio is sufficiently high (e.g. $\ell = 1$ for RCDs), the amount of contingent convertible debt is smaller than the amount of straight debt ($P_C < P_D$) and the maturity of the debt is sufficiently long (e.g. $\frac{1}{m} > 1$), Assumption 3 holds.

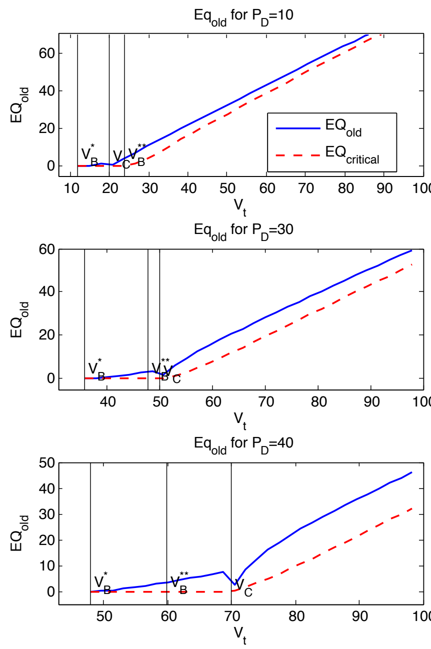

Figure 15: Testing Assumption 3 for FSC-CBs. The parameters are $\ell = 1, m = 1, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 50, 70$.

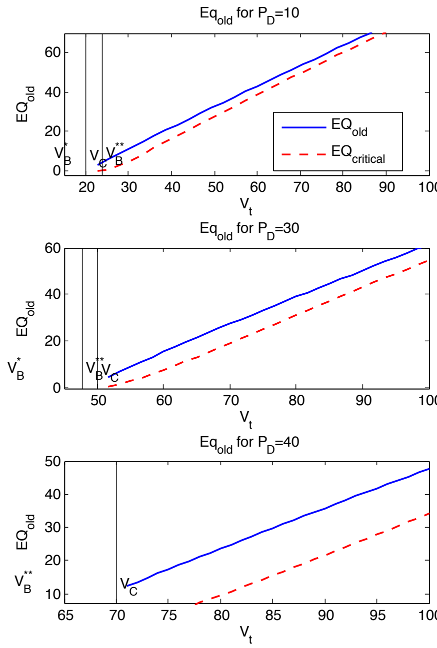

Figure 16: Testing Assumption 3 for RCD2s. The parameters are $\ell = 1, m = 1, P_C = 10, P_D = 10, 30, 40$ and $V_C = 20, 50, 70$.

7.3 Agency Costs

In the basic Merton (1974) model, equity can be regarded as a Call option on the firm's assets. Therefore, equity holders always want to increase the risk (volatility), while debt holders would like to decrease the risk. By including a rolling debt structure and tax benefits, the risk incentives of equity holders and debt holders are in general not opposing any more. As Leland (1994b) has pointed out, for short maturity debt, equity holders and debt holders do both prefer not to scale up the risk. In our model with jumps and CCBs, the risk incentives are more complex.

We consider three different firms, that have the same amount of total risk as measured by the quadratic variation. The default barrier V_B is chosen optimally and the conversion barrier is set 20% higher than the default barrier. Following the second stage optimization as described in Section 10, the amount of straight debt P_D is chosen to maximize the total value of the firm for a fixed amount of contingent convertible bonds $P_C = 10$. Keeping the amount of debt constant, we want to analyze which agents will profit or suffer from increasing the risk. In our model we have two types of risk: continuous risk (measured by the volatility σ) and jump risk (measured by the jump intensity λ). We will change the total amount of risk (the quadratic variation) by either increasing the volatility or the jump intensity. Figure 17 shows by how many percentage points the equity value, debt value or CCB value changes, if we increase the total risk by 1 percentage point. There are three main observations:

1. Contingent convertible bonds have very similar risk incentives as straight debt. The

agency costs between equity holders and CCB holders are slightly larger than for debt holders.

2. For short-term debt, the incentives of equity holders have the same sign as the incentives for debt and CCB holders. This can be interpreted as lower agency costs for short maturities.

3. The agency costs related to jump risk for a firm that is mainly exposed to continuous risk are very small. Vice versa, the agency costs for continuous risk for a firm with higher jump risk are also much less pronounced than for a firm with mainly continuous risk.

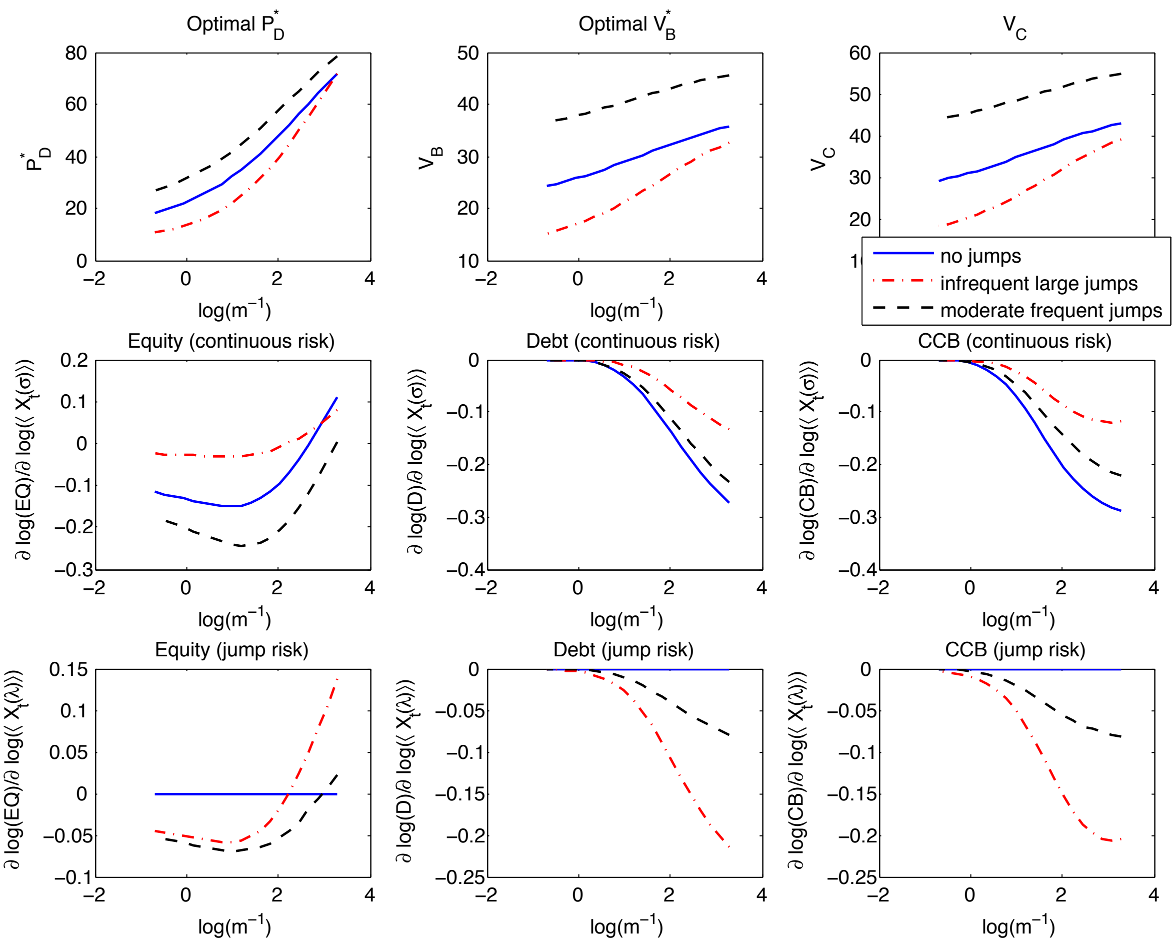

Figure 17: Risk incentives for RCD1 contracts with optimal debt. The parameters are $\ell = 1, P_C = 10, c_D = c_C = 0.8, V_C = 1.2 \cdot V_B$. We plot the relative change in equity and debt values for a relative change in total risk (quadratic variation).

8 How should CCBs be designed?

We have presented and completely characterized two different CCB contracts: FSCCBs and RCD2s. The contract RCD1 is a special case of FSCCB. The important question is which should be used in practice. We will analyze the different contracts with respect to manipulation, noise trading and multiple equilibria.

8.1 Manipulation

We think of manipulation as spreading "good" or "bad" news that will influence the stock price and the credit spread. Furthermore, we assume that conversion is based on the credit spread as described in Section 6. Spreading "good" news will temporarily increase the stock price S_t and lower the credit spread π_t. Spreading "bad" news results in the opposite movements. However, after the manipulation the prices will return to their former level. Equivalently, we can think of manipulation as directly affecting the firm's value process V_t. The only interesting case is when the firm's value process V_t is lowered to a level $V_{manip} < V_C$ which triggers conversion and then returns to the former level V_t.

First we consider manipulation by contingent convertible bondholders. In the case of RCD2 contracts spreading "bad" news can trigger conversion and lead to a temporary underevaluation of the stock price S_t. As the number of shares granted to contingent convertible bondholders depends on the stock price at conversion, the temporary underevaluation has a permanent effect. The lower the bondholders can press down the price, the more shares they receive. After the price correction, the contingent convertible bondholders make a profit. More formally, if the contingent convertible bondholders can temporarily manipulate the firm's value process to any arbitrary level V_{manip} with $V_C \geq V_{manip} > V_B$ and the tax benefits are sufficiently low, the will always do so independently of the parameters of the RCD2 contract:

Proposition 20. *Assume that a firm issues RCD2s and contingent convertible bondholders can temporarily lower the firm's value to an arbitrary V_{manip} with $V_C \geq V_{manip} > V_B$. If the equity value without CoCo bonds is sufficiently high, i.e. $EQ_{debt}(V_t) > CCB(V_t)$, they will always manipulate the market for any $\ell \in (0, \infty)$.*

Proof. See Appendix 12.8. □

We show in the proof, that $EQ_{debt}(V_t) + TB_C(V_t) > CCB(V_t)$ is always satisfied. Hence, if the tax benefits are sufficiently low, $EQ_{debt}(V_t) > CCB(V_t)$ will also hold.

The same mechanism does not work with FSCCB (and hence RCD1) contracts. Here the interests of the contingent convertible bondholders are more aligned with those of the shareholders. Spreading "bad" news can trigger conversion, but will not affect the number of shares granted to the bondholders. Spreading "good" news can increase the value of the stocks, but cannot trigger conversion. Hence, FSCCB contracts offer less incentives for manipulation to the contingent convertible bondholders. More formally, there exists always a conversion parameter ℓ such that the contingent convertible bondholders do not want to manipulate the market:

Lemma 12. *Assume that a firm issues RCD1s and contingent convertible bondholders can* *temporarily lower the firm's value to an arbitrary $V_{manip} > V_B$. If ℓ is small enough such that*

$$CCB(V_t) - \ell P_C \frac{EQ_{debt}(V_t)}{EQ_{debt}(V_C)} \geq 0$$

they will not manipulate the market at time t.

Proof. If contingent convertible bondholders do not manipulate the market, they get CCB(V_t). If the manipulate the market, they obtain shares that have a value of $\ell P_C \frac{EQ_{debt}(V_t)}{EQ_{debt}(V_C)}$ after the price correction. □

Hence, RCD1 contracts can always be designed such that the bondholders do not want to manipulate the market at time t.

Second, we consider manipulation by the equity holders. We start with a RCD1 contract. Equity holders will not manipulate the market at time t if

$$EQ_{old}(V_t, V_B, V_C) - \left(EQ_{debt}(V_t) - \ell P_C \frac{EQ_{debt}(V_t)}{EQ_{debt}(V_C)} \right) \geq 0$$

The first term is the value of their equity if they do not manipulate the market. The second term is the value of their equity in the case of manipulation after conversion and after the price correction. Plugging in the definitions, the inequality is equivalent to

$$TB_C(V_t) - CCB(V_t, V_B, V_C) + \ell P_C \frac{EQ_{debt}(V_t)}{EQ_{debt}(V_C)} \geq 0$$

If $\ell = 1$, i.e. the contingent convertible bondholders receive equity at conversion that has the same market value as the face value of the CCBs, then the inequality will always be satisfied.

Proposition 21. *Assume that a firm has issued straight debt and RCD1s. For $V_t \leq V_0$ and $\ell = 1$, equity holders will never manipulate the market to trigger conversion.*

Proof. See Appendix 12.8. □

As a result, if the conversion value is sufficiently high, the equity holders will not manipulate the market to enforce conversion. A similar reasoning applies to RCD2 contracts. Intuitively, the less likely manipulation by contingent convertible bondholders, the more likely manipulation by the equity holders and vice versa. However, there exist parameters, such that neither of them wants to trigger conversion. After conversion, the tax benefits of the contingent convertible bonds are lost. If these benefits are sufficiently high and the conversion parameter ℓ is chosen accordingly, neither of them will manipulate the market.

Lemma 13. *Assume a firm issues RCD1s. If the conversion parameter ℓ is chosen such that*

$$TB_C(V_t) \geq CCV(V_t) - \ell P_C \frac{EQ_{debt}(V_t)}{EQ_{debt}(V_C)} \geq 0$$

then neither equity holders nor contingent convertible bondholder will manipulate the market.

We have seen that RCD1 contracts are more robust against manipulation than RCD2 contracts. We favor RCD1 contracts with $\ell = 1$. Equity holders will never manipulate such a contract and for sufficiently high coupon payments c_C the contingent convertible bondholders will not do it neither.

8.2 Noise trading

So far we have defined the stock price process as a function of V_t, i.e.

$$S_t = \frac{1}{n}\left(EQ(V_t) - DC(V_t)\right)$$

for $t < \tau_C$ and the only source of risk was the process V_t. We will now suppose that S_t is driven by the firm's value process V_t and an additional independent process. This makes economically sense as changes in the stock price do not necessarily solely reflect changes in the fundamental value. Additional factors, e.g. noise trading, can be captured by including a noise process.

Definition 9. *The endogenous stock price process with noise trading before the time of conversion ($t < \tau_C$) is defined as*

$$S_t = S(V_t) = \frac{EQ(V_t) - DC(V_t)}{n}\left(1 + \tilde{X}_t\right)$$

where n is the number of "old" shares and $\tilde{X}_t$ is an arbitrary martingale process, which has expectation zero, i.e. $E[\tilde{X}_t - \tilde{X}_s|\tilde{X}_s] = 0$ for $s \leq t$ and $\tilde{X}_0 = 0$, and is independent of X_t, which is driving V_t.

The intuition behind this modeling approach is that the stock price should reflect the value of the shareholders' claim on the firm's productive assets. As it is shown empirically stock prices can be more volatile than the fundamental value of the underlying assets. Hence, the noise process $\tilde{X}_t$ should capture this additional source of uncertainty.

Proposition 22. *The conversion value for FSCCBs and RCD1s under a stock price with noise trading equals the corresponding payment without exogenous shocks.*

Proof. The conversion value equals

$$\begin{aligned}
&n'E\left[S(\tau_C)e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right]\\
=&\frac{n'}{n}E\left[(EQ(V_{\tau_C}) - DC(V_{\tau_C})e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right]E\left[\left(1+\tilde{X}_{\tau_C}\right)\right]\\
=&\frac{n'}{n+n'}E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right]
\end{aligned}$$

by the assumption that $E[\tilde{X}_t] = 0$ and the independenc of $\tilde{X}$ and X. □

The situation is different for RCD2s. In this case fluctuations in the stock price affect the number of shares granted to the contingent convertible bondholders at conversion. If for example the stock price falls due to a downward shock in $\tilde{X}$, the bondholders get a higher number of shares although the fundamental value did not change. In more detail, the value of the equity used for the redistribution at the time of conversion is $n \cdot S(\tau_C) = EQ_{debt}(\tau_C) \cdot (1 + \tilde{X}_{\tau_C})$. The condition if the equity is sufficient to fully pay the promised conversion value changes to $V_{\tau_C} > T^{-1}\left(\ell P_C\left(1+\tilde{X}_{\tau_C}\right)^{-1}\right)$.

Proposition 23. *The conversion value for RCD2s under a stock price with noise trading does in general not equal the corresponding payment without exogenous shocks.*

Proof.

$$\begin{aligned} CONV =& \ell P_C E\left[e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\tau\}}\mathbb{1}_{\{V_{\tau_C}>T^{-1}\left(\ell P_C\left(1+\tilde{X}_{\tau_C}\right)^{-1}\right)\}}\right] \\ &+ E\left[EQ_{debt}(V_{\tau_C})\left(1+\tilde{X}_{\tau_C}\right)e^{-(m+r)\tau_C}\mathbb{1}_{\{V(\tau_C)>V_B\}}\mathbb{1}_{\{V_{\tau_C}\leq T^{-1}\left(\ell P_C\left(1+\tilde{X}_{\tau_C}\right)^{-1}\right)\}}\mathbb{1}_{\{\tau_C<\infty\}}\right] \end{aligned}$$

The problem is that $\tilde{X}(\tau_C)$ appears in a nonlinear way in the above formula and a closed-form solution is not available. However, it is obvious to see that in general the expected value above does not coincide with the corresponding expectation without exogenous shocks. □

This is an argument in favor of FSCCBs and RCD1s over RCD2s.

8.3 Multiple equilibria

In Section 3.4.1 we have shown for FSCCBs, that defining the number of shares n' granted to the contingent convertible bondholders at conversion in terms of $S(0)$ will lead to multiple equilibria. This result can be extended to show, that if n' is a function of any S_t with $t < \tau_C$, there can be multiple equilibria. One way to circumvent this problem is simply to avoid linking n' to any stock price. However, when writing a contract it is natural to relate n' to some market price. A very appealing alternative are RCD1s. Here, the number n' is calculated using the model to predict $S(V_C)$. In this setup, all prices are unique. Similarly, for RCD2 contracts, we also obtain unique prices. This can be seen as an argument for RCD1 and RCD2 contracts.

8.4 Optimal design of CCBs

In a perfect market environment, the different CCBs should be equivalent as all information is correctly priced. However, if we take into account manipulation and noise trading, RCD1 contracts with $\ell = 1$ are more robust against this market imperfections than the other contracts. In addition, RCD1 will always have a unique equilibrium price. In order to avoid that CCBs degenerate to a straight debt contract without recovery payment, we have to ensure that Assumption 3 holds. CCB contracts with a high maturity are more likely to satisfy this assumption. For this reason we propose RCD1 with $\ell = 1$ and a high average maturity as the "best" contract.

9 Extensions

9.1 Time-Varying Firm's Value Process

So far we have assumed that the firm's value process does not change after conversion. In particular, the proportional rate at which profit is disbursed to investors δ is constant before and after conversion. Remember, that as the firm has bondholders and shareholders, δ cannot be seen as a dividend rate. First, the coupons and principal payments have to be paid before the residual is paid out as dividends. However, one of the arguments of introducing contingent convertible debt was that after the conversion the coupon payments are lower than before, allowing the firm to recover from financial distress. A constant δ implies that after conversion the total dividend payments equal the former dividend payments plus the payments for the contingent convertible debt. This high dividend payment could be justified economically by the argument, that after conversion the number of shareholders is larger than before. Nonetheless, it seems that a high dividend payment during times of financial distress is not very common. Hence, a more realistic model should take into account that the payout rate δ decreases after conversion. In this section, we will introduce a general approach which allows all parameters of the firm's value process to change after the conversion. Notwithstanding, the focus will be on different δs.

Assume two different payout ratios:

$$\delta_1 \text{ for } [0, \tau_C] \qquad \delta_2 \text{ for } (\tau_c, \tau]$$

and

$$dV_t = V_t \left((r - \delta_1)dt + \sigma dW_t^* + d\left(\sum_{i=1}^{N_t} (Z_i - 1) \right) - \lambda \xi dt \right) \qquad \text{for } t \leq \tau_C$$

$$dV_t = V_t \left((r - \delta_2)dt + \sigma dW_t^* + d\left(\sum_{i=1}^{N_t} (Z_i - 1) \right) - \lambda \xi dt \right) \qquad \text{for } t > \tau_C.$$

By assumption we have $V_C \geq V_B$, which implies

$$\tau \geq \tau_C.$$

Note, that the probability law of τ_C does not change. However, the probability law of τ is not the same anymore. As the value of the coupon payments and the principal repayment of contingent convertible bonds depends only on certain Laplace transforms of τ_C, introducing the time-varying firm's value process does not affect these values. But the prices of straight debt coupons and the conversion values will change. In order to calculate the price of normal debt coupons we need to calculate $E\left[e^{-\rho\tau}\right]$ and $E\left[e^{X(\tau)-\rho\tau}\mathbb{1}_{\{\tau<\infty\}}\right]$, where X_t relates to V_t by $V_t = V_0 \exp(X_t)$.

Theorem 5. *The Laplace transform of the default time for a firm, whose payout ratio changes at conversion, is given by*

$$\begin{aligned} E\left[e^{-\tau\rho}\right] =& \bar{c}_1 \left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{3,\rho}} J(\log(V_C/V_0), \bar{\beta}_{3,\rho}, \log(V_C/V_B), \rho) \\ &+ \bar{c}_2 \left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{4,\rho}} J(\log(V_C/V_0), \bar{\beta}_{4,\rho}, \log(V_C/V_B), \rho) \\ &+ \left(\frac{V_B}{V_C}\right)^{\eta_2} \frac{\eta_2 - \beta_{3,\rho}}{\eta_2} \frac{\beta_{4,\rho} - \eta_2}{\beta_{4,\rho} - \beta_{3,\rho}} \left(\left(\frac{V_C}{V_0}\right)^{\beta_{3,\rho}} - \left(\frac{V_C}{V_B}\right)^{\beta_{4,\rho}} \right) \end{aligned}$$

with

$$\bar{c}_1 = \frac{\eta_2 - \bar{\beta}_{3,\rho}}{\eta_2} \frac{\bar{\beta}_{4,\rho}}{\bar{\beta}_{4,\rho} - \bar{\beta}_{3,\rho}}$$

$$\bar{c}_2 = \frac{\bar{\beta}_{4,\rho} - \eta_2}{\eta_2} \frac{\bar{\beta}_{3,\rho}}{\bar{\beta}_{4,\rho} - \bar{\beta}_{3,\rho}}$$

and $-\bar{\beta}_{3,\rho} > -\bar{\beta}_{4,\rho}$ *are the two negative roots of the equation*

$$\bar{\psi}(\beta) = \rho$$

with $\bar{\psi}$ *being the Lévy exponent of* $\bar{X}_t = (r - \delta_2)t + \sigma W_t^* + \sum_{i=1}^{N_t} Y_i$. *The functions* $-\beta_{3,\rho} > -\beta_{4,\rho}$ *are the two negative roots of the equation* $\psi(\beta) = \rho$, *where* ψ *is the Lévy exponent of* $X_t = (r - \delta_1)t + \sigma W_t^* + \sum_{i=1}^{N_t} Y_i$ *The function* J *is defined as*

$$J(x, \theta, y, \rho) = E\left[e^{-\rho\tau + \theta X_\tau} \mathbb{1}_{\{\tau < \infty, -(X_\tau - x) < y\}}\right]$$

The explicit form of $J(x, \theta, y, \rho)$ *is given in Proposition 8.*

Proof. See Appendix 12.9. □

Theorem 6. *The default time for a firm, whose payout ratio changes at conversion, satisfies the following equality for* $\theta > -\eta_2$*:*

$$\begin{aligned}
E\left[e^{-\tau\rho + \theta X_\tau} \mathbb{1}_{\{\tau < \infty\}}\right] =& \bar{d}_1 \left(\frac{V_B}{V_0}\right)^{-\theta - \bar{\beta}_{3,\rho}} J(\log(V_C/V_0), -\bar{\beta}_{3,\rho}, \log(V_C/V_B), \rho) \\
&+ \bar{d}_2 \left(\frac{V_B}{V_0}\right)^{-\theta - \bar{\beta}_{4,\rho}} J(\log(V_C/V_0), -\bar{\beta}_{4,\rho}, \log(V_C/V_B), \rho) \\
&+ \frac{\eta_2 - \beta_{3,\rho}}{\beta_{4,\rho} - \beta_{3,\rho}} \frac{\beta_{4,\rho} + \theta}{\eta_2 + \theta} \left(\frac{V_C}{V_0}\right)^{\theta + \beta_{3,\rho}} + \frac{\beta_{4,\rho} - \eta_2}{\beta_{4,\rho} - \beta_{3,\rho}} \frac{\beta_{3,\rho} + \theta}{\eta_2 + \theta} \left(\frac{V_C}{V_0}\right)^{\theta + \beta_{4,\rho}} \\
&- J(\log(V_C/V_0), \theta, \log(V_C/V_B), \rho)
\end{aligned}$$

where

$$\begin{aligned}
\bar{d}_1 &= \frac{\eta_2 - \bar{\beta}_{3,\rho}}{\bar{\beta}_{4,\rho} - \bar{\beta}_{3,\rho}} \frac{\bar{\beta}_{4,\rho} + \theta}{\eta_2 + \theta} \\
\bar{d}_2 &= \frac{\bar{\beta}_{4,\rho} - \eta_2}{\bar{\beta}_{4,\rho} - \bar{\beta}_{3,\rho}} \frac{\bar{\beta}_{3,\rho} + \theta}{\eta_2 + \theta}
\end{aligned}$$

and with the same notation as in Theorem 5 for the rest.

Proof. See Appendix 12.9. □

This allows us to calculate the price of straight debt. For the conversion value we need to make only a small change in the evaluation formulas. The value of equity after conversion $EQ(V_{\tau_C})$ has to be determined using the second process, i.e. we replace all $\beta_{3,\rho}$ and $\beta_{4,\rho}$ with $\bar{\beta}_{3,\rho}$ and $\bar{\beta}_{4,\rho}$ in the corresponding formula. The choice of the optimal default barrier is analogous to the case of a constant δ.

10 Finding an Optimal Regulation Scheme

The main question is if contingent convertible bonds can be used as a regulation instrument for banks. A “good” regulation instrument would reduce the default probability of a bank without imposing to high costs on the bank. Intuitively, the higher the amount of debt of a firm, the higher the default probability. A simple way to limit the default probability is to limit the amount of debt that a firm is allowed to have. This is equivalent to requiring the firm to hold a minimum amount of equity. However, there are a cost to this regulation, as the firm would lose tax benefits. Instead of limiting the amount of debt, the regulator could require the bank to replace part of its debt with contingent convertible bonds. For example, Flannery (2009a) proposes a scenario, in which banks can choose between holding equity equal to 6% of an asset aggregate, or holding equity equal to 4% of the asset aggregate and CCBs equal to 4% of the asset aggregate.

In the following we will first analyze the optimal capital structure of a firm without regulation. Then we show that, if Assumption 3 is satisfied, a regulation schemes that restricts the amount of straight debt and requires mandatory issuing of CCBs, strictly dominates a regulation that only limits the amount of straight debt. The CCB regulation scheme will achieve the same upper bound on the level of risk, but the total value of the firm will be the same as under no regulation. In addition, under the CCB regulation scheme the costs to the government in terms of tax benefits are lower than if no regulation is imposed. However, if Assumption 3 is violated, these results do not hold any more .

Throughout the section we make the following two assumptions. First, the conversion barrier V_C is exogenously given. Second:

Assumption 6. *The coupon payments of the contingent convertible bonds are positive:*

$$c_C > 0.$$

Note, that as long as there are bankruptcy costs the coupon of straight debt is always positive, when debt is issued at par at time 0. Assumption 6 is satisfied in all practically relevant situations. For example, for all RCD1 and RCD2 with $\ell \leq 1$, which are issued at par at time 0, the assumption is satisfied.

10.1 Optimal Capital Structure without Regulation

First we consider a firm that issues only normal debt bonds. The parameter V_B is determined endogenously while the parameters m, λ, θ, V_0, c_D and r can be assumed to be given exogenously. Hence, the only remaining choice parameter is the amount of debt P_D. This will be chosen to maximize the total value of the firm, i.e.

$$\max_{P_D} G_{debt} = \max_{P_D}(V + TB_D - BC).$$

Chen and Kou (2009) show that $G_{debt}(P_D)$ is a strictly concave function in P_D. Hence, for any given V, there exists a unique P_D that maximizes $G_{debt}(P_D)$. As the initial value of the firm V is given, the optimal choice of P_D is a tradeoff between tax benefits TB_D and bankruptcy costs BC. Based on our endogenously determined parameters we solve

$$\frac{\partial TB_D}{\partial P_D} = \frac{\partial BC}{\partial P_D}.$$

If we allow the firm to issue CCBs in addition to normal debt, the optimization problem changes to

$$\max_{P_D, P_C} (V + TB_D + TB_C - BC).$$

We consider two cases: Either Assumption 3 is satisfied or not. The tricky part is, that this assumption depends on the amount of debt issued by a firm. Hence, we will first analyze the optimal capital structure conditioned on satisfying this constraint. Second, we determine the optimal amount of debt under the restriction that the assumption is violated. Finally, the firm picks the one of the two combinations $\{P_D, P_C\}$, which yields a higher total value of the firm.

Case 1: Assumption 3 satisfied:

Under Assumption 3, the optimal barrier level V_B equals V_B^* which is independent of any features of CCBs. The optimization problem becomes

$$\max_{P_D, P_C} (V + TB_D(P_D) + TB_C(P_C) - BC(P_D)).$$

The FOC for P_D is then

$$\frac{\partial TB_D(V, V_B^*)}{\partial P_D} = \frac{\partial BC(V, V_B^*)}{\partial P_D},$$

which coincides with the case without CCBs. The optimal level of straight debt does not depend on any characteristics of the CCBs. The next lemma implies, that there exists a unique value of P_D that maximizes the total value of the firm.

Lemma 14. *The total value of the firm $G(P_D)$ is a strictly concave function in P_D, if $V_B = V_B^*$.*

Proof. The only difference between $G(P_D)$ and $G_{debt}(P_D)$ are the tax benefits $T_C(P_C)$, which do not depend on P_D. Chen and Kou (2009) have proven the strict concavity of $G_{debt}(P_D)$. □

The total value of the firm depends on P_C only through the tax benefits TB_C, which are monotonically increasing in P_C.

Corollary 7. *If the coupon payments c_C are positive, then the total value of the firm $G(P_C)$ is increasing in the value of contingent convertible debt P_C.*

Proof. The total value of the firm is defined as

$$G = V + TB_D + TB_C - BC.$$

As the amount of contingent convertible bonds only affects the tax benefits TB_C which are defined by $\frac{\bar{c}c_C P_C}{r} E[1 - e^{-r\tau_C}]$ and the coupons c_C are not negative, the statement follows. □

As a consequence of Corollary 7, equity will be crowded out by CCBs one-to-one as long as Assumption 3 is satisfied. As we have seen the firm as a whole will always profit from issuing CCBs, while the taxpayer pays the cost of the additional tax shield. We will in the following assume that a firm can only issue CCBs as certain fraction of its debt. This assumption is implicit or explicit in various proposals to use CBBs for banking regulation. Hence, there will be only tax benefits for CBBs issued as part of a regulation requirement.

Case 2: Assumption 3 violated:

If Assumption 3 is violated, the optimal default barrier is the maximum of V_B^{**} and V_C. First, we will focus on the case $V_B = V_B^{**}$, which depends on P_D and P_C. The optimization problem becomes

$$\max_{P_D, P_C} (V + TB_D(P_D, P_C) + TB_C(P_D, P_C) - BC(P_D, P_C)).$$

The FOC for P_D and P_C are then

$$\begin{aligned} \frac{\partial TB_D(V, V_B^{**}) + TB_C(V, V_B^{**})}{\partial P_D} &= \frac{\partial BC(V, V_B^{**})}{\partial P_D} \\ \frac{\partial TB_D(V, V_B^{**}) + TB_C(V, V_B^{**})}{\partial P_C} &= \frac{\partial BC(V, V_B^{**})}{\partial P_C}. \end{aligned}$$

We start with the special case $c_D = c_C$, i.e. the coupon payments for straight debt and contingent convertible debt are the same. In this case, P_D and P_C are perfect "substitutes" for the firm as the total value of the firm will only be influenced by $P_D + P_C$. The total value of the firm is the same as for a firm that issues only straight debt in the amount of $P_D + P_C$.

Corollary 8. *If $c_D = c_C$, then the firm would like to choose an amount of debt such the default barrier is the same as in case 1.*

Proof. Define $\tilde{P} = P_D + P_C$ and $\tilde{c} = c_D = c_C$. Obviously it holds $\tilde{C} = \tilde{c}\tilde{P} = c_D P_D + c_C P_C = C_D + C_C$ which implies that $EQ_{debt}(V, V_B, P_D + P_C, C_D + C_C) = EQ(V, V_B, \tilde{P}, \tilde{C})$ and $\mathrm{G}_{debt}(\tilde{P}, \tilde{C}) = G_{debt}(P_D + P_C, C_D + C_C)$, i.e. the total value of the firm and the equity value of the old shareholders is only influenced by the sum $P_D + P_C$. Hence, the optimal default barrier $V_B^{**}(\tilde{P}, \tilde{C})$ will be the same as in the case where only straight debt is issued. □

The hypothetically optimal V_B^{**} is not feasible, as it would be smaller than V_C. This leads to the following conclusion:

Corollary 9. *Assume that $c_D = c_C$. The optimal debt choice $\{P_D, P_C\}$ is any combination of P_D and P_C such that $V_B^*(P_D + P_C) = V_C$.*

If $c_D \neq c_C$, straight debt and the degenerated contingent convertible debt are not perfect substitutes any more. We assume that at time zero all the debt is issued at par. At conversion, which coincides with default, the contingent convertible bondholders receive nothing, while the straight debt bondholders get the recovery payment. Thus, a higher coupon $c_C > c_D$ is needed to compensate the contingent convertible bondholders. Hence, we will assume that the coupon c_C for the contingent convertible debt has to be higher than c_D. The optimization problem of the firm is then

$$\max_{P_D, P_C} G_{debt}(\tilde{P}, \tilde{C}) \qquad \text{subject to } \tilde{P} = P_D + P_C \text{ and } \tilde{C} = c_D P_D + c_C P_C.$$

This problem is equivalent to

$$\max_{\tilde{P}, \tilde{c}} G_{debt}(\tilde{P}, \tilde{c}\tilde{P}) \qquad \text{subject to } \tilde{c} \in [c_D, c_C].$$

We split this two-dimensional problem into a two-stage optimization problem. In the first stage, for any $\tilde{c} \in [c_D, c_C]$ we solve the problem and obtain a unique optimal amount of debt $\tilde{P}(\tilde{c})$ and optimal default barrier $V_B^*(\tilde{c})$ (for simplicity we express the optimal default barrier only in terms of the remaining choice variable $\tilde{c}$). In a second stage, the coupon $\tilde{c}$ is chosen that maximizes the total value of the firm. We denote the optimal coupon by c^*:

$$c^* = \arg \max_{\tilde{c} \in [c_D, c_C]} G_{debt}(\tilde{P}(\tilde{c}))$$

Proposition 24. *Assume that $c_D < c_C$. If $V_B^*(c^*) \geq V_C$, then the optimal debt choice $\{P_D, P_C\}$ is the combination of P_D and P_C that satisfies*

$$P_D + P_C = \tilde{P}(c^*) \qquad \text{and} \qquad c^*(P_D + P_C) = P_D c_D + P_C c_C$$

If $V_B^*(c^*) < V_C$, then the optimal debt choice $\{P_D, P_C\}$ is the highest amount of P_C such that two conditions are satisfied: 1. $V_B^{**}(P_D, P_C) = V_C$ and 2. Assumption 3 is violated.

Proof. See Appendix 12.10. □

The key result of this section is the following. If Assumption 3 is satisfied, the default barrier will be strictly smaller, than in the other case. In addition, if Assumption 3 is violated, the default barrier is to some extent unresponsive to restrictions in the maximal amount of straight debt, as straight debt and the degenerated contingent convertible debt become (perfect or imperfect) substitutes. In the next section we will discuss regulation. Intuitively speaking, a regulator wants to enforce a small default barrier, because this will imply a lower default probability. Based on this section we will conclude that the regulator wants to require that only contingent convertible bonds satisfying Assumption 3 are issued.

10.2 Optimal capital structure with regulation

In this section we discuss different regulation schemes. The regulator will impose restrictions on the capital structure of a bank, such that the "risk" does not exceed a pre-specified level. We will formalize the concept of "risk" from the perspective of a regulator, but intuitively the regulator wants to enforce a low default barrier. The lower the default barrier, the lower the probability of default.

Definition 10. *The probability that default happens before t as a function of P_D is defined as*

$$\Upsilon(t, P_D) = \mathbb{P}(\tau \leq t).$$

We assume that the regulator uses a specific risk measure:

Definition 11. *Denote the parameter space of the choice variables as Θ. We are only interested in $\Theta = \{(P_D, P_C) \in [0, \infty)^2\}$. A risk measure $\chi_i(P_D, P_C)$ is a mapping from Θ to $[0, \infty)$. A regulation scheme is defined as a restriction of the parameters of our model to the set $\tilde{\Theta}$ such that $\chi_i(P_D, P_C) \leq \omega$ for all $(P_D, P_C) \in \tilde{\Theta}$ and a fixed risk level ω. If we consider only one choice variable we suppress the other in the notation of χ_i.*

We make the additional assumption that the risk measure has the property that if the default probability $\mathbb{P}(\tau \leq t)$ is higher under (P_D, P_C) than under $(\tilde{P}_D, \tilde{P}_C)$ for all t, then $\chi_i(P_D, P_C) \geq \chi_i(\tilde{P}_D, \tilde{P}_C)$.

Definition 12. *A capital requirement ρ_i is defined as the maximum amount of debt P_D that a firm is allowed to include in its capital structure such that risk measure $\chi_i(P_D)$ is always smaller than some critical value ω:*

$$\rho_i = \sup\{P_D \in [0,\infty) : \chi_i(\tilde{P}_D) \leq \omega \ \forall \tilde{P}_D \leq P_D\}.$$

Example 1. *We define*

$$\rho_1(\omega) = \sup\{P_D \in [0,\infty) : \Upsilon(1, P_D) \leq \omega\}$$

The measure ρ_2 is not restricted to the time period 1:

$$\rho_2(\omega) = \sup\left\{P_D \in [0,\infty) : \int_0^\infty s(t)\Upsilon(t, P_D)dt \leq \omega\right\}$$

where $s(t)$ is a weighting function satisfying $\int_0^\infty s(t)dt = 1$.

The intuition behind the two capital requirements ρ_i is that by setting ω sufficiently low, the default probability is restricted from above in a certain sense. Our model allows us to calculate the Laplace transform of the default time in closed form. Applying Laplace inversion we can numerically calculate the two capital requirement regulation schemes.

If we want to compare different regulation schemes, we have to specify what a "good" regulation means. Regulation can be costly to the firm and the taxpayer. The taxpayers are affected by the amount of tax benefits that they are granting to the firm, while a not "optimal" amount of debt can lower the total value of the firm G.

Definition 13. *If two regulation schemes have the same maximum amount of risk ω as specified by the risk measure χ_i, the first regulation scheme is said to be more efficient if the total value of the firm is strictly higher than under the second scheme.*

We want to show that requiring a firm to replace a certain amount of its straight debt by contingent convertible debt can be a more efficient regulation scheme than using only maximal capital requirements.

Throughout this section we assume that Assumption 3 is satisfied. This has the following consequences:

1. $G(P_D)$ is strictly concave and there exists a unique optimal amount of debt, which we will denote by P_D^*.
2. The optimal default barrier is V_B^*.

We can conclude the following:

Corollary 10. *The optimal default barrier V_B^* is a function of P_D but not P_C.*

Corollary 11. *The optimal amount of debt P_D is independent of any features of CCBs.*

Corollary 12. *The default probability $\Upsilon(t, P_D)$ is strictly increasing in P_D.*

Corollary 13. *The default probability $\Upsilon(t, P_D)$ is independent of P_C.*

Corollary 14. *Any risk measure* χ_i *is increasing in* P_D.

Corollary 15. *The risk measures* χ_1 *and* χ_2 *are independent of* P_C.

Now we want to compare a capital requirement regulation scheme ρ_i with a regulation scheme that requires the mandatory issuing of CCBs.

Definition 14. *A CCB regulation scheme is a tuple* $\phi_i = (\phi_i^D, \phi_i^C)$ *of an upper bound on the amount of straight debt* ϕ_i^D *and a fixed amount of CCBs* ϕ_i^C *such that*

$$\chi_i(\tilde{P}_D, \tilde{P}_C) \leq \omega \quad \forall \tilde{P}_D \leq \phi_i^D, \quad \tilde{P}_C = \phi_i^C.$$

In order to prove rigorously that a CCB regulation scheme is more efficient than a capital requirement regulation scheme, we have required Assumption 3, which can be tested. The economic intuition behind our regulation approach is straightforward. First, we assume that there exists an optimal level of leverage of straight debt. This makes sense as a firm issuing straight debt faces the tradeoff between tax benefits and bankruptcy costs and the optimal leverage should set the marginal gains of tax benefits equal to the marginal costs of bankruptcy costs. Next, it is also intuitive to assume that a higher amount of straight debt increases the default probability. Hence, if the optimal leverage of a firm implies a too high default probability from the point of view of the regulator, one way to reduce it is to require the firm to lower its level of debt. This would also lower the tax benefits associated with the straight debt. As the new level of leverage is not optimal for the firm any more the total value of the firm will be lower under such a regulation. However, the firm as a whole would benefit from issuing CCBs as it profits from the tax benefits. The amount of CCBs can be chosen such that its tax benefits exactly compensate for the loss due to the capital requirement.

Proposition 25. *Consider first a firm without any regulation. Its optimal amount of debt is* P_D^* *and the maximal total value of the firm is* $G(P_D^*)$. *Second consider a capital requirement* ρ_i. *The risk measured by* χ_i *for* $i = 1, 2$ *under this scheme is limited to* ω *and the loss in total value to the firm is* $G(P_D^*) - G(\rho_i)$. *Third, we define a CCB regulation scheme as* (ϕ_i^D, ϕ_i^C), *where* $\phi_i^D = \rho_i$ *and* ϕ_i^C *is such that* $TB(\phi_i^C) = G(P_D^*) - G(\rho_i)$. *The risk under the CCB regulation scheme is bounded by* ω *and the total value of the firm is equal to the value under no regulation, i.e. it is efficient compared to the capital requirement and the firm is indifferent between the CCB regulation and no regulation.*

Note that the above regulation scheme is in a certain sense equivalent to a regulation where existing straight debt is partly replaced by CCBs. Hence, if the optimal amount of debt P_D^* without regulation is known, requiring the firm to replace a certain fraction of the optimal debt amount by CCBs and hence ending up with a lower level of straight debt, will yield exactly the same outcome.

We make an additional assumption:

Assumption 7. *The value of the contingent convertible debt is larger than the related tax benefits:* $CB > TB_C$.

For a realistic tax rebate rate, this assumption will always be satisfied.

Lemma 15. *The maximal total tax benefits under the CCB regulation scheme are lower than the maximal tax benefits under no regulation.*

Proof. See Appendix 12.10. □

Definition 15. *The total leverage is defined as*

$$TL = \frac{P_D + P_C}{G}.$$

Lemma 16. *The maximal possible total leverage under the CCB regulation scheme is higher than the maximal possible total leverage under a pure capital requirement regulation scheme, if Assumption 7 is satisfied.*

Proof. See Appendix 12.10. □

The above CCB regulation scheme yields the same total value for the firm as the case where no regulation is imposed. Hence, the firm as a whole does not suffer. Next, the tax deduction costs for the taxpayer are lower compared to the case without regulation. Therefore, the taxpayer is better off. Most importantly, the default probability is lower than in the case without regulation.

The above results hold only if Assumption 3 is satisfied. If this assumption is violated, the optimal default barrier is at least V_C, which is by definition larger than V_B^*. Furthermore, if Assumption 3 is violated, the default barrier can increase in the amount of contingent convertible bonds P_C. Hence, requiring a bank to issue CoCo bonds can actually increase its risk. Next, as in this case straight debt and degenerated contingent convertible bonds become (perfect or imperfect) substitutes, any capital requirement on the straight debt can be circumvented by issuing more CoCos. The traditional capital requirement regulation would become ineffective. Hence, a regulator would always impose restrictions such that Assumption 3 holds.

10.3 TBTF Firms

In this section we analyze firms that are "too big to fail" (TBTF). As bankruptcy of such firms might result in a crisis of the overall financial system, the government will not let them fail. At the time of default of a TBTF firm the government will take over its assets and its obligations to make payments to debt holders. Hence, the debt holders of TBTF firms have an implicit government guarantee on their debt contract, which makes their debt basically risk-free.

We will first model formally a TBTF firm, that issues only straight debt. As its debt is risk-free, the value equals:

$$D_{debt}^{TBTF} = \frac{C_D + mP_D}{m + r}.$$

This comes at a cost to the government. At the time of default, the government steps in and obtains assets worth $V(\tau)$. In return, the government takes over the obligation to make the coupon payments and repayments of the face value of debt forever. Therefore, the value of the government subsidy for the firm is

$$SUB^{TBTF}(V, V_B) = \frac{C_D + mP_D}{m + r} E\left[e^{-(m+r)\tau}\right] - E\left[V(\tau)e^{-r\tau}\right].$$

The total value of the firm equals the value of the firm's assets plus the tax benefits and the government subsidy. Because of the potential government bailout, the bankruptcy costs do not appear in total value of the firm.

$$\begin{aligned} G^{TBTF}_{debt}(V, V_B) &= V + TB_D(V, V_B) + SUB^{TBTF}(V, V_B) \\ &= V + \frac{\bar{c}C_D}{r} E\left[1 - e^{-r\tau}\right] + \frac{C_D + mP_D}{m+r} E\left[e^{-(m+r)\tau}\right] - E\left[V(\tau)e^{-r\tau}\right] \end{aligned}$$

The equity value is the residual claim of the total value of the firm after the value of the debt is subtracted:

$$EQ^{TBTF}_{debt}(V, V_B) = G^{TBTF}_{debt}(V, V_B) - D^{TBTF}_{debt}$$

Albul, Jaffee and Tchistyi (2010) consider TBTF firms in their model with infinite maturity bonds. They show, that if only consol bonds are issued, the value of the equity and the optimal default barrier are the same as for a normal firm. However, in our model with a rolling debt structure this result does not hold any more.

Proposition 26. *The optimal default barrier of a TBTF firm equals*

$$V_B^{***} = \frac{\frac{C_D + mP_D}{r+m}\beta_{3,r+m}\beta_{4,r+m} - \frac{\bar{c}C_D}{r}\beta_{3,r}\beta_{4,r}}{(\beta_{3,r}+1)(\beta_{4,r}+1)} \frac{\eta_2 + 1}{\eta_2}.$$

Proof. V_B^{***} is simply the solution to the smooth pasting condition:

$$\left(\frac{\partial (EQ^{TBTF}_{debt})(V, V_B)}{\partial V} |_{V=V_B} \right) = 0$$

□

A TBTF firm does not face the tradeoff between tax benefits and bankruptcy costs. As long as $V_B = V_B^{***}$ the firm will issue as much debt as possible.

Proposition 27. *Assume that $V_B = V_B^{***}$. The total value of a TBTF firm is strictly increasing in the amount of straight debt:*

$$\frac{\partial G^{TBTF}_{debt}}{\partial P_D} > 0$$

Hence, the regulator should restrict the total amount of debt, that a TBTF firm is allowed to issue. Assume that the regulator wants to limit the risk of all banks to specific level. According to our definition of risk, this is equivalent to imposing an upper bound on the default barrier V_B. Let's denote this target default barrier by $\bar{V}_B$. Assume, that we have two firms that are identical, but one is considered TBTF, while the other does not profit from an implicit government guarantee. What is the upper bound on the amount of straight debt for these two firms, that ensures that the default barrier is below $\bar{V}_B$? Proposition 13 and Proposition 26 imply that the optimal default barriers are proportional to P_D:

Proposition 28. *The optimal default barriers for a normal firm and a TBTF firm can be written as*

$$V_B^* = \kappa^* P_D \qquad V_B^{***} = \kappa^{***} P_D.$$

It holds $\kappa^* \leq \kappa^{***}$. Therefore, in order to enforce that the default barrier is below the critical level $\bar{V}_B$, the regulator has to use a stricter capital requirement for TBTF firms ($P_D \leq \bar{V}_B/\kappa^{***}$), than for a normal firm ($P_D \leq \bar{V}_B/\kappa^*$).

This proposition says that the default risk is increasing faster in the amount of straight debt P_D for a TBTF firm than for a normal firm. Extending the evaluation formulas for CCBs to a TBTF firm is straightforward. We just need to replace EQ_{debt} by EQ_{debt}^{TBTF} and V_B^* by V_B^{***}. The regulator can apply a similar CCB regulation scheme to a TBTF firm as described in the last subsection. The CCBs can be used to compensate the firm for its loss in the total value due to the capital requirement. As the TBTF already profits from the government subsidy SUB^{TBTF}, it will usually need less tax benefits from the CCBs to obtain the same total value as a firm without this subsidy. The main takeaway of this subsection is that a TBTF firm will always have a lower amount of straight debt than a comparable normal firm under regulation.

11 Conclusion

In the aftermath of the financial crisis of 2008 contingent convertible bonds were discussed as regulation instruments for banks. CCBs are new debt instruments that automatically convert to equity when the issuing firm or bank reaches a specified level of financial distress. We conceptualize the modeling of CCBs and we are the first to present a formal model for this new hybrid security, which incorporates jumps in the firm's value process and allows for a rolling debt structure. We extend Chen and Kou's model to incorporate contingent convertible debt by introducing a second barrier which triggers conversion. We are able to completely characterize two different types of CCBs: In the first case the number of shares granted at conversion is fixed a priori. In the second specification the number of shares granted at conversion is chosen a posteriori such that the value of the shares equals a specified value. We are the first to determine the dilution costs to the old shareholders for the two types of CCBs. Our analysis shows that CCBs behave similarly to straight debt in many ways: The credit spread as a function of maturity is humped-shaped and the limiting credit spread for a maturity approaching zero is generally non-zero.

However, the specification of the conversion payment has huge effects on the features of CCBs. In order to obtain a unique equilibrium price for FSCCBs, certain restrictions have to be imposed on the design of this debt contract. There are two different conceptional approaches to modeling RCDs. In one approach RCDs and FSCCBs can be incorporated into the same unified framework. In the other approach, FSCCBs and RCDs have very distinct properties. We also explain how to evaluate the model if the parameters of the firm's value process change after conversion.

We discuss whether conversion can be based on observable market prices. We show that the conversion event can be specified in terms of credit spreads or risk premiums for CDSs, leading to the same pricing formulas that we have obtained when conversion was triggered by movements in the firm's value process. Hence, our evaluation formulas can be applied in practice with a trigger event based on observable market prices.

An important question concerns the optimal design of CCBs. We show that FSCCBs and RCD1s are not affected by noise in the stock price process. Furthermore, FSCCB and RCD1 contracts are more robust against manipulation by the contingent convertible bondholders. If in the case of conversion bond holders of RCD1s get shares that have the same value as the face value of debt (i.e. the conversion parameter ℓ is equal to 1), then equity holders will never manipulate the market. Therefore, we favor RCD1 contracts with $\ell = 1$.

Last but not least we analyze the potential of CCBs as a regulation instrument. If the no-early-default condition is satisfied a regulation combining leverage restrictions and the requirement of issuing a certain fraction of CCBs can efficiently lower the default probability without reducing the total value of the firm. However, if the no-early-default condition is violated, a CCB regulation can actually increase the risk. In order to ensure that this condition holds, only CCBs with a long maturity should be issued.

12 Appendix

12.1 Proofs for Section 2

Proposition 2. *The total value of the firm equals*

$$G_{debt}(V,V_B) = V + \frac{\bar{c}C_D}{r} E\left[1 - e^{-r\tau}\right] - \alpha E\left[V(\tau)e^{-r\tau}\mathbb{1}_{\{\tau<\infty\}}\right]. \tag{12.1}$$

Proof. By definition, the total value of the firm equals

$$\begin{aligned}
G_{debt}(V,V_B) &= V + TB_D(V,V_B) - BC(V,V_B) \\
&= V + \bar{c}C_D E\left[\int_0^\tau e^{-rt}dt\right] - \alpha E\left[V(\tau)e^{-r\tau}\mathbb{1}_{\{\tau<\infty\}}\right] \\
&= V + \frac{\bar{c}C_D}{r} E\left[1 - e^{-r\tau}\right] - \alpha E\left[V(\tau)e^{-r\tau}\mathbb{1}_{\{\tau<\infty\}}\right].
\end{aligned}$$

□

12.2 Proofs for Section 3

Proposition 3. *The total value of all outstanding convertible debt equals*

$$\begin{aligned}
CB(V,V_B,V_C) &= \int_0^\infty p\Psi(t)d_C(V,V_B,V_C,t)dt \\
&= \left(\frac{c_C P_C + mP_C}{m+r}\right) E\left[1 - e^{-(m+r)\tau_C}\right] + CONV(V,V_B,V_C)
\end{aligned}$$

where $CONV$ is the total conversion value

$$CONV(V,V_B,V_C) = \int_0^\infty p_C \cdot \Psi(t) \cdot conv(V,V_B,V_C,t)dt$$

Proof. First note that

$$\Psi(s) = \int_s^\infty \varphi(y)dy = \int_s^\infty me^{-my}dy = e^{-sm}$$

Hence, it follows that $P_C = p_C \int_0^\infty \Psi(s)ds = p_C \int_0^\infty e^{-ms}ds = \frac{p_C}{m}$. Therefore, we get

$$\begin{aligned}
CB(V,V_B,V_C) = &\int_0^\infty p\Psi(t)d_C(V,V_B,V_C,t)dt \\
=&p_C c_C E\left[\int_0^{\tau_C} e^{-rt}\int_t^\infty \Psi(s)dsdt\right] + p_C E\left[\int_0^{\tau_C} e^{-rs}\Psi(t)dt\right] \\
&+ \int_0^\infty p_C \cdot \Psi(t) \cdot conv(V,V_B,V_C,t)dt \\
=&p_C c_C E\left[\int_0^{\tau_C} e^{-rt}\frac{1}{m}e^{-mt}dt\right] + p_C E\left[\int_0^{\tau_C} e^{-rt}e^{-mt}dt\right] + CONV(V,V_B,V_C) \\
=&\left(\frac{p_C c_C}{m} + p_C\right) E\left[\frac{1}{-(r+m)}\left(e^{-(r+m)\tau_C} - 1\right)\right] + CONV(V,V_B,V_C) \\
=&\frac{c_C P_C + mP_C}{m+r} E\left[1 - e^{-(m+r)\tau_C}\right] + CONV(V,V_B,V_C).
\end{aligned}$$

□

Proposition 4. *If the value of the shares, that holders of a single contingent convertible bond with face value 1 receive at conversion, is $n'S(\tau)/P_C$, then the value of the individual bond satisfies*

$$d_C(V, V_B, V_C, t) = E\left[\int_0^{t\wedge\tau_C} c_C e^{-rs} ds\right] + E\left[e^{-rt}\mathbb{1}_{\{t<\tau_C\}}\right] + \frac{n'}{P_C} E\left[S(\tau_C)e^{-r\tau_C}\mathbb{1}_{\{\tau_C\leq t\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right].$$

Under the assumption of an exponential maturity profile $\varphi(t) = me^{-mt}$ the total value of the convertible debt CB is given by:

$$CB(V, V_B, V_C) = \left(\frac{c_C P_C + mP_C}{m+r}\right) E\left[1 - e^{-(m+r)\tau_C}\right] + n'E\left[S(\tau_C)e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right].$$

Proof. We only need to calculate the total conversion value:

$$\begin{aligned} CONV &= \int_0^\infty p_C \Psi(t) d_C(V, V_B, V_C, t)dt \\ &= \frac{n'}{P_C}\int_0^\infty p_C e^{-mt} E\left[S(\tau_C)e^{-r\tau_C}\mathbb{1}_{\{\tau_C\leq t\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right] dt \\ &= \frac{n'}{P_C} p_C E\left[S(\tau_C)e^{-r\tau_C}\int_{\tau_C}^\infty e^{-ms}ds\mathbb{1}_{\{V_{\tau_C}>V_B\}}\mathbb{1}_{\{\tau_C<\infty\}}\right] \\ &= \frac{n'}{P_C} p_C E\left[S(\tau_C)e^{-r\tau_C}\frac{1}{m}e^{-m\tau_C}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\mathbb{1}_{\{\tau_C<\infty\}}\right] \\ &= \frac{n'}{P_C}\frac{p_C}{m} E\left[S(\tau_C)e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right]. \end{aligned}$$

□

Proposition 5. *If $n' = \frac{P_C \ell}{S_0}$, then there exist two different combinations of prices for $\{S_0, DC(0)\}$, which satisfy the consistency and equilibrium conditions for FSCCBs. The dilution costs $DC(V_0)$ at time $t = 0$ for a contingent convertible bond with such a fixed number of shares equal:*

$$DC(V_0) = \frac{EQ(V_0) + n'S_0}{2} \pm \sqrt{\left(\frac{EQ(V_0) + n'S_0}{2}\right)^2 - E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] n'S_0}.$$

Proof. Equation 3.3 has to be satified for V_0, which implies

$$DC(V_0) = \frac{n'S_0}{EQ(V_0) - DC(V_0) + n'S_0} E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right]$$

Solving for $DC(V_0)$ yields

$$DC(V_0) = \frac{EQ(V_0) + n'S_0}{2} \pm \sqrt{\left(\frac{EQ(V_0) + n'S_0}{2}\right)^2 - E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] n'S_0}.$$

As the price of FSCCBs at time zero is a function of $DC(0)$, multiplicity of the market value of the dilution costs results in multiple equilibrium prices. □

Proposition 6. *If the value of the shares given to holders of contingent convertible bonds at conversion is ℓ, the values of the individual bonds of RCD1 under Assumption 2 satisfy*

$$d_C(V, V_B, V_C, t) = E\left[\int_0^{t\wedge\tau_C} c_C e^{-rs} ds\right] + E\left[e^{-rt}\mathbb{1}_{\{t<\tau_C\}}\right] + \ell E\left[\frac{S(\tau_C)}{S(V_C)}e^{-r\tau_C}\mathbb{1}_{\{\tau_C\leq t\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right].$$

For an exponential maturity profile $\varphi(t) = me^{-mt}$ the total value of the convertible debt CB for RCD1 under Assumption 2 is given by:

$$CB(V, V_B, V_C) = \left(\frac{c_C P_C + mP_C}{m+r}\right) E\left[1 - e^{-(m+r)\tau_C}\right] + \ell P_C E\left[\frac{EQ(V_{\tau_C})}{EQ(V_C)}e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{V_{\tau_C}>V_B\}}\right].$$

Proof. Recall that under Assumption 2 n and n' are set as

$$n = \frac{EQ(V_0) - DC(V_0)}{S_0} \qquad \text{and} \qquad n' = \frac{\ell P_C}{S(V_C)}.$$

At the time of conversion the stock price $S(V_C)$ equals the equity value divided by the number of old and new shares:

$$S(V_C) = \frac{EQ(V_C)}{n+n'} = \frac{EQ(V_C)}{n + \frac{\ell P_C}{S(V_C)}}$$

which is equivalent to

$$\begin{aligned} S(V_C)n + \ell P_C &= EQ(V_C) && \Leftrightarrow \\ S(V_C) &= \frac{EQ(V_C) - \ell P_C}{n} && \Leftrightarrow \\ \frac{S(V_C)}{S_0} &= \frac{EQ(V_C) - \ell P_C}{EQ(V_0) - DC(V_0)}. \end{aligned}$$

Hence, the dilution costs simplify to

$$\begin{aligned} DC(V_0) =& \frac{\ell P_C}{(EQ(V_0) - DC(V_0))S(V_C)/S(0) + \ell P_C} E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] \\ =& \ell P_C E\left[\frac{EQ(V_{\tau_C})}{EQ(V_C)}e^{-(r+m)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{\tau_C<\tau\}}\right]. \end{aligned} \tag{12.2}$$

By Lemma 1 it follows

$$CONV(V_0) = DC(V_0).$$

Note, that the value of the equity after conversion is independent of any features of the contingent convertible debt. In particular, the dilution costs do not appear on the RHS of equation 12.2. □

12.3 Modeling RCD2s

In this section we present the details for modeling RCD2s. The number of shares granted at conversion is determined based on the stock price $S(V_{\tau_C})$, i.e. the contingent convertible bondholders receive $n' = \frac{\ell P_C}{S(V_{\tau_C})}$ shares. We do not require Assumption 2 to be satisfied.

Assume that the value of equity for $V_{\tau_C} = V_C$ is sufficient to pay the conversion value. Then, in a model without jumps and $\ell = 1$ the contingent convertible bond with face value 1 has the same features as a riskless bond with face value 1. If we include jumps the contingent convertible bondholders face the additional risks that conversion and bankruptcy happen simultaneously or that the value of the equity after conversion is not sufficient to pay the promised conversion value. Hence, jumps introduce two additional sources of risk.

12.3.1 Evaluation of RCD2s

The conversion value for RCD2s requires us to distinguish several cases. If $\tau_C < \tau$, i.e. the downward movement of V_{τ_C} is not sufficient to trigger bankruptcy, the contingent convertible bondholders receive a payment. If on the one hand the value of the equity is sufficiently large, they get a number of stocks such that the value of the total payment equals ℓP_C. If on the other hand the value of the equity is insufficient to make the promised payment to the contingent convertible bond holders, they take possession of the whole equity and the old shareholders are completely diluted out. We assume that the face value of all contingent convertible debt is P_C and thus a bondholder with a bond with face value 1 gets a fraction $1/P_C$ of the value of the equity $EQ(V_{\tau_C})$ after conversion in this case. Recall proposition 7.

Proposition 7. *If the payment to holders of contingent convertible bonds at conversion is ℓ, the values of the individual bonds satisfy*

$$\begin{aligned}
d_C(V, V_B, V_C, t) =& E\left[\int_0^{t\wedge\tau_C} c_C e^{-rs} ds\right] + E\left[e^{-rt}\mathbb{1}_{\{t<\tau_C\}}\right] + \ell E\left[e^{-r\tau_C}\mathbb{1}_{\{\tau_C\leq t\}}\mathbb{1}_{\{\ell P_C\leq EQ(V_{\tau_C})\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] \\
&+ \frac{1}{P_C}E\left[e^{-r\tau_C}EQ(V_{\tau_C})\mathbb{1}_{\{\tau_C\leq t\}}\mathbb{1}_{\{\tau_C<\tau\}}\mathbb{1}_{\{\ell P_C>EQ(V_{\tau_C})\}}\right]
\end{aligned}$$

For an exponential maturity profile $\varphi(t) = me^{-mt}$ the total value of the convertible debt CB is given by:

$$\begin{aligned}
CB(V, V_B, V_C) =& \left(\frac{c_C P_C + mP_C}{m+r}\right) E\left[1 - e^{-(m+r)\tau_C}\right] + \ell P_C E\left[e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\tau\}}\mathbb{1}_{\{\ell P_C\leq EQ(V_{\tau_C})\}}\right] \\
&+ E\left[EQ(V_{\tau_C})e^{-(m+r)\tau_C}\mathbb{1}_{\{V(\tau_C)>V_B\}}\mathbb{1}_{\{\ell P_C>EQ(V_{\tau_C})\}}\mathbb{1}_{\{\tau_C<\infty\}}\right].
\end{aligned}$$

Proof. We only need to do the calculations for the conversion value:

$$
\begin{aligned}
CONV =& \int_0^\infty p_C \Psi(t) d_C(V, V_B, V_C, t) dt \\
=& \ell p_C E\left[e^{-r\tau_C} \int_{\tau_C}^\infty e^{-mt} dt \mathbb{1}_{\{\ell P_C \le EQ(V_{\tau_C})\}} \mathbb{1}_{\{\tau_C < \tau\}}\right] \\
&+ \frac{p_C}{P_C} E\left[e^{-r\tau_C} EQ(V_{\tau_C}) \int_{\tau_C}^\infty e^{-mt} dt \mathbb{1}_{\{\tau_C < \tau\}} \mathbb{1}_{\{\ell P_C > EQ(V_{\tau_C})\}}\right] \\
=& \ell \frac{p_C}{m} E\left[e^{-(r+m)\tau_C} \mathbb{1}_{\{\ell P_C \le EQ(V_{\tau_C})\}} \mathbb{1}_{\{\tau_C < \tau\}}\right] \\
&+ \frac{p_C}{m P_C} E\left[e^{-(r+m)\tau_C} EQ(V_{\tau_C}) \mathbb{1}_{\{\tau_C < \tau\}} \mathbb{1}_{\{\ell P_C > EQ(V_{\tau_C})\}} \mathbb{1}_{\{\tau_C < \infty\}}\right]
\end{aligned}
$$

□

Equipped with the results of Section 4, we can derive the price of RCD2s.

Theorem 7. *The price of the RCD2s equals*

$$
\begin{aligned}
CB =& \frac{C_C + m P_C}{r+m} \left(1 - \frac{\beta_{4,r+m}}{\eta_2} \frac{\eta_2 - \beta_{3,r+m}}{\beta_{4,r+m} - \beta_{3,r+m}} \left(\frac{V_C}{V}\right)^{\beta_{3,r+m}} - \frac{\beta_{3,r+m}}{\eta_2} \frac{\beta_{4,r+m} - \eta_2}{\beta_{4,r+m} - \beta_{3,r+m}} \left(\frac{V_C}{V}\right)^{\beta_{4,r+m}}\right) \\
&+ \ell P_C \cdot G\left(\log\left(\frac{V_C}{V_0}\right), \log\left(\frac{V_C}{\max(T^{-1}(\ell P_C), V_B)}\right), m+r\right) \mathbb{1}_{\{V_C > T^{-1}(\ell P_C)\}} \\
&+ \sum \alpha_i V_0^{\theta_i} \left(J\left(\log\left(\frac{V_C}{V_0}\right), \theta_i, \log\left(\frac{V_C}{V_B}\right), m+r\right)\right. \\
&\left. - J\left(\log\left(\frac{V_C}{V_0}\right), \theta_i, \log\left(\frac{V_C}{T^{-1}(\ell P_C)}\right), m+r\right) \mathbb{1}_{\{V_C > T^{-1}(\ell P_C)\}}\right) \mathbb{1}_{\{V_B < T^{-1}(\ell P_C)\}}
\end{aligned}
$$

Proof. We only need to prove the formula for the conversion value.

$$
\begin{aligned}
CONV =& \ell P_C E\left[e^{-(m+r)\tau_C} \mathbb{1}_{\{\tau_C < \tau\}} \mathbb{1}_{\{\ell P_C \le EQ(V_{\tau_C})\}}\right] \\
&+ E\left[EQ(V_{\tau_C}) e^{-(m+r)\tau_C} \mathbb{1}_{\{\tau_C < \tau\}} \mathbb{1}_{\{\ell P_C > EQ(V_{\tau_C})\}} \mathbb{1}_{\{\tau_C < \infty\}}\right] \\
=& \ell P_C E\left[e^{-(m+r)\tau_C} \mathbb{1}_{\{V_{\tau_C} > \max(T^{-1}(\ell P_C), V_B)\}}\right] \\
&+ E\left[EQ(V_{\tau_C}) e^{-(m+r)\tau_C} \mathbb{1}_{\{V_B < V_{\tau_C} \le T^{-1}(\ell P_C)\}} \mathbb{1}_{\{\tau_C < \infty\}}\right] \mathbb{1}_{\{V_B < T^{-1}(\ell P_C)\}} \\
=& \ell P_C E\left[e^{-(m+r)\tau_C} \mathbb{1}_{\left\{-(X(\tau_C) - x_C) < -\log\left(\frac{\max(T^{-1}(\ell P_C), V_B)}{V_C}\right)\right\}}\right] \\
&+ \sum \alpha_i V_0^{\theta_i} E\left[e^{-(m+r)\tau_C + \theta_i X(\tau_C)} \mathbb{1}_{\{V_B < V_{\tau_C} \le T^{-1}(\ell P_C)\}} \mathbb{1}_{\{\tau_C < \infty\}}\right] \mathbb{1}_{\{V_B < T^{-1}(\ell P_C)\}} \\
=& \ell P_C E\left[e^{-(m+r)\tau_C} \mathbb{1}_{\left\{-(X(\tau_C) - x_C) < -\log\left(\frac{\max(T^{-1}(\ell P_C), V_B)}{V_C}\right)\right\}}\right] \\
&+ \sum \alpha_i V_0^{\theta_i} E\Bigg[e^{-(m+r)\tau_C + \theta_i X(\tau_C)} \left(\mathbb{1}_{\{-(X_{\tau_C} - x_C) < \log(V_C/V_B)\}} - \mathbb{1}_{\{-(X_{\tau_C} - x_C) < \log(V_C/T^{-1}(\ell P_C))\}}\right) \\
&\mathbb{1}_{\{\tau_C < \infty\}}\Bigg] \mathbb{1}_{\{V_B < T^{-1}(\ell P_C)\}}
\end{aligned}
$$

□

12.3.2 Costs of Dilution for RCD2s

For contingent convertible bonds with a flexible number of shares at conversion, the dilution costs are much more complicated. Here, the number of shares depends on the stock price value $S(\tau_C)$ at the time of conversion. In more detail, the number of shares of the old shareholders are

$$n = \frac{EQ(V_0) - DC(V_0)}{S(0)}$$

while the number of new shares of the bondholders equal

$$n' = \frac{\ell P_C}{S(\tau_C)}$$

if the equity value $EQ(V_{\tau_C})$ is sufficiently high. Otherwise, contingent convertible bondholders own the whole remaining equity. Note, that n' is a random variable at time $t = 0$. Therefore the dilution costs DC at time $t = 0$ are

$$\begin{aligned} DC(V_0) = E &\left[EQ(V_{\tau_C}) e^{-(r+m)\tau_C} \frac{\ell P_C}{(EQ(V_0) - DC(V_0))S(\tau_C)/S(0) + \ell P_C} \mathbb{1}_{\{\tau_C < \infty\}} \mathbb{1}_{\{EQ(V_{\tau_C}) \geq \ell P_C\}} \right] \\ &+ E \left[EQ(V_{\tau_C}) e^{-(m+r)\tau_C} \mathbb{1}_{\{\tau_C < \infty\}} \mathbb{1}_{\{EQ(V_{\tau_C}) < \ell P_C\}} \right]. \end{aligned}$$

Under certain assumptions on the stock price process this equation boils down to $CONV(V_0)$.

12.3.3 Modeling the Stock Price Process for RCD2s

If RCD2s are included in the capital structure, there is the possibility that old shareholders are completely diluted out before bankruptcy. Hence, the stock price of the "old" shares can be zero, although the company has not defaulted. However, there will be the "new" shares of the former contingent convertible bondholders with a positive value. Hence, we need to distinguish between "old" and "new" shares:

Definition 16. *The endogenous stock price for the old shares is defined as*

$$S_{old}(t) = S_{old}(V_t) = \begin{cases} \frac{EQ(V_t) - DC(V_t)}{n} & \text{if } t > \tau_C \\ \frac{EQ(V_t)}{n+n'} & \text{if } \tau_C \leq t < \tau \text{ and } EQ(\tau_C) \geq \ell P_C \\ 0 & \text{if } \tau_C \geq t \text{ and } EQ(\tau_C) < \ell P_C \text{ or if } \tau \leq t \end{cases}$$

and the price for the new shares is

$$S_{new}(t) = S_{new}(V_t) = \begin{cases} S_{old}(t) & \text{if } t < \tau_C \\ S_{old}(t) & \text{if } \tau_C \leq t < \tau \text{ and } EQ(\tau_C) \geq \ell P_C \\ EQ(V_t) & \text{if } \tau_C \leq t < \tau \text{ and } EQ(\tau_C) < \ell P_C \text{ (here we have normalized } n' = 1) \end{cases}$$

where n is the number of "old" shares

$$n = \frac{EQ(V_0) - DC(V_0)}{S_{old}(0)}$$

and n' is the number of "new" shares issued at conversion

$$n' = \frac{\ell P_C}{S_{old}(\tau_C)}.$$

In the case of RCD2s the conversion value and dilution costs must also coincide:

Proposition 29. *The conversion value for RCD2s equals the dilution costs:*

$$CONV(V_t) = DC(V_t)$$

Proof. Recall that n and n' are set as

$$n = \frac{EQ(V_0) - DC(V_0)}{S_{old}(0)}$$

and

$$n' = \frac{\ell P_C}{S_{old}(\tau_C)}.$$

At the time of conversion the stock price $S_{old}(\tau_C)$ equals the equity value divided by the number of old and new shares:

$$\begin{aligned} S_{old}(\tau_C)\mathbb{1}_{\{EQ(V_{\tau_C})\geq \ell P_C\}} &= \frac{EQ(V_{\tau_C})}{n+n'}\mathbb{1}_{\{EQ(V_{\tau_C})\geq \ell P_C\}} \\ &= \frac{EQ(V_{\tau_C})}{n+\frac{\ell P_C}{S_{old}(\tau_C)}}\mathbb{1}_{\{EQ(V_{\tau_C})\geq \ell P_C\}} \end{aligned}$$

which is equivalent to

$$\begin{aligned} (S_{old}(\tau_C)n + \ell P_C)\mathbb{1}_{\{EQ(V_{\tau_C}\geq \ell P_C\}} &= EQ(V_{\tau_C})\mathbb{1}_{\{EQ(V_{\tau_C}\geq \ell P_C\}} && \Leftrightarrow \\ S_{old}(\tau_C)\mathbb{1}_{\{EQ(V_{\tau_C}\geq \ell P_C\}} &= \frac{EQ(V_{\tau_C}) - \ell P_C}{n}\mathbb{1}_{\{EQ(V_{\tau_C}\geq \ell P_C\}} && \Leftrightarrow \\ \frac{S_{old}(\tau_C)}{S_{old}(0)}\mathbb{1}_{\{EQ(V_{\tau_C}\geq \ell P_C\}} &= \frac{EQ(V_{\tau_C}) - \ell P_C}{EQ(V_0) - DC(V_0)}\mathbb{1}_{\{EQ(V_{\tau_C}\geq \ell P_C\}}. \end{aligned}$$

Hence, the dilution costs simplify to

$$\begin{aligned} DC(V_0) =& E\left[EQ(V_{\tau_C})e^{-(r+m)\tau_C}\frac{\ell P_C}{(EQ(V_0) - DC(V_0))S(\tau_C)/S(0) + \ell P_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{EQ(V_{\tau_C}\geq \ell P_C\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] \\ &+ E\left[EQ(V_{\tau_C})e^{-(m+r)\tau_C}\mathbb{1}_{\{\tau_C<\infty\}}\mathbb{1}_{\{EQ(V_{\tau_C})<\ell P_C\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] \\ =& \ell P_C E\left[e^{-(r+m)\tau_C}\mathbb{1}_{\{\ell P_C\leq EQ(V_{\tau_C})\}}\mathbb{1}_{\{\tau_C<\tau\}}\right] \\ &+ E\left[e^{-(r+m)\tau_C}EQ(V_{\tau_C})\mathbb{1}_{\{\tau_C<\tau\}}\mathbb{1}_{\{\ell P_C> EQ(V_{\tau_C})\}}\mathbb{1}_{\{\tau_C<\infty\}}\right] \\ =& CONV(V_t). \end{aligned}$$

□

12.4 Comparing RCDs and FSCCBs

The two contracts RCD and FSCCB differ in the specification of the number of shares n' granted to the contingent convertible shareholders in the event of conversion. We have seen that RCD1s are actually a particular version of FSCCBs. Here, we want to analyze whether we can make any statement about n' in the both cases.

The old shareholders own a number of shares n that is fixed at time $t = 0$ and that is equal to the value of equity to them divided by the price of the stock at time $t = 0$:

$$n = \frac{EQ(V_0) - DC(V_0)}{S(0)}$$

Assume that in the case of FSCCBs the new shareholders (i.e. the holders of contingent convertible bonds) receive a fixed number of shares n' that satisfies a posteriori the following condition:

$$n'_{FSCCB} = \frac{P_C \ell}{S_0} = \frac{n\ell P_C}{EQ(V_0) - DC(V_0)}.$$

In the case of RCD1s the corresponding number is

$$n'_{RCD1} = \frac{\ell P_C}{S(V_C)} = \frac{n\ell P_C}{EQ(V_C) - \ell P_C}.$$

Note that the last equality follows from the assumption that

$$\ell P_C = n'_{RCD1} S(V_C) = \frac{n'_{RCD1}}{n + n'_{RCD1}} EQ(V_C).$$

How do the two numbers n'_{FSCCB} and n'_{RCD1} relate to each other? Denote the contract parameters for the two specifications by ℓ_{FSCCB} respectively ℓ_{RCD1}. Assume that $\ell_{RCD1} \geq \ell_{FSCCB}$. By the definition of n we can conclude

$$\begin{aligned} n'_{RCD1} &= \frac{n\ell_{RCD1} P_C}{EQ(V_C) - \ell P_C} \\ &= \frac{EQ(V_0) - DC(V_0)}{EQ(V_C) - \ell_{RCD1} P_C} \frac{\ell_{FSCCB} P_C}{S_0} \frac{\ell_{RCD1}}{\ell_{FSCCB}} \\ &= \frac{EQ(V_0) - DC(V_0)}{EQ(V_C) - \ell_{RCD1} P_C} n'_{FSCCB} \frac{\ell_{RCD1}}{\ell_{FSCCB}}. \end{aligned}$$

Hence

$$\frac{n'_{RCD1}}{n'_{FSCCB}} = \underbrace{\frac{EQ(V_0) - DC(V_0)}{EQ(V_C) - \ell_{RCD1} P_C}}_{>1} \frac{\ell_{RCD1}}{\ell_{FSCCB}}.$$

The inequality $\frac{EQ(V_0) - DC(V_0)}{EQ(V_C) - \ell_{RCD1} P_C} > 1$ holds in our model because we will show later that $EQ(.)$ is a strictly increasing function and that $DC(V_0) < \ell_{FSCCB} P_C$.

Lemma 17. *If $\ell_{RCD1} \geq \ell_{FSCCB}$ then $n'_{RCD1} > n'_{FSCCB}$.*

It is important to note, that the numbers n'_{RCD1} and n'_{FSCCB} are both constants independently of future realizations of V_t.

12.5 Proofs for Section 4

Proposition 8. *Assume that X_t follows a Kou process and τ denotes the first passage time to $x<0$, i.e. $\tau=\inf(0\leq t: X_t\leq x)$. It holds that for $y>0$, $\theta>-\eta_2$ and $\rho>0$:*

$$\begin{aligned}
&E\left[e^{-\rho\tau+\theta X_\tau}\mathbb{1}_{\{\tau<\infty,-(X_\tau-x)<y\}}\right]\\
&=\left(\frac{\eta_2-\beta_{3,\rho}}{\beta_{4,\rho}-\beta_{3,\rho}}e^{x\beta_{3,\rho}}+\frac{\beta_{4,\rho}-\eta_2}{\beta_{4,\rho}-\beta_{3,\rho}}e^{x\beta_{4,\rho}}\right)e^{\theta x}\\
&\quad+e^{\theta x}\frac{\eta_2}{\theta+\eta_2}\frac{\left(1-e^{-(\theta+\eta_2)y}\right)}{(1-e^{-\eta_2 y})}\left(\frac{e^{x\beta_{3,\rho}}}{\beta_{4,\rho}-\beta_{3\rho}}\left(\frac{\eta_2-\beta_{3,\rho}}{\eta_2}\beta_{4,\rho}-(\eta_2-\beta_{3,\rho})-e^{-\eta_2 y}\frac{(\eta_2-\beta_{3,\rho})(\beta_{4,\rho}-\eta_2)}{\eta_2}\right)\right.\\
&\quad\left.+\frac{e^{x\beta_{4,\rho}}}{\beta_{4,\rho}-\beta_{3,\rho}}\left(\frac{\beta_{4,\rho}-\eta_2}{\eta_2}\beta_{3,\rho}-(\beta_{4,\rho}-\eta_2)+e^{-\eta_2 y}\frac{\eta_2-\beta_{3,\rho}}{\eta_2}(\beta_{4,\rho}-\eta_2)\right)\right)
\end{aligned}$$

where $-\beta_{3,\rho}>-\beta_{4,\rho}$ are the two negative roots of the equation $\psi(\beta)=\rho$.

Proof.

$$\begin{aligned}
&E\left[e^{-\rho\tau+\theta X_\tau}\mathbb{1}_{\{\tau<\infty,-(X_\tau-x)<y\}}\right]\\
=&E\left[e^{\rho\tau+\theta X_\tau}\mathbb{1}_{\{\tau<\infty,X_\tau=x\}}\right]+e^{\theta x}E\left[e^{-\rho\tau+\theta(X_\tau-x)}\mathbb{1}_{\{\tau<\infty,0<-(X_\tau-x)<y\}}\right]\\
=&e^{\theta x}E\left[e^{-\rho\tau}\mathbb{1}_{\{\tau<\infty,X_\tau=x\}}\right]+e^{\theta x}E\left[e^{\rho\tau}\mathbb{1}_{\{\tau<\infty,0<-(X_\tau-x)<y\}}\right]\frac{\int_{-y}^0 e^{\theta Y}\eta_2 e^{\eta_2 Y}dY}{\int_{-y}^0\eta_2 e^{\eta_2 Y}dY}\\
=&e^{\theta x}E\left[e^{-\rho\tau}\mathbb{1}_{\{\tau<\infty,X_\tau=x\}}\right]+e^{\theta x}E\left[e^{-\rho\tau}\mathbb{1}_{\{\tau<\infty,0<-(X_\tau-x)<y\}}\right]\frac{\eta_2}{\theta+\eta_2}\frac{\left(1-e^{-(\theta+\eta_2)y}\right)}{(1-e^{-\eta_2 y})}
\end{aligned}$$

Note that

$$E\left[e^{-\rho\tau}\mathbb{1}_{\{\tau<\infty,0<-(X_\tau-x)<y\}}\right]=E\left[e^{-\rho\tau}\right]-E\left[e^{-\rho\tau}\mathbb{1}_{\{X_\tau=x\}}\right]-E\left[e^{-\rho\tau}\mathbb{1}_{\{-(X_\tau-x)>y\}}\right]$$

Using the following results from Kou and Wang (2003) we can finish the proof:

$$E\left[e^{-\rho\tau}\mathbb{1}_{\{X_\tau=x\}}\right]=\frac{\eta_2-\beta_{3,\rho}}{\beta_{4,\rho}-\beta_{3,\rho}}e^{x\beta_{3,\rho}}+\frac{\beta_{4,\rho}-\eta_2}{\beta_{4,\rho}-\beta_{3,\rho}}e^{x\beta_{4,\rho}} \tag{12.3}$$

$$E\left[e^{-\rho\tau}\mathbb{1}_{\{-(X_\tau-x)\geq y\}}\right]=e^{-\eta_2 y}\frac{\eta_2-\beta_{3,\rho}}{\eta_2}\frac{\beta_{4,\rho}-\eta_2}{\beta_{4,\rho}-\beta_{3,\rho}}\left(e^{x\beta_{3,\rho}}-e^{x\beta_{4,\rho}}\right)\qquad y>0 \tag{12.4}$$

□

Lemma 4. *The total value of the equity at conversion $EQ(V_{\tau_C})$ satisfies*

$$EQ(V_{\tau_C})=EQ_{debt}(V_{\tau_C})=\sum_i\alpha_i V_{\tau_C}^{\theta_i}=\sum_i V_0^{\theta_i}\alpha_i e^{X(\tau_C)\theta_i}$$

with

$$\alpha_1 = 1 \qquad \theta_1 = 1$$
$$\alpha_2 = -\frac{\bar{c}C_D}{r}\frac{\beta_{4,r}}{\eta_2}\frac{\eta_2-\beta_{3,r}}{\beta_{4,r}-\beta_{3,r}}(V_B)^{\beta_{3,r}} \qquad \theta_2 = -\beta_{3,r}$$
$$\alpha_3 = -\frac{\bar{c}C_D}{r}\frac{\beta_{3,r}}{\eta_2}\frac{\beta_{4,r}-\eta_2}{\beta_{4,r}-\beta_{3,r}}(V_B)^{\beta_{4,r}} \qquad \theta_3 = -\beta_{4,r}$$
$$\alpha_4 = -\alpha V_B\frac{\beta_{4,r}+1}{\eta_2+1}\frac{\eta_2-\beta_{3,r}}{\beta_{4,r}-\beta_{3,r}}(V_B)^{\beta_{3,r}} \qquad \theta_4 = -\beta_{3,r}$$
$$\alpha_5 = -\alpha V_B\frac{\beta_{3,r}+1}{\eta_2+1}\frac{\beta_{4,r}-\eta_2}{\beta_{4,r}-\beta_{3,r}}(V_B)^{\beta_{4,r}} \qquad \theta_5 = -\beta_{4,r}$$
$$\alpha_6 = \frac{C_D+mP}{r+m}\frac{\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}(V_B)^{\beta_{3,r+m}} \qquad \theta_6 = -\beta_{3,r+m}$$
$$\alpha_7 = \frac{C_D+mP}{r+m}\frac{\beta_{3,r+m}}{\eta_2}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}(V_B)^{\beta_{4,r+m}} \qquad \theta_7 = -\beta_{4,r+m}$$
$$\alpha_8 = -(1-\alpha)V_B\frac{\beta_{4,r+m}+1}{\eta_2+1}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}(V_B)^{\beta_{3,r+m}} \qquad \theta_8 = -\beta_{3,r+m}$$
$$\alpha_9 = -(1-\alpha)V_B\frac{\beta_{3,r+m}+1}{\eta_2+1}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}(V_B)^{\beta_{4,r+m}} \qquad \theta_9 = -\beta_{4,r+m}$$
$$\alpha_{10} = \frac{\bar{c}C_D}{r} - \frac{C_D+mP}{r+m} \qquad \theta_{10} = 0.$$

Proof. The total equity value is the difference between the total value of the firm and the value of actual debt payments:

$$EQ(V_t) = G(V_t) - D(V_t) - CB(V_t) + CONV(V_t)$$

Hence, we conclude that

$$
\begin{aligned}
&EQ(V) = \\
&V + \frac{\bar{c}C_D}{r}\left(1 - \frac{\beta_{4,r}}{\eta_2}\frac{\eta_2-\beta_{3,r}}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_B}{V}\right)^{\beta_{3,r}} - \frac{\beta_{3,r}}{\eta_2}\frac{\beta_{4,r}-\eta_2}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_B}{V}\right)^{\beta_{4,r}}\right) \\
&+ \frac{\bar{c}C_C}{r}\left(1 - \frac{\beta_{4,r}}{\eta_2}\frac{\eta_2-\beta_{3,r}}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_C}{V}\right)^{\beta_{3,r}} - \frac{\beta_{3,r}}{\eta_2}\frac{\beta_{4,r}-\eta_2}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_C}{V}\right)^{\beta_{4,r}}\right) \\
&- \alpha V_B\left(\frac{\beta_{4,r}+1}{\eta_2+1}\frac{\eta_2-\beta_{3,r}}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_B}{V}\right)^{\beta_{3,r}} + \frac{\beta_{3,r}+1}{\eta_2+1}\frac{\beta_{4,r}-\eta_2}{\beta_{4,r}-\beta_{3,r}}\left(\frac{V_B}{V}\right)^{\beta_{4,r}}\right) \\
&- \frac{C_D+mP}{r+m}\left(1 - \frac{\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_B}{V}\right)^{\beta_{3,r+m}} - \frac{\beta_{3,r+m}}{\eta_2}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_B}{V}\right)^{\beta_{4,r+m}}\right) \\
&- (1-\alpha)V_B\left(\frac{\beta_{4,r+m}+1}{\eta_2+1}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_B}{V}\right)^{\beta_{3,r+m}} + \frac{\beta_{3,r+m}+1}{\eta_2+1}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_B}{V}\right)^{\beta_{4,r+m}}\right) \\
&- \frac{C_C+mP_C}{r+m}\left(1 - \frac{\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_C}{V}\right)^{\beta_{3,r+m}} - \frac{\beta_{3,r+m}}{\eta_2}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}\left(\frac{V_C}{V}\right)^{\beta_{4,r+m}}\right)
\end{aligned}
$$

This has a structure of the form

$$EQ(V_t) = \sum_i \alpha_i V_t^{\theta_i}$$

where V_t is the only time dependent variable. Note that at the time of conversion τ_C the value of the tax benefits of the contingent convertible bonds and the value of $CB - CONV$ are zero. Hence, the corresponding terms in $EQ(V_{\tau_C})$ disappear. □

12.6 Proofs for Section 5

Theorem 3. *There are only two possible solutions for the optimal default barrier. Either the optimal default barrier coincides with the optimal default barrier with only straight debt or it equals the maximum of the conversion barrier and $V_B^{**}: V_B = V_B^*$ or $V_B = \max(V_C, V_B^{**})$, where*

$$V_B^{**} = \frac{\frac{C_D+C_C+m(P_D+P_C)}{r+m}\beta_{3,r+m}\beta_{4,r+m} - \frac{\bar{c}(C_D+C_C)}{r}\beta_{3,r}\beta_{4,r}}{\alpha(\beta_{3,r}+1)(\beta_{4,r}+1)+(1-\alpha)(\beta_{3,r+m}+1)(\beta_{4,r+m}+1)}\frac{\eta_2+1}{\eta_2}.$$

V_B^{**} equals the optimal default barrier of a firm with only straight debt with face value P_D+P_C and coupon C_D+C_C. If

$$EQ_{old}(V, V_B^*, V_C) \geq EQ_{debt}(V, V_B^{**}, P_D+P_C, C_D+C_C) \qquad \text{for all } V \geq V_C$$

for $V_B^{**} > V_C > V_B^*$ or

$$EQ_{old}(V, V_B^*, V_C) \geq 0 \qquad \text{for all } V \geq V_C.$$

for $V_B^* < V_B^{**} \leq V_C$ then

$$V_B = V_B^*$$

otherwise

$$V_B = \max(V_C, V_B^{**}).$$

Proof. We will first show when V_B^* is optimal. Assume the following five conditions are all satisfied:

1. Assume that first

$$EQ_{old}(V, V_B^*, V_C) \geq 0 \qquad \text{for all } V \geq V_C$$

i.e. the default barrier V_B^* satisfies the limited liability constraint.

2. Second, the equity of the old shareholders for $V_B > V_C$ is given by

$$EQ_{old}(V, V_B, V_C, P_D, P_C, C_D, C_C) = EQ_{debt}(V, V_B, P_D+P_C, C_D+C_C),$$

i.e. the equity value of the older shareholders for $V_B > V_C$ is the same as for a firm that issues only straight debt in the amount $P_D + P_C$ with coupon $C_D + C_C$.

3. Third, if the shareholders want to default before conversion the optimal default barrier is V_B^{**} if $V_B^{**} > V_C$ and V_C otherwise.

4. Fourth, it should hold that

$$EQ_{old}(V, V_B^*, V_C) \geq EQ_{old}(V, V_B^{**}, V_B^{**}) \qquad \text{for } V_B^{**} > V_C \text{ and for all } V \geq V_C$$

i.e. the older shareholders do not want to default before conversion for $V_B^{**} > V_C$.

5. Fifth,

$$EQ_{old}(V, V_B^*, V_C) \geq EQ_{old}(V, V_C, V_C) \qquad \text{for all } V > V_C,$$

i.e. the old shareholders prefer the default barrier V_B^* to the default barrier V_C.

The fourth and the fifth condition imply that the shareholders will never choose a default barrier higher than or equal to V_C, because this would result in a lower equity value. By the commitment condition, the optimal default barrier after conversion is V_B^*. This is a valid solution to our optimization problem, if and only if the limited liability constraint is satisfied. The first condition ensures that the limited liability constraint is satisfied for $V \geq V_C$. After conversion V_B^* trivially satisfies the limited liability constraint.

The first and fourth condition are stated as assumptions in our theorem. Now we need to show that the second, third and fifth condition are always satisfied. We start with the second statement. Recall that if the conversion barrier is smaller than the default barrier, we can treat this case as if the default and conversion barrier are the same. Hence, for all $V_B = y \geq V_C$

$$\begin{aligned}
&EQ_{old}(V, y, V_C, P_D, C_D, P_C, C_C) \\
=&EQ_{debt}(V, y, P_D, C_D) + TB_C(V, y) - CCB(V, y, y, P_C, C_C) \\
=&V + \frac{\bar{c}C_D}{r} E\left[1 - e^{-r\tau}\right] + \alpha E\left[V(\tau)e^{-r\tau}\mathbb{1}_{\{\tau<\infty\}}\right] \\
&- \left(\frac{cP_D + mP_D}{m+r} E\left[1 - e^{-(m+r)\tau}\right] + (1-\alpha)E\left[V(\tau)e^{-(m+r)\tau}\mathbb{1}_{\{\tau<\infty\}}\right]\right) \\
&+ \frac{\bar{c}C_C}{r} E\left[1 - e^{-r\tau}\right] - \left(\left(\frac{cP_C + mP_C}{m+r}\right) E\left[1 - e^{-(m+r)\tau}\right] + 0\right) \\
=&EQ_{debt}(V, y, P_D + P_C, C_D + C_C)
\end{aligned}$$

This means that the equity value of the old shareholders is the same as in the case with only straight debt, but with a higher face and coupon value. Chen and Kou (2009) have shown the $EQ_{debt}(V, V_B)$ is strictly decreasing in V_B for $V \geq V_B$ and $V_B \geq V_B^{**}$. Hence,

$$EQ_{debt}(V, V_B^{**}) \geq EQ_{debt}(V, y) \qquad \text{for all } y \geq V_B^{**}, \text{ for all } V \geq V_B^{**}.$$

We have already shown, that in the case of only straight debt V_B^{**} is the optimal default barrier for an amount of debt $P_D + P_C$ and a coupon value of $C_D + C_C$. However, if $V_B^{**} < V_C$, the commitment problem rules out V_B^{**} as a solution. The old shareholders would maximize their equity value by choosing $V_B = V_C$. This proves the third statement. In order to show the fifth statement, we have to reformulate it:

$$\begin{aligned}
&EQ_{old}(V, V_B^*, V_C) - EQ_{old}(V, V_C, V_C) \\
=&EQ_{debt}(V, V_B^*) + TB_C(V, V_C) - CCB(V, V_B^*, V_C) - EQ_{debt}(V, V_C) - TB_C(V, V_C) + CCB(V, V_C, V_C) \\
=&EQ_{debt}(V, V_B^*) - EQ_{debt}(V, V_C) - CONV(V, V_B^*, V_C)
\end{aligned}$$

The equality holds as the conversion value for $V_B = V_C$ is equal to zero: $CONV(V, V_C, V_C) = 0$. Assume first, that at conversion the old shareholders are completely diluted out, i.e. all the equity is given to the new shareholders. In this case, the event of conversion is like the default event for the old shareholders and they are indifferent between V_B^* and V_C:

$$EQ_{old}(V, V_B^*, V_C) = EQ_{old}(V, V_C, V_C)$$

which is equivalent to

$$EQ_{debt}(V, V_B^*) - EQ_{debt}(V, V_C) = CONV(V, V_B^*, V_C)$$

i.e. the conversion value equals exactly the gain in the equity value due to a lower default barrier. Obviously, complete dilution gives the highest possible conversion value and hence establishes an upper bound on $CONV(V, V_B^*, V_C)$. Therefore, for an arbitrary amount of shares granted at conversion the following inequality has to hold:

$$EQ_{debt}(V, V_B^*) - EQ_{debt}(V, V_C) \geq CONV(V, V_B^*, V_C)$$

which is equivalent to

$$EQ_{old}(V, V_B^*, V_C) \geq EQ_{old}(V, V_C, V_C)$$

and thus proves the statement.

What happens, if the first condition (limited liability for V_B^*) is violated? The value of the equity will be zero before conversion and hence, default will be triggered for a value of the firm's assets that is larger than V_C. Anticipating this, the old shareholders will choose a default barrier $V_B > V_C$ such that the value of their equity is maximized. As we have shown before, this problem is equivalent to maximizing

$$\max_{V_B \geq V_C} EQ_{debt}(V, V_B, P_D + P_C, C_D, C_C) \qquad \text{s.t. } EQ_{debt}(V', V_B, P_D + P_C, C_D + C_C) > 0 \ \forall \ V' > V_B$$

A firm with only straight debt, that has face value $P_D + P_C$ and coupons $C_D + C_C$, would ideally choose V_B^{**}. However, if $V_B^{**} < V_C$, the commitment problem does not allow it to take V_B^{**}. As EQ_{debt} is strictly decreasing in the default barrier, the firm would choose the smallest possible default barrier such that $V_B > V_C$, which is obviously V_C. The limited liability constraint is trivially satisfied as EQ_{debt} is strictly increasing in the firm's value. A similar reasoning applies to the case where the fourth condition is violated. $\square$

Proposition 15. *Assume that the shareholders choose $\{V_B, V_C\}$ according to the two stage optimization problem in order to maximize EQ_{old}. If*

$$EQ_{old}(V, V_B^*, V_C^*) \geq EQ_{debt}(V, V_B^{**}, V_B^{**}) \text{ for all } V > V_B^{**}$$

then the optimal solution is

$$V_B = V_B^* \text{ and } V_C = V_C^*,$$

otherwise

$$V_B = V_B^{**} \text{ and } V_C = V_B^{**}$$

Proof. From Theorem 3 we already know, that the optimal default barrier is either V_B^* or $\max\{V_B^{**}, V_C\}$. In the case, where $V_B = V_B^*$, the old shareholders' equity can be at most $EQ_{old}(V, V_B^*, V_C^*)$, as V_C^* is chosen such that is maximizes their equity value. In the other case the old shareholders will choose a conversion and default barrier, such that default happens before conversion, i.e. $V_B \geq V_C$. As we have seen in the proof of Theorem 3 in this case $EQ_{old}(V, V_B, V_C, P_D, C_D, P_C, C_C) = EQ_{debt}(V, V_B, P_D + P_C, C_D + C_C)$. The optimal default barrier is then V_B^{**}. In order to ensure, that default happens before conversion the conversion barrier must be smaller than V_B^{**}, i.e $V_C \leq V_B^{**}$. As V_C is a choice variable, the old shareholders can always ensure that this condition holds by setting $V_C = V_B^{**}$. Hence, the old shareholders can get at most $EQ_{debt}(V, V_B^{**}, V_B^{**})$ in this case. Comparing the two maximal values yields the optimal choice. □

Proposition 16. *Assume that V_C is chosen to maximize the total value of the firm in a second stage. If $G(V, V_B^*, \bar{V}_C) > G(V, V_B^{**}, V_B^{**})$ for all $V > V_B^{**}$, then the optimal solution is*

$$V_B = V_B^* \text{ and } V_C = \bar{V}_C,$$

otherwise

$$V_B = V_B^{**} \text{ and } V_C = V_B^{**}$$

Proof. For $V \geq V_C \geq V_B$ the total value of the firm is a strictly decreasing function in the conversion barrier:

$$\frac{\partial G(V, V_B, V_C)}{\partial V_C} < 0$$

This is due to the fact that

$$G(V, V_B, V_C) = V + TB_D(V, V_B) + TB_C(V, V_C) - BC(V, V_B)$$

and hence the total value of the firm is only affected by V_C through the tax benefits. It is easy to verify that

$$\frac{\partial TB_C(V, V_C)}{\partial V_C} = \frac{\partial \frac{\bar{c} C_C}{r} E\left[1 - e^{-r\tau_C}\right]}{\partial V_C} < 0$$

for any Markov process V. Therefore, the firm would always choose the lowest possible conversion barrier for a given V_B in the first stage, which is $V_C = V_B$. However, by choosing $V_C = V_B$ the default barrier does not stay fixed, but can change as well. As we have seen in the proof of Theorem 3 in this case $EQ_{old}(V, V_B, V_C, P_D, C_D, P_C, C_C) = EQ_{debt}(V, V_B, P_D + P_C, C_D + C_C)$. The optimal default barrier is then V_B^{**} and the resulting total value of the firm is $G(V, V_B^{**}, V_B^{**})$. If the firm chooses the lowest possible value of $\bar{V}_C$, such that V_B^* satisfies the limited liability constraint, then the total value of the firm equals $G(V, V_B^*, \bar{V}_C)$. Comparing the optimal total values of the firm for the two cases yields the optimal solution. □

12.7 Proofs for Section 6

Lemma 9. *Assumption 5 implies that the value of the total straight debt, if it was risk free, is larger than the value of the largest possible recovery payment:*

$$B(C_D) > D(V, V_B) \qquad \Rightarrow \qquad \frac{C_D + mP_D}{r+m} > (1-\alpha)V_B.$$

Proof.

$$\begin{aligned}
B(C_D) &\geq D(V, V_B) && \Leftrightarrow \\
\frac{C_D + mP_D}{r+m} &\geq \frac{C_D + mP_D}{r+m}\left(1 - \tilde{A} - \tilde{B}\right) + (1-\alpha)V_B\left(\tilde{C} + \tilde{D}\right) && \Leftrightarrow \\
\frac{C_D + mP_D}{r+m} &\geq (1-\alpha)V_B \frac{\tilde{C} + \tilde{D}}{\tilde{A} + \tilde{B}}
\end{aligned}$$

with

$$\begin{aligned}
\tilde{A} &= \frac{\beta_{4,r+m}}{\eta_2} \frac{\eta_2 - \beta_{3,r+m}}{\beta_{4,r+m} - \beta_{3,r+m}} \\
\tilde{B} &= \frac{\beta_{3,r+m}}{\eta_2} \frac{\beta_{4,r+m} - \eta_2}{\beta_{4,r+m} - \beta_{3,r+m}} \\
\tilde{C} &= \frac{\beta_{4,r+m} + 1}{\eta_2 + 1} \frac{\eta_2 - \beta_{3,r+m}}{\beta_{4,r+m} - \beta_{3,r+m}} \\
\tilde{D} &= \frac{\beta_{3,r+m} + 1}{\eta_2 + 1} \frac{\beta_{4,r+m} - \eta_2}{\beta_{4,r+m} - \beta_{3,r+m}}
\end{aligned}$$

All we need to show is that

$$\frac{\tilde{C} + \tilde{D}}{\tilde{A} + \tilde{B}} < 1$$

$$\begin{aligned}\frac{\tilde{C}+\tilde{D}}{\tilde{A}+\tilde{B}} &= \frac{\eta_2}{\eta_2+1}\frac{(\beta_{4,r+m}+1)(\eta_2-\beta_{3,r+m})+(\beta_{3,r+m}+1)(\beta_{4,r+m}-\eta_2)}{\beta_{4,r+m}(\eta_2-\beta_{3,r+m})+\beta_{3,r+m}(\beta_{4,r+m}-\eta_2)} \\ &= \frac{\eta_2}{\eta_2+1}\left(1+\frac{\eta_2-\beta_{3,r+m}+\beta_{3,r+m}-\eta_2}{\beta_{4,r+m}(\eta_2-\beta_{3,r+m})+\beta_{3,r+m}(\beta_{4,r+m}-\eta_2)}\right) \\ &= \frac{\eta_2}{\eta_2+1} \\ &< 1\end{aligned}$$

□

Lemma 10. *If the condition* $\frac{C_D+mP_D}{r+m} \geq \frac{\eta_2}{\eta_2+1}\frac{\beta_{3,r+m}+1}{\beta_{3,r+m}}(1-\alpha)V_B$ *is satisfied, then the value of the straight debt is an increasing function in the firm's value:*

$$\frac{\partial D(V)}{\partial V} > 0 \qquad \textit{for all } V \geq V_B.$$

Proof. Define $x = V_B/V$. Obviously, it holds

$$\frac{\partial D(V)}{\partial V} = \frac{\partial D(V)}{\partial x}\frac{\partial x}{\partial V} = \frac{\partial D(V)}{\partial x}\frac{-V_B}{V^2}.$$

Hence, we need to show $\frac{\partial D(V)}{\partial x} < 0$.

We will use Lemma B.1 from the Appendix of Chen and Kou (2009) which states the following: Consider the function $g(x) = Ax^{\alpha_1} + Bx^{\beta_1} - Cx^{\alpha_2} - Dx^{\beta_2}$, $0 \leq x \leq 1$. In the case of $0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$, $A + B \geq C + D$ and $A > C$, then $g(x) \geq 0$ for all $0 \leq x \leq 1$.

In our case the debt as a function of x is given by

$$\begin{aligned}D(x) = \frac{C_D+mP_D}{r+m}&\left(1-\frac{\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}x^{\beta_{3,r+m}}-\frac{\beta_{3,r+m}}{\eta_2}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}x^{\beta_{4,r+m}}\right) \\ +(1-\alpha)V_B&\left(\frac{\beta_{4,r+m}+1}{\eta_2+1}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}x^{\beta_{3,r+m}}+\frac{\beta_{3,r+m}+1}{\eta_2+1}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}x^{\beta_{4,r+m}}\right)\end{aligned}$$

Therefore,

$$\begin{aligned}D'(x) = &-\frac{C_D+mP_D}{r+m}\left(\frac{\beta_{3,r+m}\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}x^{\beta_{3,r+m}-1}\right. \\ &\left.+\frac{\beta_{4,r+m}\beta_{3,r+m}}{\eta_2}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}x^{\beta_{4,r+m}-1}\right) \\ &+(1-\alpha)V_B\left(\frac{\beta_{3,r+m}(\beta_{4,r+m}+1)}{\eta_2+1}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}x^{\beta_{3,r+m}-1}\right. \\ &\left.+\frac{\beta_{4,r+m}(\beta_{3,r+m}+1)}{\eta_2+1}\frac{\beta_{4,r+m}-\eta_2}{\beta_{4,r+m}-\beta_{3,r+m}}x^{\beta_{4,r+m}-1}\right) \\ =&-\left(Ax^{\alpha_1}+Bx^{\beta_1}-Cx^{\alpha_2}-Dx^{\beta_2}\right)\end{aligned}$$

with

$$
\begin{aligned}
A &= \frac{C_D + mP_D}{r+m} \frac{\beta_{3,r+m}\beta_{4,r+m}}{\eta_2} \frac{\eta_2 - \beta_{3,r+m}}{\beta_{4,r+m} - \beta_{3,r+m}} \\
B &= \frac{C_D + mP_D}{r+m} \frac{\beta_{4,r+m}\beta_{3,r+m}}{\eta_2} \frac{\beta_{4,r+m} - \eta_2}{\beta_{4,r+m} - \beta_{3,r+m}} \\
C &= (1-\alpha)V_B \frac{\beta_{3,r+m}(\beta_{4,r+m}+1)}{\eta_2 + 1} \frac{\eta_2 - \beta_{3,r+m}}{\beta_{4,r+m} - \beta_{3,r+m}} \\
D &= (1-\alpha)V_B \frac{\beta_{4,r+m}(\beta_{3,r+m}+1)}{\eta_2 + 1} \frac{\beta_{4,r+m} - \eta_2}{\beta_{4,r+m} - \beta_{3,r+m}}
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_1 = \alpha_2 &= \beta_{3,r+m} - 1 \\
\beta_1 = \beta_2 &= \beta_{4,r+m} - 1
\end{aligned}
$$

From Kou and Wang (2003) we know that $\beta_{4,r+m} > \eta_2 > \beta_{3,r+m} > 0$. Hence $0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$. Next,

$$
A + B = \frac{C_D + mP_D}{r+m} \left(\frac{\beta_{3,r+m}\beta_{4,r+m}}{\eta_2} \right)
$$

and

$$
C + D = (1-\alpha)V_B \left(\frac{\beta_{3,r+m}\beta_{4,r+m}}{\eta_2 + 1} + \frac{\beta_{3,r+m}(\eta_2 - \beta_{3,r+m}) + \beta_{4,r+m}(\beta_{4,r+m} - \eta_2)}{(\eta_2+1)(\beta_{4,r+m} - \beta_{3,r+m})} \right)
$$

We need to show that $A + B \geq C + D$. Note, that

$$
\begin{aligned}
&\frac{A+B}{C+D} \geq 1 && \Leftrightarrow \\
&\frac{\frac{C_D+mP_D}{r+m}}{(1-\alpha)V_B} \geq \frac{\frac{\beta_{3,r+m}\beta_{4,r+m}}{\eta_2+1} + \frac{\beta_{3,r+m}(\eta_2-\beta_{3,r+m})+\beta_{4,r+m}(\beta_{4,r+m}-\eta_2)}{(\eta_2+1)(\beta_{4,r+m}-\beta_{3,r+m})}}{\frac{\beta_{3,r+m}\beta_{4,r+m}}{\eta_2}} && \Leftrightarrow \\
&\frac{\frac{C_D+mP_D}{r+m}}{(1-\alpha)V_B} \geq \frac{\eta_2}{\eta_2+1} + \frac{\eta_2}{\eta_2+1} \left(\frac{\eta_2 - \beta_{3,r+m}}{\beta_{4,r+m}(\beta_{4,r+m} - \beta_{3,r+m})} + \frac{\beta_{4,r+m} - \eta_2}{\beta_{3,r+m}(\beta_{4,r+m} - \beta_{3,r+m})} \right) && \Leftrightarrow \\
&\frac{\frac{C_D+mP_D}{r+m}}{(1-\alpha)V_B} \geq \frac{\eta_2}{\eta_2+1} \left(1 + \frac{\beta_{3,r+m}(\eta_2 - \beta_{3,r+m}) + \beta_{4,r+m}(\beta_{4,r+m} - \eta_2)}{\beta_{3,r+m}\beta_{4,r+m}(\beta_{4,r+m} - \beta_{3,r+m})} \right)
\end{aligned}
$$

We know that $0 < \beta_{3,r+m} < \eta_2 < \beta_{4,r+m}$. Hence, it holds that

$$
\frac{\beta_{3,r+m}(\eta_2 - \beta_{3,r+m}) + \beta_{4,r+m}(\beta_{4,r+m} - \eta_2)}{\beta_{3,r+m}\beta_{4,r+m}(\beta_{4,r+m} - \beta_{3,r+m})} < \frac{1}{\beta_{3,r+m}}
$$

By assumption we have

$$
\frac{\frac{C_D+mP_D}{r+m}}{(1-\alpha)V_B} \geq \frac{\eta_2}{\eta_2+1} \frac{\beta_{3,r+m}+1}{\beta_{3,r+m}} = \frac{\eta_2}{\eta_2+1} \left(1 + \frac{1}{\beta_{3,r+m}} \right)
$$

Thus, we conclude

$$\frac{\frac{C_D+mP_D}{r+m}}{(1-\alpha)V_B} \geq \frac{\eta_2}{\eta_2+1}\left(1+\frac{1}{\beta_{3,r+m}}\right)$$
$$\geq \frac{\eta_2}{\eta_2+1}\left(1+\frac{\beta_{3,r+m}(\eta_2-\beta_{3,r+m})+\beta_{4,r+m}(\beta_{4,r+m}-\eta_2)}{\beta_{3,r+m}\beta_{4,r+m}(\beta_{4,r+m}-\beta_{3,r+m})}\right).$$

Last but not least, we need to show $A > C$.

$$A-C=\frac{C_D+mP_D}{r+m}\frac{\beta_{3,r+m}\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}-(1-\alpha)V_B\frac{\beta_{3,r+m}(\beta_{4,r+m}+1)}{\eta_2+1}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}$$

By assumption $\frac{C_D+mP_D}{r+m} > (1-\alpha)V_B$ as $\frac{\eta_2}{\eta_2+1}\frac{\beta_{3,r+m}+1}{\beta_{3,r+m}} > 1$. Hence, it is sufficient to show the following:

$$\begin{aligned}
&\frac{\beta_{3,r+m}\beta_{4,r+m}}{\eta_2}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}-\frac{\beta_{3,r+m}(\beta_{4,r+m}+1)}{\eta_2+1}\frac{\eta_2-\beta_{3,r+m}}{\beta_{4,r+m}-\beta_{3,r+m}}\\
=&\frac{\beta_{3,r+m}\beta_{4,r+m}(\eta_2+1)(\eta_2-\beta_{3,r+m})-\eta_2\beta_{3,r+m}(\beta_{4,r+m}+1)(\eta_2-\beta_{3,r+m})}{\eta_2(\eta_2+1)(\beta_{4,r+m}-\beta_{3,r+m})}\\
=&\frac{\beta_{3,r+m}(\eta_2-\beta_{3,r+m})(\beta_{4,r+m}-\eta_2)}{\eta_2(\eta_2+1)(\beta_{4,r+m}-\beta_{3,r+m})}\\
>&0
\end{aligned}$$

The last step follows from the fact that $0 < \beta_{3,r+m} < \eta_2 < \beta_{4,r+m}$.

Therefore, all the conditions for Lemma B.1 from Chen and Kou (2009) are satisfied and thus $D'(x) < 0$. □

Lemma 11. *If the default barrier is chosen optimally as V_B^* (i.e. V_B is chosen optimally by the shareholders as in the case with only straight debt), then the condition*

$$\frac{C_D+mP_D}{r+m} \geq \frac{\eta_2}{\eta_2+1}\frac{\beta_{3,r+m}+1}{\beta_{3,r+m}}(1-\alpha)V_B$$

is always satisfied.

Proof. Plugging in the expression for V_B^* yields

$$\frac{C_D+mP_D}{r+m} \geq \frac{\eta_2}{\eta_2+1}\frac{\beta_{3,r+m}+1}{\beta_{3,r+m}}(1-\alpha)V_B \quad \Leftrightarrow$$
$$\frac{C_D+mP_D}{r+m} \geq \frac{\beta_{3,r+m}+1}{\beta_{3,r+m}}(1-\alpha)\frac{\frac{C_D+mP_D}{r+m}\beta_{3,r+m}\beta_{4,r+m}-\frac{\bar{c}C_D}{r}\beta_{3,r}\beta_{4,r}}{\alpha(\beta_{3,r}+1)(\beta_{4,r}+1)+(1-\alpha)(\beta_{3,r+m}+1)(\beta_{4,r+m}+1)}$$

This expression is equivalen to

$$\frac{C_D+mP_D}{r+m}\underbrace{\left(\frac{\beta_{3,r+m}}{\beta_{3,r+m}+1}\alpha(\beta_{3,r}+1)(\beta_{4,r}+1)+(1-\alpha)\beta_{3,r+m}\right)}_{>0} \geq -\frac{\bar{c}C_D}{r}\underbrace{(1-\alpha)\beta_{3,r}\beta_{4,r}}_{>0}$$

As long as $C_D > 0$, i.e. the firm has to make positive coupon payments, the above expression will always hold. However, Assumption 5 implies that $C_D > 0$. □

12.8 Proofs for Section 8

Proposition 20. *Assume that a firm issues RCD2s and contingent convertible bondholders can temporarily lower the firm's value to an arbitrary V_{manip} with $V_C \geq V_{manip} > V_B$. If the equity value without CoCo bonds is sufficiently high, i.e. $EQ_{debt}(V_t) > CCB(V_t)$, they will always manipulate the market for any $\ell \in (0, \infty)$.*

Proof. By lowering the firm's value process to V_{manip} the contingent convertible bondholder enforce conversion and will receive equity with the market value $\min\{\ell P_C, EQ(V_{manip})\}$. The total equity $EQ(V) = EQ_{debt}(V)$ is a continuous and strictly monotonic function for $V \in (V_B, V_C)$. Hence for any given ℓ there exists a V_{manip} such that $\ell P_C > EQ(V_{manip})$. Hence, by manipulating the market the contingent convertible bondholders can always completely dilute out the old shareholders and take control over the firm. We know that before conversion $EQ_{old}(V_t) > 0$. This is equivalent to $EQ_{debt}(V_t) + TB_C(V_t) > CCB(V_t)$. If the tax benefits are sufficiently low, the equity value after manipulation $EQ(V_t) = EQ_{debt}(V_t)$, which is then owned only by the contingent convertible bondholders, is larger than the bondholder's value without manipulation $CCB(V_t)$. In this case, manipulation is profitable. □

Proposition 21. *Assume that a firm has issued straight debt and RCD1s. For $V_t \leq V_0$ and $\ell = 1$, equity holders will not manipulate the market to trigger conversion.*

Proof. At time 0, the contingent convertible bonds sell at par:

$$CCB(V_0, V_C) = P_C$$

Hence, the inequality is satisfied

$$\underbrace{TB_C(V_0)}_{\geq 0} + P_C \underbrace{\frac{EQ_{debt}(V_0)}{EQ_{debt}(V_C)}}_{\geq 1} \geq P_C$$

As the conversion value is less than the face value, i.e. $CONV(V_t, V_B, V_C) \leq P_C$ for $V_0 \geq V_t \geq V_C$, it holds that

$$CCB(V_t, V_B, V_C) \leq P_C \qquad \text{for } V_0 \geq V_t \geq V_C.$$

Therefore

$$TB_C(V_t) + P_C \frac{EQ_{debt}(V_t)}{EQ_{debt}(V_C)} \geq CCB(V_t, V_B, V_C).$$

□

12.9 Proofs for Section 9

Theorem 5. *The Laplace transform of the default time for a firm, whose payout ratio changes at conversion, is given by*

$$\begin{aligned} E\left[e^{-\tau\rho}\right] =&\bar{c}_1\left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{3,\rho}} J(\log(V_C/V_0), \bar{\beta}_{3,\rho}, \log(V_C/V_B), \rho) \\ &+ \bar{c}_2\left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{4,\rho}} J(\log(V_C/V_0), \bar{\beta}_{4,\rho}, \log(V_C/V_B), \rho) \\ &+ \left(\frac{V_B}{V_C}\right)^{\eta_2} \frac{\eta_2-\beta_{3,\rho}}{\eta_2}\frac{\beta_{4,\rho}-\eta_2}{\beta_{4,\rho}-\beta_{3,\rho}}\left(\left(\frac{V_C}{V_0}\right)^{\beta_{3,\rho}} - \left(\frac{V_C}{V_B}\right)^{\beta_{4,\rho}}\right) \end{aligned}$$

with

$$\begin{aligned} \bar{c}_1 &= \frac{\eta_2-\bar{\beta}_{3,\rho}}{\eta_2}\frac{\bar{\beta}_{4,\rho}}{\bar{\beta}_{4,\rho}-\bar{\beta}_{3,\rho}} \\ \bar{c}_2 &= \frac{\bar{\beta}_{4,\rho}-\eta_2}{\eta_2}\frac{\bar{\beta}_{3,\rho}}{\bar{\beta}_{4,\rho}-\bar{\beta}_{3,\rho}} \end{aligned}$$

and $-\bar{\beta}_{3,\rho} > -\bar{\beta}_{4,\rho}$ *are the two negative roots of the equation*

$$\bar{\psi}(\beta) = \rho$$

with $\bar{\psi}$ *being the Lévy exponent of* $\bar{X}_t = (r-\delta_2)t + \sigma W_t^* + \sum_{i=1}^{N_t} Y_i$. *The functions* $-\beta_{3,\rho} > -\beta_{4,\rho}$ *are the two negative roots of the equation* $\psi(\beta) = \rho$, *where* ψ *is the Lévy exponent of* $X_t = (r-\delta_1)t + \sigma W_t^* + \sum_{i=1}^{N_t} Y_i$ *The function* J *is defined as*

$$J(x,\theta,y,\rho) = E\left[e^{-\rho\tau+\theta X_\tau}\mathbb{1}_{\{\tau<\infty, -(X_\tau - x)<y\}}\right]$$

The explicit form of $J(x,\theta,y,\rho)$ *is given in Proposition 8.*

Proof.

$$\begin{aligned} E\left[e^{-\tau\rho}\right] &= E\left[e^{-\tau_C\rho}e^{-(\tau-\tau_C)\rho}\mathbb{1}_{\{\tau>\tau_C\}} + e^{-\tau_C\rho}\mathbb{1}_{\{\tau=\tau_C\}}\right] \\ &= E\left[e^{-\tau_C\rho}E\left[e^{-(\tau-\tau_C)\rho}\mathbb{1}_{\{\tau>\tau_C\}}|X_{\tau_C},\tau_C\right] + e^{-\tau_C\rho}\mathbb{1}_{\{\tau=\tau_C\}}\right] \end{aligned} \tag{12.5}$$

We will first consider the conditional expectation:

$$E\left[e^{-(\tau-\tau_C)\rho}\mathbb{1}_{\{\tau>\tau_C\}}|X_{\tau_C},\tau_C\right] = E\left[e^{-\bar{\tau}\rho}\right]$$

where $\bar{X}_t$ is defined as

$$\bar{X}_t = (r-\delta_2)t + \sigma W_t^* + \sum_{i=1}^{N_t} Y_i$$

and $\bar{\tau} = \inf\left(t\in[0,\infty) : \bar{X}_t \le \log(V_B/V(\tau_C))\right)$. The above equality is true on account of the Markov property of X_t. Hence,

$$\begin{aligned} E\left[e^{-(\tau-\tau_C)\rho}|X_{\tau_C},\tau_C\right] &= \bar{c}_1\left(\frac{V_B}{V(\tau_C)}\right)^{\bar{\beta}_{3,\rho}} + \bar{c}_2\left(\frac{V_B}{V(\tau_C)}\right)^{\bar{\beta}_{4,\rho}} \\ &= \bar{c}_1\left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{3,\rho}} e^{-X(\tau_C)\bar{\beta}_{3,\rho}} + \bar{c}_2\left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{4,\rho}} e^{-X(\tau_C)\bar{\beta}_{4,\rho}} \end{aligned}$$

where

$$\bar{c}_1 = \frac{\eta_2 - \bar{\beta}_{3,\rho}}{\eta_2} \frac{\bar{\beta}_{4,\rho}}{\bar{\beta}_{4,\rho} - \bar{\beta}_{3,\rho}}$$
$$\bar{c}_2 = \frac{\bar{\beta}_{4,\rho} - \eta_2}{\eta_2} \frac{\bar{\beta}_{3,\rho}}{\bar{\beta}_{4,\rho} - \bar{\beta}_{3,\rho}}$$

and $-\bar{\beta}_{3,\rho} > -\bar{\beta}_{4,\rho}$ are the two negative roots of the equation

$$\bar{\psi}(\beta) = \rho$$

with $\bar{\psi}$ being the Lévy exponent of $\bar{X}_t$. The first expectation in 12.5 equals

$$\begin{aligned}
&E\left[e^{-\tau_C\rho}\left(\bar{c}_1\left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{3,\rho}} e^{-X(\tau_C)\bar{\beta}_{3,\rho}} + \bar{c}_2\left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{4,\rho}} e^{-X(\tau_C)\bar{\beta}_{4,\rho}}\right)\mathbb{1}_{\{\tau>\tau_C\}}\right] \\
=&E\left[e^{-\tau_C\rho-\bar{\beta}_{3,\rho}X(\tau_C)}\mathbb{1}_{\{x_C-X(\tau_C)<x_C-x_B\}}\right]\bar{c}_1\left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{3,\rho}} \\
&+E\left[e^{-\tau_C\rho-\bar{\beta}_{4,\rho}X(\tau_C)}\mathbb{1}_{\{x_C-X(\tau_C)<x_C-x_B\}}\right]\bar{c}_2\left(\frac{V_B}{V_0}\right)^{\bar{\beta}_{4,\rho}}
\end{aligned}$$

where $x_C = \log(V_C/V_0)$ and $x_B = \log(V_B/V_0)$. The condition $\mathbb{1}_{\{x_C-X(\tau_C)<x_C-x_B\}}$ ensures that the downward jumps are not large enough to trigger conversion and bankruptcy. Now we can apply Proposition 8 to the two expectations. The second expectation in equation 12.5 equals

$$\begin{aligned}
E\left[e^{-\tau_C\rho}\mathbb{1}_{\{\tau=\tau_C\}}\right] &= E\left[e^{\tau_C\rho}\mathbb{1}_{\{x_C-X(\tau_C)>x_C-x_B\}}\right] \\
&= e^{-\eta_2(x_C-x_B)}\frac{\eta_2-\beta_{3,\rho}}{\eta_2}\frac{\beta_{4,\rho}-\eta_2}{\beta_{4,\rho}-\beta_{3,\rho}}\left(e^{x_C\beta_{3,\rho}} - e^{x_C\beta_{4,\rho}}\right) \\
&= \left(\frac{V_B}{V_C}\right)^{\eta_2}\frac{\eta_2-\beta_{3,\rho}}{\eta_2}\frac{\beta_{4,\rho}-\eta_2}{\beta_{4,\rho}-\beta_{3,\rho}}\left(\left(\frac{V_C}{V_0}\right)^{\beta_{3,\rho}} - \left(\frac{V_C}{V_0}\right)^{\beta_{4,\rho}}\right)
\end{aligned}$$

where we have applied equation 12.4 in the second line. □

Theorem 6. *The default time for a firm, whose payout ratio changes at conversion, satisfies the following equality for $\theta > -\eta_2$:*

$$\begin{aligned}
E\left[e^{-\tau\rho+\theta X_\tau}\mathbb{1}_{\{\tau<\infty\}}\right] =& \bar{d}_1\left(\frac{V_B}{V_0}\right)^{-\theta-\bar{\beta}_{3,\rho}} J(\log(V_C/V_0), -\bar{\beta}_{3,\rho}, \log(V_C/V_B), \rho) \\
&+\bar{d}_2\left(\frac{V_B}{V_0}\right)^{-\theta-\bar{\beta}_{4,\rho}} J(\log(V_C/V_0), -\bar{\beta}_{4,\rho}, \log(V_C/V_B), \rho) \\
&+\frac{\eta_2-\beta_{3,\rho}}{\beta_{4,\rho}-\beta_{3,\rho}}\frac{\beta_{4,\rho}+\theta}{\eta_2+\theta}\left(\frac{V_C}{V_0}\right)^{\theta+\beta_{3,\rho}} + \frac{\beta_{4,\rho}-\eta_2}{\beta_{4,\rho}-\beta_{3,\rho}}\frac{\beta_{3,\rho}+\theta}{\eta_2+\theta}\left(\frac{V_C}{V_0}\right)^{\theta+\beta_{4,\rho}} \\
&- J(\log(V_C/V_0), \theta, \log(V_C/V_B), \rho)
\end{aligned}$$

where

$$\bar{d}_1 = \frac{\eta_2 - \bar{\beta}_{3,\rho}}{\bar{\beta}_{4,\rho} - \bar{\beta}_{3,\rho}} \frac{\bar{\beta}_{4,\rho} + \theta}{\eta_2 + \theta}$$
$$\bar{d}_2 = \frac{\bar{\beta}_{4,\rho} - \eta_2}{\bar{\beta}_{4,\rho} - \bar{\beta}_{3,\rho}} \frac{\bar{\beta}_{3,\rho} + \theta}{\eta_2 + \theta}$$

and with the same notation as in Theorem 5 for the rest.

Proof.

$$\begin{aligned}
&E\left[e^{-\tau\rho+\theta X_\tau}\mathbb{1}_{\{\tau<\infty\}}\right] \\
=&E\left[e^{-\rho\tau_C+\theta X_{\tau_C}} E\left[e^{-\rho(\tau-\tau_C)+\theta(X_\tau - X_{\tau_C})}|X(\tau_C),\tau_C\right]\mathbb{1}_{\{\tau>\tau_C\}} + e^{-\rho\tau_C+\theta X(\tau_C)}\mathbb{1}_{\{\tau=\tau_C\}}\right] \\
=&E\left[e^{-\rho\tau_C+\theta X_{\tau_C}}\left(\bar{d}_1\left(\frac{V_B}{V(\tau_C)}\right)^{\theta+\bar{\beta}_{3,\rho}} + \bar{d}_2\left(\frac{V_B}{V(\tau_C)}\right)^{\theta+\bar{\beta}_{4,\rho}}\right)\mathbb{1}_{\{\tau<\infty,\tau>\tau_C\}} + e^{-\rho\tau_C+\theta X(\tau_C)}\mathbb{1}_{\{\tau<\infty,\tau=\tau_C\}}\right] \\
=&E\left[\left(e^{-\rho\tau_C-\bar{\beta}_{3,\rho}X(\tau_C)}\bar{d}_1\left(\frac{V_B}{V_0}\right)^{-\theta-\bar{\beta}_{3,\rho}} + e^{-\rho\tau_C-\bar{\beta}_{4,\rho}X(\tau_C)}\bar{d}_2\left(\frac{V_B}{V_0}\right)^{-\theta-\bar{\beta}_{4,\rho}}\right)\mathbb{1}_{\{\tau<\infty,\tau>\tau_C\}}\right] \\
&+E\left[e^{-\rho\tau_C+\theta X(\tau_C)}\mathbb{1}_{\{\tau<\infty,\tau=\tau_C\}}\right]
\end{aligned}$$

The first expectation can be calculated using Proposition 8. The second expectation equals

$$\begin{aligned}
E\left[e^{-\rho\tau_C+\theta X(\tau_C)}\mathbb{1}_{\{\tau_C<\infty,\tau=\tau_C\}}\right] &= E\left[e^{-\rho\tau_C+\theta X(\tau_C)}\mathbb{1}_{\{\tau_C<\infty,-(X(\tau_C)-x_C)\geq x_C-x_B\}}\right] \\
&= E\left[e^{-\rho\tau_C+\theta X(\tau_C)}\mathbb{1}_{\{\tau_C<\infty\}}\right] - E\left[e^{-\rho\tau_C+\theta X(\tau_C)}\mathbb{1}_{\{-(X(\tau_C)-x_C)<x_C-x_B\}}\right]
\end{aligned}$$

which can also be calculated using Proposition 8. □

12.10 Proofs for Section 10

Proposition 24. *Assume that $c_D < c_C$. If $V_B^*(c^*) \geq V_C$, then the optimal debt choice $\{P_D, P_C\}$ is the combination of P_D and P_C that satisfies*

$$P_D + P_C = \tilde{P}(c^*) \qquad \text{and} \qquad c^*(P_D + P_C) = P_D c_D + P_C c_C$$

If $V_B^*(c^*) < V_C$, then the optimal debt choice $\{P_D, P_C\}$ is the highest amount of P_C such that two conditions are satisfied: 1. $V_B^{**}(P_D, P_C) = V_C$ and 2. Assumption 3 is violated.

Proof. If $V_B^*(c^*) \geq V_C$, then $V_B^*(c^*)$ is feasible. Hence, the firm will choose a combination of P_D and P_C which leads to $c^*(P_D+P_C) = P_D c_D + P_C c_C$. However, because of the commitment problem, $V_B = V_B^*(c^*) < V_C$ is not a feasible solution. By the strict concavity of $G_{debt}(\tilde{P})$ we conclude, that the optimal default barrier will be $V_B = V_C$. Hence, the firm will choose $\{P_D, P_C\}$ such that $V_B^{**} = V_C$. As long as $V_B = V_C$ the total value of the firm is a strictly increasing function in P_D and P_C. The firm solves the following problem:

$$\max_{P_D,P_C} V + TB_D(V_C) + TB_C(V_C) - BC(V_C)$$

Therefore, the firm choses the highest values for P_D and P_C such that $V_B^{**}(P_D, P_C) = V_C$. As $c_D < c_C$ the marginal increase in tax benefits for contingent convertible debt is higher than for straight debt. Hence, the optimal debt choice $\{P_D, P_C\}$ is the highest amount of P_C such that $V_B^{**}(P_D, P_C) = V_C$ under the constraint that Assumption 3 is not satisfied. $\square$

Lemma 15. *The maximal total tax benefits under the CCB regulation scheme are lower than the maximal tax benefits under no regulation.*

Proof. The total value of the firm for debt P_D and no CCBs is $G_{debt}(P_D)$. By assumption

$$\begin{aligned} G_{debt}(P_D^*) &= V + TB_D(P_D^*) - BC(P_D^*) \\ &= G_{debt}(\rho_i) + TB_C(\phi_i^C) \end{aligned}$$

which is equivalent to

$$BC(P_D^*) - BC(\rho_i) = TB_D(P_D^*) - TB_D(\rho_i) - TB_C(\phi_i^C).$$

As $P_D^* \geq \rho_i$, and the default barrier is strictly increasing in the amount of debt, it holds that $BC(P_D^*) > BC(\rho_i)$, which yields

$$TB_D(P_D^*) > TB_D(\rho_i) + TB_C(\phi_i^C).$$

$\square$

Lemma 16. *The maximal possible total leverage under the CCB regulation scheme is higher than the maximal possible total leverage under a pure capital requirement regulation scheme, if Assumption 7 is satisfied.*

Proof. The total leverage with the CCB regulation scheme is

$$TL_1 = \frac{\rho_i + \phi_i^C}{G_D(\rho_i) + TB_C(\phi_i^C)}.$$

The total leverage for the regulation without contingent convertible bonds equals

$$TL_2 = \frac{\rho_i}{G_D(\rho_i)}.$$

Obviously, we have $\frac{\rho_i}{G_D(\rho_i)} < 1$. By Assumption 7 it holds $\frac{\phi_i^C}{TB_C(\phi_i^C)} > 1$. Hence, we conclude $\frac{\phi_i^C}{TB_C(\phi_i^C)} > \frac{\rho_i}{G_D(\rho_i)}$. This is equivalent to $TL_1 > TL_2$ as the following chain of equivalent statements shows.

$$
\begin{aligned}
\frac{\rho_i + \phi_i^C}{G_D(\rho_i) + TB_C(\phi_i^C)} &> \frac{\rho_i}{G_D(\rho_i)} && \Leftrightarrow \\
\frac{\rho_i}{G_D(\rho_i)} \frac{G_D(\rho_i)}{G_D(\rho_i) + TB_C(\phi_i^C)} + \frac{\phi_i^C}{G_D(\rho_i) + TB_C(\phi_i^C)} &> \frac{\rho_i}{G_D(\rho_i)} && \Leftrightarrow \\
\frac{\rho_i}{G_D(\rho_i)} \left(\frac{G_D(\rho_i)}{G_D(\rho_i) + TB_C(\phi_i^C)} - 1 \right) + \frac{\phi_i^C}{G_D(\rho_i) + TB_C(\phi_i^C)} &> 0 && \Leftrightarrow \\
\frac{\phi_i^C}{G_D(\rho_i) + TB_C(\phi_i^C)} \frac{G_D(\rho_i) + TB_C(\phi_i^C)}{TB_C(\phi_i^C)} &> \frac{\rho_i}{G_D(\rho_i)} && \Leftrightarrow \\
\frac{\phi_i^C}{TB_C(\phi_i^C)} &> \frac{\rho_i}{G_D(\rho_i)}.
\end{aligned}
$$

□

12.11 Pure Diffusion Case

In the pure diffusion case the value of the firm's assets which follows a geometric Brownian motion is given by

$$dV_t = V_t((r - \delta)dt + \sigma dW_t).$$

Thus the process X is simply

$$X_t = \left(r - \delta - \frac{1}{2}\sigma^2 \right) t + \sigma W_t.$$

and its Laplace exponent is given by

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \left(r - \delta - \frac{1}{2}\sigma^2 \right) z.$$

The default time τ is defined as $\tau = \tau_x = \inf(t \geq 0 : X(t) \leq x)$ with $x = \log(V_B/V)$ and $\tau_C = \tau_{x_C} = \inf(t \geq 0 : X(t) \leq x_C)$ with $x_C = \log(V_C/V)$. The Laplace exponent of τ is calculated for example in Duffie (2001):

Lemma 17. *The Laplace exponent of τ_x for the pure diffusion process X equals*

$$E\left[e^{-\lambda \tau_x}\right] = e^{\beta_\lambda x}$$

where $\beta_\lambda = \frac{\gamma + \sqrt{\gamma^2 + 2\sigma^2 \lambda}}{\sigma^2}$ and $\gamma = r - \delta - \frac{1}{2}\sigma^2$. Note that $-\beta_\lambda$ is the negative root of the equation $\psi(z) = \lambda$.

The evaluation of the straight debt is presented in Leland (1994b):

Proposition 30. *The value of the debt equals*

$$D = D(V, V_B) = \frac{C_D + mP_D}{r+m}\left(1 - \left(\frac{V_B}{V}\right)^{\beta_{r+m}}\right) + (1-\alpha)V_B\left(\frac{V_B}{V}\right)^{\beta_{r+m}}$$

while the total value of the firm is

$$G_{debt}(V, V_B) = V + \frac{\bar{c}C_D}{r}\left(1 - \left(\frac{V_B}{V}\right)^{\beta_r}\right) - \alpha V_B\left(\frac{V_B}{V}\right)^{\beta_r}.$$

The law of the first passage time equals:

$$\mathbb{P}(\tau \leq t) = \Phi(h_1) + \exp\left(\frac{2\mu x}{\sigma^2}\right)\Phi(h_2),$$

where $h_1 = \frac{x-\gamma t}{\sigma\sqrt{t}}$, $h_2 = \frac{x+\gamma t}{\sigma\sqrt{t}}$ *and* $x = \log(\frac{V_B}{V_0})$. *Finally, the optimal barrier level is:*

$$V_B^* = \frac{\frac{C_D+mP_D}{r+m}\beta_{r+m} - \frac{\bar{c}C_D}{r}\beta_r}{1 + \alpha\beta_r + (1-\alpha)\beta_{r+m}}.$$

12.11.1 RCDs in a Pure Diffusion Model

The distinction between RCD1s and RCD2s was due to the possibility of jumps. In a pure diffusion model both contracts coincide. Here we require that Assumption 2 is satisfied, i.e. we consider only contracts where the value of the equity after conversion is sufficient to make the promised payment.

Proposition 31. *In a pure diffusion model the two contracts RCD1 and RCD2 are identical. The price of the CCBs is given by*

$$CB(V, V_C) = \left(\frac{c_C P_C + mP_C}{m+r}\right) + P_C\left(\frac{(m+r)\ell - c_C - m}{m+r}\right)\left(\frac{V_C}{V}\right)^{\beta_{m+r}}.$$

The price of RCDs is completely independent of any features of the straight debt.

Proof. First note that $V_{\tau_C} = V_C$ and hence $EQ(V_{\tau_C}) = EQ(V_C)$.

$$\begin{aligned} CB(V, V_C) &= \frac{cP_C + mP_C}{m+r} E\left[1 - e^{-(m+r)\tau_C}\right] + \ell P_C E\left[e^{-(m+r)\tau_C}\right] \\ &= \frac{cP_C + mP_C}{m+r} + P_C\left(\frac{(m+r)\ell - c - m}{m+r}\right) E\left[e^{-(m+r)\tau_C}\right]. \end{aligned}$$

□

Lemma 18. *The limit for $m \to 0$ corresponds to the case where only consol bonds are issued. The price of RCDs simplifies to*

$$CB(V, V_C) = \frac{c_C P_C}{r} + P_C\left(\frac{r\ell - c_C}{r}\right)\left(\frac{V_C}{V}\right)^{\beta_r}.$$

Remark 1. *Albul, Jaffee and Tchistyi's (2010) model is the special case for $m \to 0$.*

12.11.2 FSCCBs in a Pure Diffusion Model

Proposition 32. *The price of FSCCBs in a pure diffusion model equals*

$$CB(V,V_B,V_C) = \frac{c_C P_C + mP_C}{m+r}\left(1-\left(\frac{V_C}{V}\right)^{\beta_{m+r}}\right) + \frac{n'}{n+n'}EQ_{debt}(V_C)\left(\frac{V_C}{V}\right)^{\beta_{m+r}}$$

where

$$\begin{aligned} EQ_{debt}(V_C) =& V_C + \frac{\bar{c}C_D}{r}\left(1-\left(\frac{V_B}{V_C}\right)^{\beta_r}\right) - \alpha V_B\left(\frac{V_B}{V_C}\right)^{\beta_r} \\ & - \frac{C_D + mP_D}{r+m}\left(1-\left(\frac{V_B}{V_C}\right)^{\beta_{r+m}}\right) + (1-\alpha)V_B\left(\frac{V_B}{V_C}\right)^{\beta_{r+m}}. \end{aligned}$$

Proof. First note that $V_{\tau_C} = V_C$ and hence $EQ(V_{\tau_C}) = EQ(V_C)$.

$$CB(V,V_B,V_C) = \frac{c_C P_C + mP_C}{m+r}\left(1-E\left[e^{-(m+r)\tau_C}\right]\right) + \frac{n'}{n+n'}EQ_{debt}(V_C)E\left[e^{-(m+r)\tau_C}\right].$$

□

In contrast to RCDs the price of FSCCBs explicitly depends on V_B and thus on the features of the straight debt.

Lemma 19. *For $m \to 0$ we obtain the special case of consol bonds. The pricing formula simplifies to*

$$CB(V,V_B,V_C) = \frac{c_C P_C}{r}\left(1-\left(\frac{V_C}{V}\right)^{\beta_r}\right) + \frac{n'}{n+n'}EQ_{debt}(V_C)\left(\frac{V_C}{V}\right)^{\beta_r}$$

where

$$EQ_{debt}(V,V_B,V_C) = V + \frac{\bar{c}C_D}{r} + \frac{\bar{c}C_D}{r} - \frac{\bar{c}C_D + \bar{c}C_C + r\ell P_C - c_C P_C}{r}\left(\frac{V_C}{V}\right)^{\beta_r}.$$

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