COLEMAN FUNG
RISK MANAGEMENT
RESEARCH CENTER

Working Paper # 2010 -05

The Pricing of " Troubled" Assets

Konstantin Magin, UC Berkeley

September 3, 2010

University of California
Berkeley

The Pricing of "Troubled" Assets*

Konstantin Magin

September 3, 2010

Abstract

The Troubled Assets Relief Program (TARP), authorized and appropriated by Congress in 2008, enables the Secretary of the Treasury to purchase on the government's behalf so-called "troubled" assets the purchase of which is necessary in order to promote economic stability. However, virtually none of the money was used in that way; instead, the money was injected into banks and various companies (principally in preferred stock) to recapitalize them. High uncertainty on the valuation of the assets led to a large gap between bid and ask prices and the virtual shutdown of the market. Therefore, it is of vital importance to determine how to establish the appropriate values at which these securities should be purchased. This paper develops a methodology to price "troubled" assets. It starts with pricing individual "troubled" mortgages. Then it proceeds to pricing pools of "troubled" mortgages. This paper finds that single family mortgages in default in Freddie Mac and Fannie Mae portfolios can be settled as high as 81-98 cents on the dollar.

Keywords: Asset Pricing, Case-Shiller Home Price Index, Implied Volatility, Troubled assets, Delinquent Mortgages, Mortgage-Backed Securities, Subprime Mortgages, LTV, TARP, Housing Derivatives, Futures, Futures Options, Chicago Mercantile Exchange, Freddie Mac, Fannie Mae

---

*This research was supported by a grant from the Coleman Fung Risk Management Research Center. I am very grateful to Bob Anderson, Brad DeLong and Lori Aldinger for their kind support.

1. Introduction

The Troubled Assets Relief Program (TARP), authorized and appropriated by Congress in 2008, enables the Secretary of the Treasury to purchase on the government's behalf so-called "troubled" assets created on or before March 14, 2008, the purchase of which is necessary in order to promote economic stability.[1] However, virtually none of the money was used in that way; instead, the money was injected into banks and various companies (principally in preferred stock) to recapitalize them. High uncertainty on the valuation of the assets led to a large gap between bid and ask prices and the virtual shutdown of the market. Therefore, it is of vital importance to determine how to establish the appropriate values at which these securities should be purchased.

The main goal of this paper is to price "troubled" assets. It starts with the pricing of individual "troubled" mortgages, then proceeds to the pricing of pools of "troubled" mortgages.

Fortunately for this project, in 2006 the Chicago Mercantile Exchange (CME) launched futures and futures options written on the S&P/Case-Shiller Home Price Indexes (CSI). The Case-Shiller Home Price Indexes are aimed at tracking the price path of single family homes using the repeat sales technique developed by Karl Case and Robert Shiller. The indexes measure 10 metropolitan areas throughout the U.S. : Boston, Chicago, Denver, Las Vegas, Los Angeles, Miami, New York (commuter index), San Diego, San Francisco and Washington, D.C. In addition, new investment vehicles are currently being created. Standard & Poors licensed MacroMarkets to create products based on the Case-Shiller Home Price Indexes. The CSI-based ETF began trading on the NYSE Arca in Fall 2009.[2] The modelling and pricing of these new vehicles was pioneered by Case, Shiller and Weiss (1993), (1998), Shiller and Weiss (1999).[3]

The first step in this paper will be to apply the standard Black-Scholes methodology to derive the general formula for pricing portfolios of delinquent

---

[1] Troubled assets include mortgages, pools of mortgages, mortgage-backed securities, and further derivative securities.

[2] http://online.wsj.com/article/SB123897667301591301.html.

[3] Index-Based Futures and Options Markets in Real Estate. Case,-Karl-E, Jr; Shiller,-Robert-J; Weiss,-Allan-N, Journal-of-Portfolio-Management. Winter 1993; 19(2): 83-92.

Home Equity Insurance. Shiller,-Robert-J; Weiss,-Allan-N, Journal of Real Estate Finance and Economics, 19:1, 21-47 (1999).

single family mortgages, taking into consideration that the mark-to-market LTV ratio does not have to be constant even within any given geographical area, as outlined in the model section below. What prices for "troubled" assets would be consistent with futures and futures options prices for the CSI as traded on the CME? Using market prices of futures and futures options contacts written on the S&P/Case-Shiller Home Price Indexes traded on the CME, one can obtain the implied market volatility to estimate the variances of these indexes. Then I will use this estimate to calculate the Black-Scholes price of pools of single family mortgages in default in Freddie Mac and Fannie Mae portfolios for 5 different geographical regions and various mark-to-market LTVs. I found that single family mortgages in default in Freddie Mac and Fannie Mae portfolios can be settled as high as 81-98 cents on the dollar.

2. **The Model**

Definition: We call a loan a "troubled" asset if the borrower has not made minimum payment for more than 3 months.

The following theorem below proposes hedging strategy to replicate and price "troubled" assets.

Theorem: Suppose that an investor maximizes

$$U = E\left[\frac{1}{1-\lambda} C_t^{1-\lambda} + b \frac{1}{1-\lambda} C_{t+1}^{1-\lambda}\right] \; .$$

Let PL_t be the price of troubled asset at period t, p_t be the price of the loan's collateral at period t. Suppose also that $x = \ln\left(\frac{p_{t+1}}{p_t}\right)$ and $y = \ln\left(b\left(\frac{c_{t+1}}{c_t}\right)^{-\lambda}\right)$ are bivariate normally distributed with expectations

$$\left(E\left[x\right],\, E\left[y\right]\right) = \left(\mu_x,\, \mu_y\right)$$

and the variance-covariance matrix

$$V = \begin{pmatrix} \sigma_x & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y \end{pmatrix}.$$

Suppose also that $PR_{t+1}-$ the loans's principal balance left at period $t+1$ is known at period t.

Then

$$\boxed{PL_t = p_t\left(1 - N(Z+\sigma)\right) + \frac{PR_{t+1}}{1+r_f} N(Z),}$$

where

$$N(z) = \int\limits_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} \, dv$$

and

$$Z = \frac{\ln \frac{p_t}{PR_{t+1}} + \ln R_f}{\sigma} - \frac{1}{2}\sigma.$$

Proof : Note first that the lender's period $t+1$ payoff on a troubled loan originated at period t is $\min\left[PR_{t+1},\ p_{t+1}\right]$, where PR_{t+1} is the loans's principal balance left at period $t+1$ and p_{t+1} is the price of the loan's collateral at period $t+1$. We know that the price of this risky bond PL_t with return $R = 1 + r = \min\left[PR_{t+1},\ p_{t+1}\right]$ is given by

$$PL_t = p_t - Call(p_t,\ PR_{t+1}).$$

We also know that

$$Call(p_t, PR_{t+1}) = p_t N(Z+\sigma) - \frac{PR_{t+1}}{1+rf} N(Z),$$

where

$$Z = \frac{\ln \frac{p_t}{PR_{t+1}} + \ln R_f}{\sigma} - \frac{1}{2}\sigma.$$

Hence,

$$PL_t = p_t - p_t N(Z+\sigma) + \frac{PR_{t+1}}{1+rf} N(Z) =$$

$$= p_t\left(1 - N(Z+\sigma)\right) + \frac{PR_{t+1}}{1+rf} N(Z) \ \blacksquare.$$

We are now ready to price portfolios of delinquent single family mortgages. Consider a pool of n different mortgages in default. Note first that the lender's period $t+1$ payoff on a troubled loan k originated at period t is $\min\left[PR_{kt+1},\ p_{kt+1}\right]$, where PR_{kt+1} is the loans's k principal balance left at period $t+1$ and p_{kt+1} is the price of the loan's k collateral at period $t+1$. We know that the price of this pool of n different mortgages in default is given by

$$PL_t = \sum_{k=1}^{n} PL_{kt} = \sum_{k=1}^{n} \left(p_{kt} - Call(p_{kt},\quad PR_{kt+1})\right).$$

$\Pr oposition\ 1: Let\ \frac{PR_{kt+1}}{p_{kt+1}} = \lambda_{t+1}$[4] *for all* $k = 1, ..., n$. *Then*

$$PL_t = \sum_{k=1}^{n} \left(\underbrace{p_{kt} - Call(p_{kt},\quad PR_{kt+1})}_{\substack{PL_{kt} \\ \Pr ice\ of\ Mortgage\ k \\ in\ Default}} \right) = \underbrace{\sum_{k=1}^{n} p_{kt} - Call\left(\sum_{k=1}^{n} p_{kt},\ \sum_{k=1}^{n} PR_{kt+1}\right)}_{\substack{Price\ of\ RMBS \\ Based\ on\ Portfolio\ of\ n\ Loans \\ in\ Default}}.$$

$Proof:$ Let

$$\Omega_1 = \{\omega \in \Omega \mid \lambda_{t+1}(\omega) \geqslant 1\}$$
$$\Omega_2 = \{\omega \in \Omega \mid \lambda_{t+1}(\omega) < 1\}.$$

Clearly,

$$\Omega = \Omega_1 \cup \Omega_2.$$

So, we have two cases to consider here:

[4] In the current environment second liens are hardly ever satisfied. It is therefore appropriate to use LTV not CLTV for pricing delinquent mortgages.

Period $t+1$ Payoff for the Portfolio of n Different Mortgages in Default

State of Nature	Realization	Period $t+1$ Payoff for Loan k	Period $t+1$ Payoff for Portfolio of n Loans in Default
$\omega \in \Omega_1$	$p_{kt+1}(\omega) \leqslant PR_{kt+1}$	$p_{kt+1}(\omega)$	$\sum_{k=1}^{n} p_{kt+1}(\omega)$
$\omega \in \Omega_2$	$p_{kt+1}(\omega) > PR_{kt+1}$	PR_{kt+1}	$\sum_{k=1}^{n} PR_{kt+1}$

Period $t+1$ Payoff for RMBS based on Portfolio of n Loans in Default

State of Nature	Realization	Period $t+1$ Payoff for RMBS Based on Portfolio of n Loans in Default
$\omega \in \Omega_1$	$\sum_{k=1}^{n} p_{kt+1}(\omega) \leqslant \sum_{k=1}^{n} PR_{kt+1}$	$\sum_{k=1}^{n} p_{kt+1}(\omega)$
$\omega \in \Omega_2$	$\sum_{k=1}^{n} p_{kt+1}(\omega) > \sum_{k=1}^{n} PR_{kt+1}$	$\sum_{k=1}^{n} PR_{kt+1}$

So, we can conclude by No Arbitrage Condition that
$PL_t = \sum_{k=1}^{n} \left(p_{kt} - Call(p_{kt}, \quad PR_{kt+1}) \right) = \sum_{k=1}^{n} p_{kt} - Call(\sum_{k=1}^{n} p_{kt}, \ \sum_{k=1}^{n} PR_{kt+1}).$ ■

However, we cannot generally assume that the LTV ratio λ_{t+1} is constant even within any given geographical area. So, we need to expand the model to take this into account.

The probability density function for a continuous uniform distribution on the interval $[a, b]$ is given by

$$f(x) = \begin{cases} 0 \ for \ x < a \\ \frac{1}{b-a} \ for \ a \leqslant x \leqslant b \\ 0 \ for \ x > b \end{cases} .$$

Pr *oposition* $2:$ *Let random variable* x *be uniformly distributed on the interval* $[a, b]$. *Then for any integrable function* g *on* $]-\infty, \ \infty[$ *we have*

$$E\left[g\left(x\right)\right]=\int\limits_{-\infty}^{\infty}g\left(x\right)f\left(x\right)dx=\frac{1}{b-a}\int\limits_{a}^{b}g\left(x\right)\,dx.$$

$Proof:$ Indeed,

$$E\left[g\left(x\right)\right]=\int\limits_{-\infty}^{\infty}g\left(x\right)f(x)dx=$$

$$=\int\limits_{-\infty}^{a}g\left(x\right)f(x)dx+\int\limits_{a}^{b}g\left(x\right)f(x)dx+\int\limits_{b}^{\infty}g\left(x\right)f(x)dx=$$

$$=\frac{1}{b-a}\int\limits_{a}^{b}g\left(x\right)dx.\ \blacksquare$$

So, using this result we obtain the following proposition:

$\Pr oposition\ 2:$ *Suppose all mortgages from a set* S_i *of mortgages in default have the same LTV ratio* λ_{t+1}. *Suppose also that the LTV ratio* λ_{t+1} *is uniformly distributed on the interval* $\left[a,b\right]$.

Then

$$E\left[PL_t\right]=\sum_{k\in S_i}PR_{kt+1}\left[\frac{1}{b-a}\left(\ln(b)-\ln(a)\right)-\frac{1}{b-a}\int\limits_{a}^{b}Call(\ \frac{1}{\lambda_{t+1}},\ 1)d\lambda_{t+1}\right].$$

$Proof$: Indeed, we have that

$$PL_t=\sum_{k\in S_i}\left(p_{kt}-Call(p_{kt},\quad PR_{kt+1})\right).\ \bigstar$$

But

$$p_{kt}=\frac{PR_{kt+1}}{\lambda_{t+1}}.$$

Substituting it into $\bigstar$ we obtain

$$PL_t=\sum_{k\in S_i}\left(\frac{PR_{kt+1}}{\lambda_{t+1}}-Call(\frac{PR_{kt+1}}{\lambda_{t+1}},\quad PR_{kt+1})\right).$$

By Proposition 1 from above we get

$$PL_t = \frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}} - Call\left(\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}},\ \sum_{k\in S_i} PR_{kt+1}\right).$$

Taking expectations of both sides of the above equation, we obtain

$$E\left[PL_t\right] = \int\limits_{-\infty}^{\infty}\left[\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}} - Call\left(\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}},\ \sum_{k\in S_i} PR_{kt+1}\right)\right] f(\lambda_{t+1})d\lambda_{t+1}.$$

Now, using Proposition 2 we get that

$$\int\limits_{-\infty}^{\infty}\left[\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}} - Call\left(\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}},\ \sum_{k\in S_i} PR_{kt+1}\right)\right] f(\lambda_{t+1})d\lambda_{t+1} =$$

$$\int\limits_{a}^{b}\left[\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}} - Call\left(\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}},\ \sum_{k\in S_i} PR_{kt+1}\right)\right] \frac{1}{b-a}d\lambda_{t+1}.$$

Then, clearly

$$\int\limits_{a}^{b}\left[\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}} - Call\left(\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}},\ \sum_{k\in S_i} PR_{kt+1}\right)\right] \frac{1}{b-a}d\lambda_{t+1} =$$

$$\frac{1}{b-a}\int\limits_{a}^{b}\left[\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}}\right] d\lambda_{t+1} - \frac{1}{b-a}\int\limits_{a}^{b}\left[Call\left(\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{\lambda_{t+1}},\ \sum_{k\in S_i} PR_{kt+1}\right)\right] d\lambda_{t+1} =$$

$$\frac{\sum\limits_{k\in S_i} PR_{kt+1}}{b-a}\left(\ln(b)-\ln(a)\right) - \frac{\sum\limits_{k\in S_i} PR_{kt+1}}{b-a}\int\limits_{a}^{b}\left[Call\left(\frac{1}{\lambda_{t+1}},\ 1\right)\right] d\lambda_{t+1} =$$

$$\sum_{k\in S_i} PR_{kt+1}\left[\frac{1}{b-a}\left(\ln(b)-\ln(a)\right) - \frac{1}{b-a}\int\limits_{a}^{b} Call\left(\frac{1}{\lambda_{t+1}},\ 1\right)d\lambda_{t+1}\right]. \blacksquare$$

3. **Calculating Implied Volatility of Case-Shiller Indexes**

We next calculate the implied volatility σ_k of the underlying Case-Shiller Index k using data for housing futures and futures options traded on the CME in from 2006 to 2008.[5] The results are given in Table 1 below.

Table 1: Implied Volatilities of Case-Shiller Indexes, 2007-2008

Index	Put/Call	Year Implied Volatility Calculated	Implied Volatility of Futures Contracts, %
$LXXR$	$Call$	2008	15.38
$SFXR$	$Call$	2008	11.41
$SDXR$	Put	2008	17.43
$DNXR$	$Call$	2007	8.38
$LVXR$	$Call$	2007	9.02
$MIXR$	$Call$	2007	9.48
$CHXR$	$Call$	2008	15.89
$BOXR$	Put	2008	14.40
$WDXR$	Put	2008	16.82
$NYXR$	$Call$	2007	9.10

Because housing futures options contracts were not traded after 2008 and futures options for some of the indexes were not traded after 2007, I had to

[5] The implied volatility was calculated using the Black-Scholes formula for futures options contracts. For futures call options it is

$$Call(F,S) = \frac{1}{1+rf} F \cdot N(Z_{ks} + \sigma_k) - \frac{S}{1+rf} N(Z_{ks}),$$

where

$$Z_{ks} = \frac{\ln \frac{F}{S}}{\sigma_k} - \frac{1}{2}\sigma_k$$

and F is the futures price.
The volatility of the futures contract is

$$\sigma_F = \frac{1}{1+rf}\sigma_k.$$

But given that rf is very close to 0, we can assume that the volatility of futures contracts σ_F is equal to the volatility σ_k of the underlying asset k.

develop a technique to estimate the implied volatility for 2009. I assumed that the ratio of the implied volatility of the index to the current volatility stays constant over time, i.e., the ratio $\frac{\sigma_{IV\ t}}{\sigma_{CV\ t}}$ is constant over time. So I first calculated the current volatilities of Case-Shiller Indexes. See Table 2 below.

Table 2: Current Volatilities of Case-Shiller Indexes (Blank cells are data that were not needed for the calculations.)

Index	Current Volatility of Futures Contracts, % 2007, $\sigma_{CV\ 2007}$	Current Volatility of Futures Contracts, % 2008, $\sigma_{CV\ 2008}$	Current Volatility of Futures Contracts, % 2009, $\sigma_{CV\ 2009}$
$LXXR$		2.94	4.37
$SFXR$		4.14	7.47
$SDXR$		2.24	3.98
$DNXR$	4.17		4.45
$LVXR$	3.32		5.22
$MIXR$	2.69		5.61
$CHXR$		3.98	6.92
$BOXR$		4.01	4.62
$WDXR$		2.81	4.90
$NYXR$	0.96		3.84

Then I multiplied the current volatility $\sigma_{CV\ 2009}$ of each index for 2009 by the ratio $\frac{\sigma_{IV\ t}}{\sigma_{CV\ t}}$ to obtain the estimated Implied volatilities $\sigma_{IV\ 2009}$ of the Case-Shiller Indexes for 2009. See Table 3 below. As expected, the implied volatilities are much higher than current volatilities.

Table 3: Estimated Implied Volatilities of Case-Shiller Indexes for 2009

Index	Put/Call	Estimated Implied Volatility of Futures Contracts, % 2009
$LXXR$	*Call*	22.92
$SFXR$	*Call*	20.54
$SDXR$	*Put*	31.03
$DNXR$	*Call*	8.97
$LVXR$	*Call*	14.16
$MIXR$	*Call*	19.81
$CHXR$	*Call*	27.65
$BOXR$	*Put*	16.56
$WDXR$	*Put*	29.27
$NYXR$	*Call*	36.40

Once I had calculated the implied volatilities of the 10 Case-Shiller indexes, I then calculated implied the volatilities for the 5 regions in the Freddie Mac and Fannie Mae single family mortgage portfolios as weighted averages of the implied volatilities of the respective indexes. Weights are taken from the Case-Shiller Composite 10 index and normalized so they add up to 1 for each region.

Table 4: Weights of Case-Shiller Indexes Used in Calculating Regional Implied Volatilities (Weights are given for the metropolitan areas for each region.)

Region	LXXR	SFXR	SDXR	LVXR	DNXR	MIXR	CHXR	BOXR	WDXR	NYXR	Regional Implied Volatility, σ, %
North Central							1.00				27.65
Northeast								0.17	0.18	0.65	31.74
Southeast						1.00					19.81
Southwest					1.00						8.97
West	0.53	0.30	0.14	0.03							23.08

5. Pricing Delinquent Single Family Mortgages in the Freddie Mac Portfolio

Fortunately, the Freddie Mac 10-K form for 2009 provides all the data necessary for our calculations broken down by region and LTV.

Table 5: Mark-to-Market LTV Distribution as a Percentage of Total Unpaid Principal Balance in the Freddie Mac Single Family Portfolio, %[6]

Region	$0\% < LTV \leqslant 80\%$	$80\% < LTV \leqslant 95\%$	$95\% < LTV \leqslant 125\%$
North Central	9.1	4.7	4.5
Northeast	16.0	5.2	3.4
Southeast	8.8	3.9	5.2
Southwest	7.7	3.3	1.3
West	14.2	4.6	8.1

Given that the total unpaid principal balance of the Freddie Mac single family portfolio is $1,903 billion one can produce mark-to-market LTV distribution as the total unpaid principal balance in the Freddie Mac single family portfolio, in $ billions. See Table 6 below.

Table 6: Mark-to-Market LTV Distribution as Total Unpaid Principal Balance in the Freddie Mac Single Family Portfolio, in $ billions

Region	$0\% < LTV \leqslant 80\%$	$80\% < LTV \leqslant 95\%$	$95\% < LTV \leqslant 125\%$
North Central	173.1730	89.4410	85.6350
Northeast	304.4800	98.9560	64.7020
Southeast	167.4640	74.2170	98.9560
Southwest	146.5310	62.7990	24.7390
West	270.2260	87.5380	154.1430

The 2009 Freddie Mac 10-K form also provides delinquency rates as a percentage of delinquent single family mortgages as a function of mark-to-market LTV in the Freddie Mac single family portfolio, in %. See Table 7 below.

[6] Table 61, p. 157-158 of 2009 Freddie Mac 10-K form.

Table 7: Delinquency Rates as a Percentage of Delinquent Single Family Mortgages as a Function of Mark-To-Market LTV in the Freddie Mac Single Family Portfolio, in %[7]

Region	$0\% < LTV \leqslant 80\%$	$80\% < LTV \leqslant 95\%$	$95\% < LTV \leqslant 125\%$
North Central	1.3	3.5	8.9
Northeast	1.5	4.9	11.9
Southeast	2.0	4.7	15.8
Southwest	1.3	3.1	7.3
West	1.1	4.3	16.4

So, multiplying each element of Table 6 by the respective element from Table 7 we obtain the unpaid principal balance for each of the 5 geographical regions and 3 mark-to-market LTV intervals. See Table 8 below.

Table 8: LTV Distribution as Total Unpaid Principal Balance PR_t of Delinquent Single Family Mortgages in the Freddie Mac Single Family Portfolio, in $ billions

Region	$0\% < LTV \leqslant 80\%$	$80\% < LTV \leqslant 95\%$	$95\% < LTV \leqslant 125\%$
North Central	2.251	3.130	7.622
Northeast	4.567	4.849	7.700
Southeast	3.349	3.488	15.635
Southwest	1.905	1.947	1.806
West	2.972	3.764	25.279

Therefore, we have now each geographical region divided into 3 LTV intervals: $[0, 0.8]$, $[0.8, 0.95]$, $[0.95, 1.25]$. We then assume that the LTV ratio λ_{t+1} is uniformly distributed on each of these 3 segments. Now, applying Proposition 3 we obtain the ratio $\frac{PL_t}{PR_t}$ for delinquent single family mortgages in the Freddie Mac single family portfolio. See Table 9 below.

[7] Table 61, p. 157-158 of the 2009 Freddie Mac 10-K form.

Table 9: Ratio $\frac{PL_t}{PR_t}$ for Delinquent Single Family Mortgages in the Freddie Mac Single Family Portfolio

State	$0\% < LTV \leqslant 80\%$	$80\% < LTV \leqslant 95\%$	$95\% < LTV \leqslant 125\%$
North Central	0.9763	0.9265	0.8360
Northeast	0.9738	0.9118	0.8217
Southeast	0.9793	0.9526	0.8628
Southwest	0.9802	0.9772	0.8946
West	0.9783	0.9422	0.8518

So, single family mortgages in default in the Freddie Mac portfolio can be settled as high as 82-98 cents on the dollar.

Now, multiplying each element of Table 8 by the respective element of Table 9 we obtain the Black-Scholes prices PL_t of delinquent single family mortgages in the Freddie Mac single family portfolio. See Table 10 below.

Table 10: Black-Scholes Prices PL_t of Delinquent Single Family Mortgages in the Freddie Mac Single Family Portfolio, in $ billions

Region	$0\% < LTV \leqslant 80\%$	$80\% < LTV \leqslant 95\%$	$95\% < LTV \leqslant 125\%$
North Central	2.1977	2.8999	6.3720
Northeast	4.4473	4.4213	6.3271
Southeast	3.2797	3.3227	13.4899
Southwest	1.8673	1.9026	1.6156
West	2.9075	3.5464	21.5327

6. Pricing Delinquent Single Family Mortgages in the Fannie Mae Portfolio

Given that the total unpaid principal balance of the Fannie Mae single family portfolio is $2,796,5 billion, one can produce geographical distribution as total unpaid principal balance in the Fannie Mae single family portfolio, in $ billions. See Table 11 below.

Table 11: Geographical Distribution as Total Unpaid Principal Balance in the Fannie Mae Single Family Portfolio, in $ billions[8]

Region	%	**Unpaid Principal Balance**
Midwest	16	447.440
Northeast	19	531.335
Southeast	24	671.160
Southwest	15	419.475
West	26	727.090

Unlike Freddie Mac, Fannie Mae provides only the aggregate (not region-by-region) mark-to-market LTV distribution for its single family mortgage portfolio.[9] So, multiplying unpaid principal balance from Table 11 by the aggregate mark-to-market LTV distribution we obtain mark-to-market LTV distribution as total unpaid principal balance in the Fannie Mae single family portfolio, in $ billions. This yields Table 12 below.

Table 12: Mark-To-Market LTV Distribution as Total Unpaid Principal Balance in the Fannie Mae Single Family Portfolio, in $ billions

Region	0%<LTV⩽80%	80%<LTV⩽90%	90%<LTV⩽100%	100%<LTV⩽125%
Midwest	281.8872	62.6416	40.2696	62.6416
Northeast	334.7411	74.3869	47.8201	74.3869
Southeast	422.8308	93.9624	60.4044	93.9624
Southwest	264.2693	58.7265	37.7527	58.7265
West	458.0667	101.7926	65.4381	101.7926

The relationship between mark-to-market LTV and delinquency rates is a macroeconomic one and likely to be the same for Fannie Mae as Freddie Mac. I therefore used the data from Table 7 for the Freddie Mac single family portfolio to calculate delinquency rates as a percentage of delinquent single family mortgages as a function of mark-to-market LTV for the Fannie Mae single family portfolio. See Table 13 below.

[8] Table 42, p. 151-153 of the 2009 Fannie Mae 10-K form.
[9] Table 42, p. 151-153 of the 2009 Fannie Mae 10-K form.

Table 13: Delinquency Rates as a Percentage of Delinquent Single Family Mortgages as a Function of Mark-To-Market LTV in the Fannie Mae Single Family Portfolio, in %[10]

Region	0%<LTV⩽80%	80%<LTV⩽90%	90%<LTV⩽100%	100%<LTV⩽125%
Midwest	1.3	3.5	6.2	8.9
Northeast	1.5	4.9	8.4	11.9
Southeast	2.0	4.7	10.25	15.8
Southwest	1.3	3.1	5.2	7.3
West	1.1	4.3	10.35	16.4

So, multiplying each element from Table 12 by the respective element from Table 13 we obtain the unpaid principal balance for each of the 5 geographical areas and 4 mark-to-market LTV intervals. See Table 14 below.

Table 14: LTV Distribution as Total Unpaid Principal Balance PR_t of Delinquent Single Family Mortgages in the Fannie Mae Single Family Portfolio, in $ billions

Region	0%<LTV⩽80%	80%<LTV⩽90%	90%<LTV⩽100%	100%<LTV⩽125%
Midwest	3.6645	2.1925	2.4967	5.5751
Northeast	5.0211	3.6450	4.0169	8.8520
Southeast	8.4566	4.4162	6.1915	14.8461
Southwest	3.4355	1.8205	1.9631	4.2870
West	5.0387	4.3771	6.7728	16.6940

Therefore, we have now each geographical area divided into 4 mark-to-market LTV intervals: [0, 0.8], [0.8, 0.90], [0.90, 1.00], [1.00, 1.25]. As with Freddie Mac we then assume that the LTV ratio λ_{t+1} is uniformly distributed on each of these 4 segments. Now, applying Proposition 3 we obtain the ratio $\frac{PL_t}{PR_t}$ for delinquent single family mortgages in the Fannie Mae single family portfolio. See Table 15 below.

Table 15: Ratio $\frac{PL_t}{PR_t}$ for Delinquent Single Family Mortgages in Fannie Mae Single Family Portfolio

Region	0%<LTV⩽80%	80%<LTV⩽90%	90%<LTV⩽100%	100%<LTV⩽125%
Midwest	0.9763	0.9347	0.9007	0.8250
Northeast	0.9738	0.9204	0.8848	0.8110
Southeast	0.9793	0.9591	0.9305	0.8510
Southwest	0.9802	0.9791	0.9678	0.8811
West	0.9783	0.9495	0.9182	0.8404

[10] Table 61, p. 157-158 of 2009 Freddie Mac 10-K form.

So, single family mortgages in default in Fannie Mae portfolios can be settled as high as 81-98 cents on the dollar.

Now, multiplying each element of Table 14 by the respective element of Table 15 we obtain the Black-Scholes prices PL_t of delinquent single family mortgages in the Fannie Mae portfolio. See Table 16 below.

Table 16: Black-Scholes Prices PL_t of Delinquent Single Family Mortgages in the Fannie Mae Single Family Portfolio, in \$ billions

Region	0%<LTV⩽80%	80%<LTV⩽90%	90%<LTV⩽100%	100%<LTV⩽125%
Midwest	3.5777	2.0493	2.2488	4.5995
Northeast	4.8895	3.3549	3.5542	7.1790
Southeast	8.2815	4.2356	5.7612	12.6340
Southwest	3.3675	1.7825	1.8999	3.7773
West	4.9294	4.1561	6.2188	14.0296

7. Conclusion

The main goal of this paper is to develop a methodology to price "troubled" assets. I first constructed a replicating strategy to price mortgages in default. This strategy involves going long on the underlying asset (shares of respective CSI-based ETFs) and short on call option written on this index with a strike price equal to the remaining principal of the mortgage in default. I then derived a general formula for pricing portfolios of delinquent single family mortgages, taking into consideration that the mark-to-market LTV ratio does not have to be constant even within any given geographical area. Then the implied market volatilities of futures contacts written on the S&P/Case-Shiller Home Price Indexes were calculated using futures options traded on the CME. I then used this estimate of implied market volatility to calculate the Black-Scholes price of pools of single family mortgages in default in the Freddie Mac and Fannie Mae portfolios for 5 different geographical regions and various mark-to-market LTVs. I found that single family mortgages in default in Freddie Mac and Fannie Mae portfolios can be settled as high as 81-98 cents on the dollar.