A Popularity Asset Pricing Model

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Abstract

This paper presents a formal model for theory of popularity as laid out informally by Ibbotson and Idzorek (2014). The paper does this by extending the capital asset pricing model (CAPM) to include security characteristics that different investors regard differently. This leads to an equilibrium in which: 1) The expected excess return on each security is a linear function of its beta and its *popularity loadings* which measure the popularity of the security based on its characteristics relative to the those of the beta-adjusted market portfolio; 2) Each investor holds a different portfolio based on his attitudes toward security characteristics; and 3) The market portfolio is *not* on the efficient frontier. I call this extended model the Popularity Asset Pricing Model, or PAPM for short.

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1. Introduction

The Capital Asset Pricing Model (CAPM) was developed over a half century ago. [1] Despite the distinction of the theory, subsequent empirical research has mostly failed to confirm it. Perhaps its biggest strength, expressing investor preferences solely in terms of risk, is also its limitation. Investors care about many characteristics that have little to do with risk. These include such asset characteristics as liquidity, taxability, scalability, divisibility, controllability, transparency, and the components of sustainability; namely environmental, social, and governance factors. Including non-risk characteristics may even be the way to resurrect the CAPM. This article addresses non-risk characteristics in a CAPM-like framework based on the concept of popularity introduced by Ibbotson and Idzorek (2014) with an equilibrium model that I call the Popularity Asset Pricing Model, or PAPM for short.

The idea of including security characteristics and investor attitudes towards them is not new. Ibbotson, Diermeier, and Siegel (1984) present a sketch for an equilibrium model based on characteristics and investors' attitudes towards them. They called this model a "New Equilibrium Theory" or NET for short.

The PAPM is relevant in both an individual security context as well as in an asset allocation context in which investors allocate their wealth between assets classes such as stocks, corporate and municipal bonds, real estate, etc. But even more broadly, the characteristics modeled in the PAPM can represent many of the psychological desires and preferences that are portrayed in behavioral finance, e.g. prospect theory, affect, sentiment, and attentiveness. Furthermore, in equity markets, the characteristics function like factor premiums, [2]such as value vs. growth, momentum vs. reversal, size, quality, liquidity, and even volatility since in some instances investors might even prefer riskier assets. The discovery of these premiums has led to the development of indexes that are investable as "smart" or "strategic beta" exchange traded funds. Since in the PAPM, such premiums are due to popularity effects, I call them *popularity premiums*.

The existence of premiums raises these questions:

1) Why do they exist? The existence of premiums appears to be a "free lunch" that should not exist in an efficient market.
2) Which investors are on the opposite side? If there are investors who are systematically beating the market, in order for the market to clear there must be investors systematically falling behind the market. These are the investors that Rob Arnott (quoted in Rostad 2013) calls "willing losers."

To answer these questions, we need a model in which:

1) There are premiums.
2) There are some investors who hold portfolios tilted towards the premiums and thus outperform the market; while there are other investors who tilt away from these premiums and thus underperform the market.

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[1] The CAPM is usually attributed to Sharpe (1964) and Lintner (1965), but there were several other related papers.

[2] Strictly speaking, factors are systematic drivers of returns made up with long and short combinations of securities with various characteristics, which may or may not be risk related. In the model presented here, the important aspect of the characteristics is not their risk profile, but the investor preferences for or against the characteristics, leading to the premiums in the marketplace.

Ibbotson and Idzorek (2014) and Idzorek and Ibbotson (2017) discuss what such a model might look like. They focus on characteristics of securities that they call "dimensions of popularity." The idea is that each security characteristic can be ranked along a popularity scale. For example, highly liquid stocks are regarded as popular, while less liquid stocks are regarded as unpopular. Investors who highly value popularity hold securities that rank high on the popularity scales, and are willing to give up return to do so. Investors who do not value popularity hold securities that rank low on the popularity scales and earn superior returns.

The key to understanding equilibrium pricing in securities is to recognize that securities have both risk and non-risk characteristics. In the PAPM, investors are risk averse and diversify, as in the CAPM, but they also vary in their preferences toward the other characteristics that securities embody. Securities supply the various characteristics, while investors demand them to varying degrees. Thus, the characteristics and ultimately the securities are priced according to the weighted average of investor preferences. The investors are positively weighted by their wealth, but negatively weighted by their risk aversion. Irrational preferences are not necessarily arbitraged away, since investors in the CAPM framework have a need to diversify to reduce their risk.

In this article, I present what is essentially a formal version of Ibbotson and Idzorek's theory. I do so by extending the CAPM to include security characteristics that different investors regard differently. This leads to an equilibrium in which:

1) The expected excess return on each security is a linear function of its beta and its *popularity loadings* which measure the popularity of the security based on its characteristics relative to the those of the beta-adjusted market portfolio.
2) Each investor holds a different portfolio based on his or her attitudes toward security characteristics.
3) Investor preferences determine the price of the securities.

The remainder of this article is organized as follows: In section 2, I review previous approaches to multi-factor models; namely Arbitrage Pricing Theory (APT) and NET. In section 3, I review the CAPM in detail to set the stage for the PAPM. In section 4, I present the PAPM in detail, building off the presentation of the CAPM in section 4. In section 5, I present a numerical example to illustrate the differences between the CAPM and the PAPM. Section 6 concludes.

2. Previous Multi-Factor Models

<u>APT</u>

Ross (1976), anticipating that there could be factors other than beta that command premiums, developed the multi-factor Arbitrage Pricing Theory (APT). In APT the return on each security is generated as a set of market-wide factors and a security-specific disturbance term. Under assumptions of complete markets and no-arbitrage, Ross shows that the expected excess return on each security is a linear combination of its factor exposures and a market-wide set of risk premiums.

While the APT is an elegant multi-factor model of risk premiums, it only explains premiums on risk factors, and does not allow for factors that have no apparent relationship to risk. Furthermore, it provides no theory of portfolio construction.

NET

Ibbotson, Diermeier, and Siegel (1984) present an asset pricing framework in which "[i]nvestors regard each asset as a bundle of characteristics [both risk and non-risk] for which they have various preferences and aversions. Investors translate each characteristic in to a cost, and require compensation in the form of expected returns for bearing these costs." In this way the factors that are priced are not limited to risk premia as in the APT, but can include non-risk characteristics such as liquidity.

Ibbotson, Diermeier, and Siegel sketch conceptual asset demand curves based on the notion that the investor-specific costs increase as the number of units of the asset held increases. However, they do not present a formal model of asset demand. I do so here with the PAPM.

3. Review of the CAPM

Since the PAPM is an extension of the CAPM, let us first review the CAPM. The CAPM makes the following assumptions:

1. Taxes, transaction costs, and other real world considerations can be ignored.
2. All investors use Mean-Variance Optimization (MVO) as described by Markowitz (1952, 1959, 1987) to select their portfolios.
3. All investors have the same forecasts; i.e., the same capital market assumptions (expected returns, standard deviations, and correlations.)
4. All investors can borrow and lend at the same risk-free rate without limit.

From these assumptions, the following conclusions emerge:

1. The market portfolio is on the efficient frontier
2. Each investor combines the market portfolio with the risk-free asset (long or short). Hence, investors do not actually need to perform MVO to construct optimal portfolios.
3. The expected excess return of each security is proportional to its systematic risk with respect to the market portfolio (beta).

To state assumption (2) formally, Let

n	= the number of risky securities in the market
$\vec{\mu}$	= the n-element vector of expected excess returns
$\boldsymbol{\Psi}$	= the $n\times n$ variance-covariance matrix of returns on the risky securities
$\vec{x}_i$	= the n-element vector of investor i's allocations to the risky securities[2]
λ_i	= the risk aversion parameter of investor i

Investor i's MVO problem is:

$$\max_{\vec{x}_i} \vec{\mu}'\vec{x}_i - \frac{\lambda_i}{2}\vec{x}_i{}'\boldsymbol{\Psi}\vec{x}_i \qquad (1)$$

[2] As stated in assumption (4), there is a risk-free security to which the investor allocates $1 - \sum_{j=1}^{n} x_{ij}$.

To state conclusion (2) formally, Let

m = the number of investors
w_i = the fraction of wealth held by investor i; $\sum_{i=1}^{m} w_i = 1$

Aggregating across investors, we have the market level of risk aversion and the market portfolio:

$$\lambda_M = \frac{1}{\sum_{i=1}^{m} \frac{w_i}{\lambda_i}} \tag{2}$$

$$\vec{x}_M = \sum_{i=1}^{m} w_i \vec{x}_i \tag{3}$$

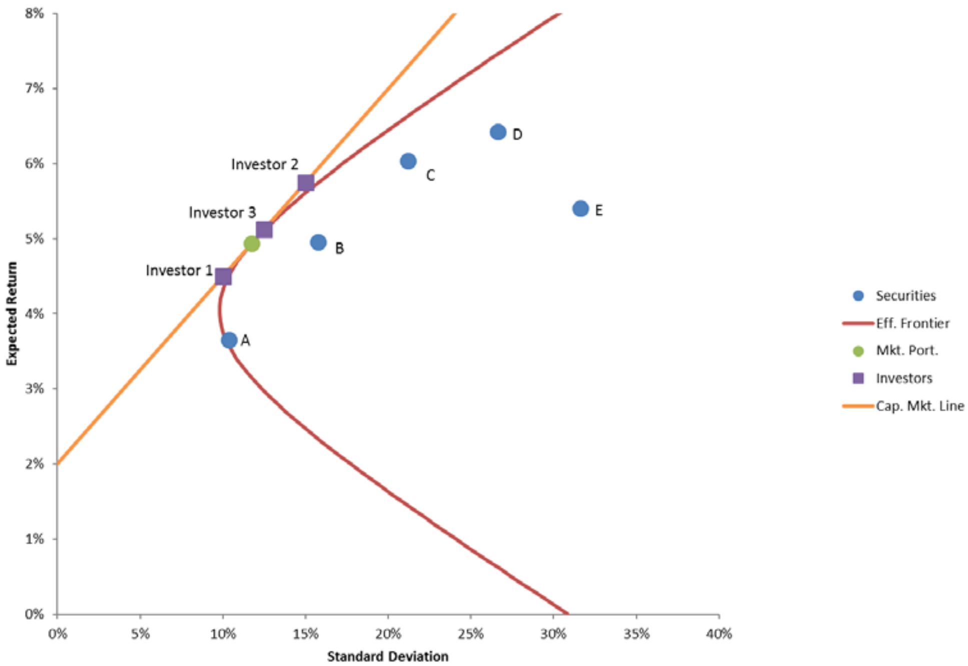
(The *M* subscript indicates aggregation to the market level.) As I show in Appendix A, each investor holds the market portfolio in proportion to the ratio of his risk tolerance to the wealth-weighted average risk tolerance:

$$\vec{x}_i = \frac{\lambda_M}{\lambda_i} \vec{x}_M \tag{4}$$

In the standard CAPM, the net supply of the risk-free asset is 0 so that $\sum_{j=1}^{n} x_{Mj}$=1. So, equation (4) tells us that if investor *i* is more risk tolerant than the average investor, he borrows at the risk-free rate and levers the market portfolio. Conversely, if investor *i* is less risk tolerant than the average investor, he holds a combination of cash the market portfolio.

Exhibit 1 illustrates conclusions (1) and (2) graphically. It shows that under the CAPM, the market portfolio is on the MVO efficient frontier. Its location is the point of tangency between the capital market line and efficient frontier. The capital market line is the line of tangency that emanates from the risk-free rate on the vertical axis. As the chart shows, not only is the market portfolio on the capital market line, but so are the portfolios of all investors. Investors who take more risk than the market portfolio have portfolios above it, indicating that they hold levered positions in it. Investors who take less risk than the market portfolio have portfolios below it, indicating that they hold delevered positions in it.

Exhibit 1: Equilibrium under the CAPM



To state conclusion (3) formally, define the expected excess return on the market portfolio as:

$$\mu_M = \vec{x}_M{}'\vec{\mu} \tag{5}$$

Define the variance of the market portfolio as:

$$\sigma_M^2 = \vec{x}_M{}'\,\boldsymbol{\Psi}\vec{x}_M \tag{6}$$

The familiar CAPM equation for expected excess returns can be written as:

$$\vec{\mu} = \vec{\beta}\mu_M \tag{7}$$

where

$$\vec{\beta} = \frac{\boldsymbol{\Psi}\vec{x}_M}{\sigma_M^2} \tag{8}$$

In other words, the expected excess return on each security is the product of its systematic risk with respect to the market portfolio (beta) and the expected excess return of the market portfolio.

Single Period Valuation under the CAPM

So far, I have given the conventional presentation of the CAPM as a model of portfolio construction and expected return. However, the CAPM is also a single period valuation model. Let

v_j = the market value of security *j*
$\tilde{y}_j$ = the exogenous random end-of-period total value of security *j*
r_f = the risk-free rate

The value of each security *j* can be written as:

$$v_j = \frac{E[\tilde{y}_j]}{1+r_f+\beta_j\mu_M} \tag{9}$$

The expected end-of-period value $E[\tilde{y}_j]$ is in a sense the "fundamental" of the security. If all securities had the same systematic risk (beta), all market values would be proportional to this fundamental. But they are not so that the market value of a security depends both on its fundamental and on its risk.

Another way to approach valuation is to risk-adjust the fundamental and discount it at the risk-free rate.

Let

$\tilde{y}_M$ = the random end-of-period value of the market as a whole
v_M = the value of the market as whole

Be definition:

$$v_M = \sum_{j=1}^{n} v_i \tag{10}$$

Let $\tilde{\vec{y}}$ denote the vector of random exogenous end-of-period total security values. The distribution of $\tilde{\vec{y}}$ constitutes the real economy. I denote the variance-covariance matrix of $\tilde{\vec{y}}$ as $\boldsymbol{\Omega}$. The systematic risk of $\tilde{y}_j$ with respect to total economic output ($\sum_{i=1}^{n} \tilde{y}_i$) is

$$\gamma_j = \frac{\sum_{i=1}^{n} \Omega_{ij}}{\sum_{i=1}^{n}\sum_{k=1}^{n} \Omega_{ik}} \tag{11}$$

The systematic risk of the fundamental γ_j, is related to the systematic risk of return β_j, as follows:

$$\gamma_j = x_j\beta_j \tag{12}$$

As I show in Appendix A, the value of security *j* can be expressed as:

$$v_j = \frac{E[\tilde{y}_j] - v_M\gamma_j\mu_M}{1+r_f} \tag{13}$$

4. The Popularity Asset Pricing Model

The PAPM is a generalization of the CAPM in which securities have characteristics other than risk and expected return that investors are concerned about. Its assumptions are:

1. Taxes, transaction costs, and other real world considerations can be ignored.[3]
2. Each security has a bundle of characteristics.
3. Investors have preferences regarding these characteristics in addition to their preferences regarding risk and expected return.
4. All investors use a generalized form of Mean-Variance Optimization (MVO) that incorporates their preferences regarding security characteristics.
5. All investors have the same forecasts; i.e., the same capital market assumptions (expected returns, standard deviations, and correlations.)
6. All investors agree on what the characteristics of the securities are.
7. All investors can borrow and lend at the same risk-free rate without limit.

The conclusions of the PAPM are:

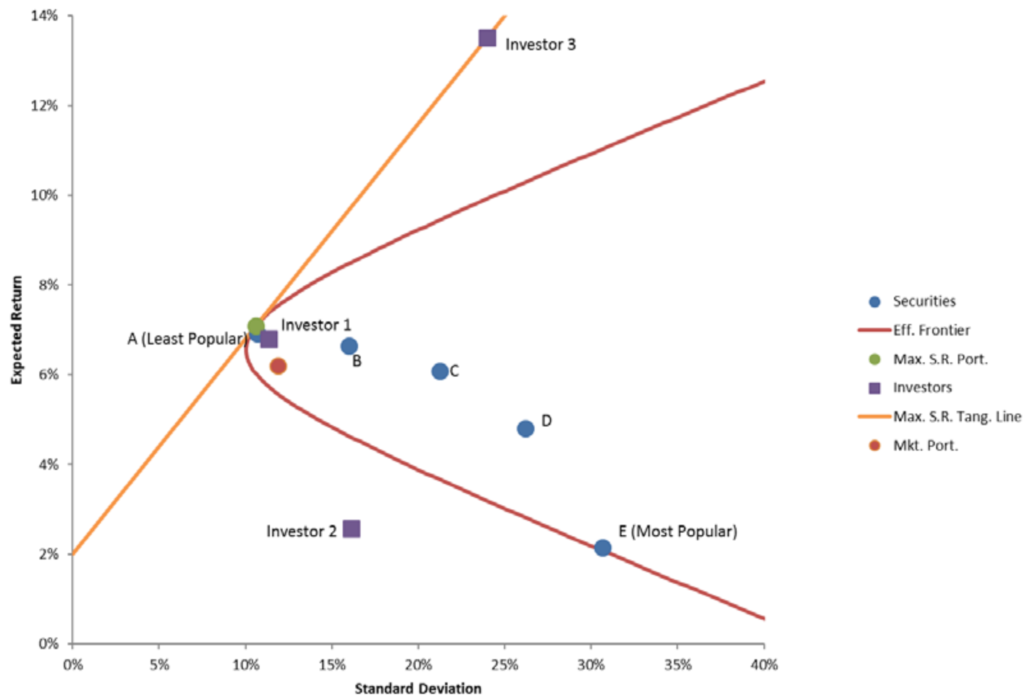
1. The market portfolio is <u>not</u> on the efficient frontier.
2. Each investor forms a customized portfolio of the risky assets that reflects his attitudes towards security characteristics. This portfolio is combined with the risk-free asset (long or short). Portfolio optimization is required to find the overall investor-specific portfolio.
3. The expected excess return of each security is a linear function of its beta and its popularity loadings which measure the popularity of the security based on its characteristics relative to the those of the beta-adjusted market portfolio. The popularity loadings are multiplied by the popularity premiums which are aggregations of the preferences of the investors regarding the characteristics. In this way, the market aggregates investor preferences in determining the influence of security characteristics on the expected returns and prices of the securities.

Note that the conclusions of the PAPM are nearly the exact opposite of those of the CAPM. Additionally, conclusion (2) is much more consistent with observed investor portfolios.

Exhibit 2 illustrates conclusions (1) and (2). As you see, the market portfolio is not on the efficient frontier. While there is a tangent line, neither the market portfolio nor all of the investor portfolios are on it, as is the case in the CAPM. However, one of the investor portfolios is on the tangent line. This is an investor who has no preference for security characteristics and therefore holds an efficient portfolio. I present the specifics of this example in section 5.

[3] While things like taxes *can be* ignored, a strength of the PAPM is that they could be easily incorporated as a characteristic.

Exhibit 2: **Equilibrium under the PAPM**



To state assumptions (2)-(4) formally, let

p = the number of characteristics (besides risk and expected excess return)
$\boldsymbol{C}$ = $n \times p$ matrix of characteristics of the securities
$\vec{\phi}_i$ = p-element vector of investor i's attitudes toward the characteristics
(The elements can be positive or negative.)

Investor i's problem is:

$$\max_{\vec{x}_i} \vec{\mu}'\vec{x}_i + \vec{\phi}_i{}'\boldsymbol{C}'\vec{x}_i - \frac{\lambda_i}{2}\vec{x}_i{}'\boldsymbol{\Psi}\vec{x}_i \qquad (14)$$

This extension of MVO is similar to the formulation in Cooper et al. (2016). The main difference is in interpretation. In Cooper et al. (2016), the non-risk characteristics are expected social impact metrics (ESG factors); whereas in the PAPM, the non-risk characteristics can include these, but can also include any number of other security characteristics that investors might care about.

Note that since the preferences for characteristics enter the utility function in parallel to expected returns, they should be in the same units. For example, if ϕ_{11}=5%, 100% exposure to a security with exposure of 1 to characteristic 1, investor 1 would be indifferent between a 100% exposure to a security with exposure of 1.0 to characteristic 1, and a 5% increase in expected return.

The solution to problem (14) is:

$$\vec{x}_i = \frac{1}{\lambda_i}\boldsymbol{\Psi}^{-1}\left(\vec{\mu} + \boldsymbol{C}\vec{\phi}_i\right) \qquad (15)$$

As I show in Appendix B, each investor's portfolio can be expressed in terms of the market portfolio and the investor's attitudes towards security characteristics:

$$\vec{x}_i = \frac{\lambda_M}{\lambda_i}\vec{x}_M + \frac{1}{\lambda_i}\boldsymbol{\Psi}^{-1}\boldsymbol{C}\left(\vec{\phi}_i - \vec{\pi}\right) \qquad (16)$$

where $\vec{\pi}$ denotes the vector of the aggregation of investor attitudes toward the characteristics:

$$\vec{\pi} = \lambda_M \sum_{i=1}^{m} \frac{w_i}{\lambda_i}\vec{\phi}_i \qquad (17)$$

For reasons that will become apparent below, I call $\vec{\pi}$ the vector *of popularity premiums*.

Equation (16) shows how each investor's portfolio differs from the market portfolio based on (1) the investor's attitude towards risk and (2) his attitude towards security characteristics.

In Appendix B, I show that expected excess returns can be written as:

$$\vec{\mu} = \vec{\beta}\mu_M + \left(\vec{\beta}\vec{c}_M' - \boldsymbol{C}\right)\vec{\pi} \qquad (18)$$

Equation (B.18) looks like multifactor asset pricing model, but with the popularity premiums rather than risk premiums. Let

$$\delta_{jk} = \beta_j c_{Mj} - C_{jk} \qquad (19)$$

so we can write:

$$\mu_j = \beta_j \mu_M + \sum_{k=1}^{p} \delta_{jk}\pi_j \qquad (20)$$

Separating out the risk-free rate, we can write equation (20) as an equation for the expected total return of security *j*:

$$E\left[\tilde{r}_j\right] = r_f + \beta_j \mu_M + \sum_{k=1}^{p} \delta_{jk}\pi_j \qquad (21)$$

Equation (21) is of the same form as the equation that Idzorek and Ibbotson (2017) postulated would hold in an equilibrium in which investors care about non-risk characteristics.

I call δ_{jk} the security *j*'s *popularity loading* on a characteristic *k*. It is positive if its security *j*'s exposure to characteristic *k* is less than that of the beta-adjusted market portfolio and negative if the reverse is true. In this way, a popularity loading of a security is positive for a given characteristic if the security is unpopular with respect to the characteristic and negative if it is popular.

As a special case, it is possible that the net attitude towards characteristics is zero ($\vec{\pi}$=0) so that the CAPM equation for expected excess returns still prevails and the market portfolio is mean-variance efficient. But even in that case, each investor still tilts his portfolio towards the characteristics that he likes and away from the ones that he dislikes as described by equation (16).

<u>Valuation under the PAPM</u>
Just as the equation for expected excess returns in the CAPM can be used to derive a one-period valuation formula (equation 13), in similar fashion, equation (18) can be used to derive a valuation formula for the PAPM.

In Appendix B, I derived the following PAPM valuation formula:

$$v_j = \frac{E[\tilde{y}_j] - \gamma_j v_M(\mu_M + \vec{c}_M{}'\vec{\pi})}{1 + r_f - \vec{c}_j{}'\vec{\pi}} \qquad (22)$$

What is interesting about this formula is while the fundamental is risk-adjusted; the discount rate is adjusted for security-specific characteristics in proportion to aggregated preferences for them; i.e., the popularity premiums Hence, market values of securities depend in part on the characteristics that they possess and how much investors care about them.

5. A Numerical Example

I created Exhibits 1 and 2 using an example with five securities and three investors. Exhibit 3 presents the assumptions regarding the joint distribution of the end-of-period value five securities. This is the model of the real economy.

To keep the example simple, I assume that there is one characteristic that investors care about that I also show on this exhibit.

Exhibit 3: Assumptions on Popularity and the Real Economy

Security	Popularity	Expected Value ($)	Standard Deviation ($)	Correlation of End-of-Period Value with				
				A	B	C	D	E
A	-0.50	10	1.0	1.0				
B	-0.25	8	1.2	0.4	1.0			
C	0.00	6	1.2	0.3	0.4	1.0		
D	0.25	4	1.0	0.2	0.3	0.4	1.0	
E	0.50	1	0.3	0.1	0.2	0.3	0.4	1.0

Exhibit 4 presents the assumptions regarding the investors. Note that investor 3 has a 0-popularity preference. This is the investor in Exhibit 2 whose portfolio is on the tangent line.

Exhibit 4: Assumptions Regarding Investors

	Investor 1	Investor 2	Investor 3	Market
Fraction of Market $ Wealth (w_j)	60%	30%	10%	100%
Risk Aversion (λ_i)	2.50	1.67	2.00	2.13
Popularity Preference (ϕ_i)	5.00%	10.00%	0.00%	6.38%

In addition to the assumptions presented in Exhibits 3 and 4, I assume that the risk-free rate is 2%.

I solved the model under the assumptions of the CAPM (all Popularity preferences set to 0) and under the assumptions of the PAPM using the techniques described in Appendices A and B respectively. Exhibits 5 and 6 present the results.

Exhibit 5: Expected Returns and Valuations under the CAPM and the PAPM

Security	Expected Return		Value ($)	
	CAPM	PAPM	CAPM	PAPM
A	3.65%	6.92%	9.65	9.35
B	4.96%	6.64%	7.62	7.50
C	6.03%	6.08%	5.66	5.66
D	6.42%	4.81%	3.76	3.82
E	5.41%	2.15%	0.95	0.98
Market	4.94%	6.20%	27.64	27.31

Exhibit 6: Investor and Market Portfolios under the CAPM and the PAPM

Security/ Statistic	Investor 1		Investor 2		Investor 3		Market	
	CAPM	PAPM	CAPM	PAPM	CAPM	PAPM	CAPM	PAPM
A	29.71%	55.57%	44.57%	-59.92%	37.14%	188.86%	34.91%	34.25%
B	23.47%	25.12%	35.21%	28.24%	29.34%	39.27%	27.58%	27.47%
C	17.43%	15.55%	26.14%	34.59%	21.78%	10.05%	20.48%	20.58%
D	11.57%	9.37%	17.36%	27.76%	14.47%	6.31%	13.60%	13.98%
E	2.92%	0.30%	4.38%	15.37%	3.65%	-12.07%	3.43%	3.59%
Cash	14.89%	-5.91%	-27.66%	53.95%	-6.38%	-126.40%	0.00%	0.00%
Exp. Ret.	4.50%	6.80%	5.75%	2.58%	5.12%	13.51%	4.94%	6.20%
Std. Dev.	10.00%	11.34%	15.00%	16.13%	12.50%	23.98%	11.75%	11.89%

Exhibit 7: Maximum Sharpe Ratio Portfolios under the CAPM and the PAPM

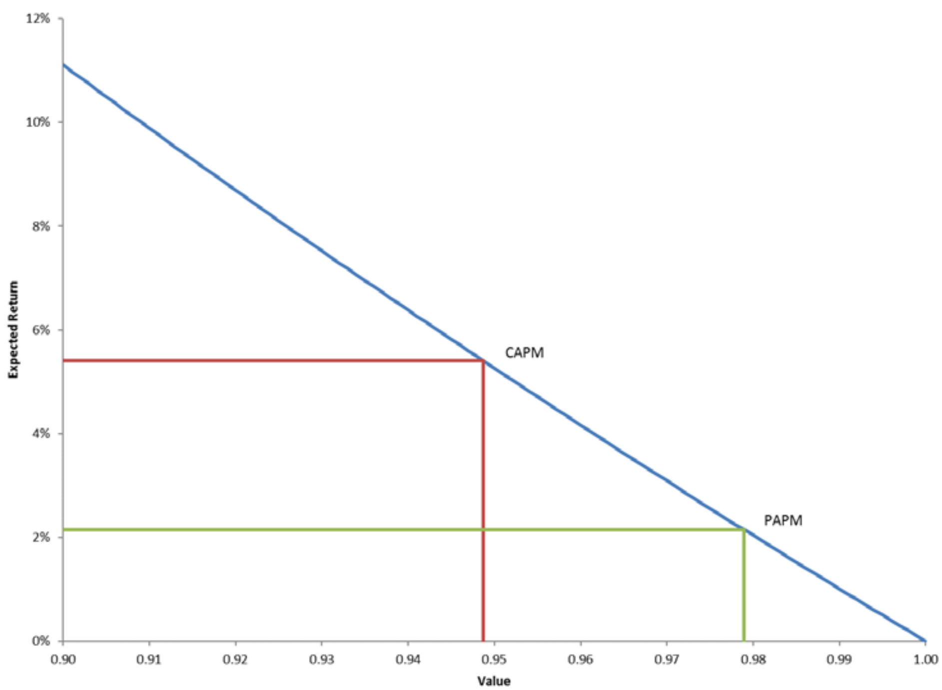
Security/ Statistic	**CAPM**	**PAPM**
A	34.91%	83.42%
B	27.58%	17.35%
C	20.48%	4.44%
D	13.60%	0.12%
E	3.43%	-5.33%
Exp. Ret.	4.94%	7.08%
Std. Dev.	11.75%	10.59%
Sharpe R.	0.25	0.48

There are two striking features of these results. First, from Exhibit 5, we see that while values of the securities are similar in both models, the expected returns are quite different. This is because the expected returns are highly sensitive to changes in value due to their inversely proportional relationship:

$$\mu_j + r_f = \frac{E[\tilde{y}_j]}{v_j} - 1 \tag{23}$$

Since $v_j < E[\tilde{y}_j]$, there the sensitivity is high. Exhibit 8 illustrates this by showing how the expected return of security E changes with changes to its value. (The curve appears to be linear rather than hyperbolic because the exhibit plots a very small part of the curve.)

Exhibit 8: Relationship between the Value and Expected Return of Security E



Second is the radical change that investor 3 undergoes from the CAPM world to the PAPM world. As you can see from Exhibit 4, investor 3 pays no attention to the non-risk characteristic and therefore focuses entirely on risk and expected return. This is why investor 3's portfolio is on the tangent line in Exhibit 2. However, whereas under the CAPM, investor 3 holds a slightly leveraged portfolio, under the PAPM, he holds a highly-levered position in security A, which is the least popular security. To see why investor 3 behaves this way, first, consider the fact that since investor 3 pays no attention to popularity, he holds a portfolio that maximizes the Sharpe ratio. This is the tangency portfolio in Exhibit 2. As Exhibit 7 shows, under the PAPM, the portfolio that maximizes the Sharpe ratio is over 83% in security A. This is because investors 1 and 2's dislike for security A drives its price down, leading to a have a high expected value and a prominent place in the portfolio that maximizes the Sharpe ratio and thus in investor 3's portfolio.

To see why investor 3's portfolio is levered so heavily, consider the problem of allocating between the tangency portfolio (which is the portfolio that maximizes the Sharpe) and cash. Let:

λ = the risk aversion of the investor
μ_T = the expected excess return of the tangency portfolio
σ_T^2 = the variance of the return on the tangency portfolio
x_T = the allocation to the tangency portfolio

The problem is:

$$\max_{x_T} \mu_T x_T - \tfrac{\lambda}{2}\sigma_T^2 x_T^2 \tag{24}$$

The solution is:

$$x_T = \frac{\mu_T}{\lambda\sigma_T^2} \tag{25}$$

Hence, in order to hold the tangency portfolio without cash or leverage (x_T=1), the investor would have to have a risk aversion of

$$\lambda_T = \frac{\mu_T}{\sigma_T^2} \tag{26}$$

Under the CAPM, this is the same the weighted average risk aversion, λ_M as defined in equation (2). From Exhibit 4, we see that in this example, this is 2.13. Since λ_3, being 2.00, is a bit less than λ_M, under the CAPM, investor 3 is a bit levered.

In contrast, under the PAPM, the tangency portfolio is not the market portfolio and therefore, the value of λ_T need not be λ_M. In fact, in this example λ_T is 4.53 which is significantly greater than λ_3. This is why under the PAPM, investor 3 holds a highly-levered portfolio.

This example illustrates how in the PAPM, where investors hold custom portfolios based on how much they weigh (or do not weigh) popularity characteristics, investors can be thought of as forming clienteles for the dimensions of popularity.

Finally, Exhibit 9 compares how the equations for expected returns under the CAPM and under the PAPM work. Note how the two least popular securities, namely, A and B have positive popularity loadings and thus can be categorized as unpopular securities. Also, note how the three most popular securities, namely, C, D and E have negative popularity loadings, and this can be categorized as popular securities. Also, note how under the PAPM, differences in popularity loadings can lead to much larger differences in expected returns than differences in betas. For example, security A has much higher expected return than security E (6.92% vs. 2.15%) even though security A has a much lower beta than security E (0.57 vs. 1.11). This is consistent with many empirical findings that differences in betas do not explain differences in returns, and demonstrates how the theory of popularity can explain this phenomenon.

Exhibit 9: Expected Return Equations under the CAPM and the PAPM

Security	**CAPM** $E[\tilde{r}_j] = r_f + \beta_j \mu_M = 2.00\% + \beta_j 2.94\%$		**PAPM** $E[\tilde{r}_j] = r_f + \beta_j \mu_M + \delta_j \pi = 2.00\% + \beta_j 4.20\% + \delta_j 6.38\%$		
	Beta	**Expected Return**	**Beta**	**Popularity Loading**	**Expected Return**
A	0.56	3.65%	0.57	0.39	6.92%
B	1.01	4.96%	1.01	0.06	6.64%
C	1.37	6.03%	1.36	-0.25	6.08%
D	1.51	6.42%	1.47	-0.52	4.81%
E	1.16	5.41%	1.11	-0.71	2.15%
Market	1.00	4.94%	1.00	0.00	6.20%

6. Conclusion

Equilibrium asset pricing models such as the CAPM and the APT predict that expected returns on securities are linear functions of systematic risk factors. However, there is a large body of empirical evidence that suggests that there are premiums related to characteristics that are not related to risk. Hence, there is a need for a new equilibrium theory that takes non-risk characteristics into account. In this article, I present such a theory which I call the Popularity Asset Pricing Model or PAPM for short. The PAPM formally derives the equation for expected return that Idzorek and Ibbotson (2017) postulate in their theory of popularity.

I form the PAPM by extending the CAPM to include preferences for non-risk security characteristics into investor objective functions. In the PAPM, an equilibrium emerges in which the expected excess return of each security is a linear function of its systematic risk with respect to the market portfolio (beta) and its popularity loadings, which measure the popularity of the security based on its characteristics relative to the those of the beta-adjusted market portfolio. As I illustrate, differences in popularity loadings can cause greater differences in expected returns than differences in betas. The coefficients on the popularity loadings, the popularity premiums, are the aggregated attitudes of investors towards the non-risk security characteristics. Furthermore, the market portfolio is not on the efficient frontier as it is merely the aggregation of the portfolios of each investor's customized portfolio.

These conclusions have important practical implications. Firstly, when estimating the equity cost of capital, adjustments need to be made for the security in question's characteristics. Secondly, they imply that it may be possible to create portfolios that are more efficient than market weighted indexes by focusing on risk and expected return only, in contrast to what investors do in this model; namely, taking into account security characteristics other than risk.

The approach that I take in constructing the PAPM can be extended to take into account other types of heterogeneity among investors. For example, investors could have heterogeneous views on the expected value of the real economic output associated with each security, much like in Lintner (1969). In general, it should be possible to derive the equilibrium of a model with investors who are heterogeneous in different respects.

Appendix A: Formal Presentation of the CAPM

The Setup of the CAPM
Investor *i*'s problem is:

$$\max_{\vec{x}_i} \vec{\mu}'\vec{x}_i - \frac{\lambda_i}{2}\vec{x}_i{}'\boldsymbol{\Psi}\vec{x}_i \tag{A.1}$$

From the first-order condition, we have:

$$\vec{\mu} = \lambda_i \boldsymbol{\Psi}\vec{x}_i \tag{A.2}$$

Solving for $\vec{x}_i$, we have:

$$\vec{x}_i = \frac{1}{\lambda_i}\boldsymbol{\Psi}^{-1}\vec{\mu} \tag{A.3}$$

Let
m = the number of investors
w_i = the fraction of wealth held by investor *i*; $\sum_{i=1}^{m} w_i = 1$

Aggregating across investors, we have the market level of risk aversion and the market portfolio:

$$\lambda_M = \frac{1}{\sum_{i=1}^{m}\frac{w_i}{\lambda_i}} \tag{A.4}$$

$$\vec{x}_M = \sum_{i=1}^{m} w_i\vec{x}_i \tag{A.5}$$

Aggregating (A.3) across investors, we have:

$$\vec{x}_M = \frac{1}{\lambda_M}\boldsymbol{\Psi}^{-1}\vec{\mu} \tag{A.6}$$

So that

$$\vec{\mu} = \lambda_M \boldsymbol{\Psi}\vec{x}_M \tag{A.7}$$

From equations (A.3) and (A.6), we can see that each investor hold the market portfolio in proportion to the ratio of the wealth-weighted average risk aversion to his risk aversion:

$$\vec{x}_i = \frac{\lambda_M}{\lambda_i}\vec{x}_M \tag{A.8}$$

In the standard CAPM, the net supply of the risk-free asset is 0 so that $\sum_{j=1}^{n} x_{Mj}$=1. So, equation (A.8) tells us that if investor *i* is less risk averse that the average investor, he borrows at the risk-free rate and levers the market portfolio. Conversely, if investor *i* is more risk averse that the average investor, he holds a combination of cash the market portfolio.

Expected Excess Returns under the CAPM

The expected excess return on the market portfolio is:

$$\mu_M = \vec{x}_M{}'\vec{\mu} \qquad \text{(A.9)}$$

Hence, multiplying equation (A.7) through by $\vec{x}_M$, yields:

$$\mu_M = \lambda_M \sigma_M^2 \qquad \text{(A.10)}$$

where $\sigma_M^2 = \vec{x}_M{}'\boldsymbol{\Psi}\vec{x}_M$. This is the variance of the market portfolio.

From equation (A.10), it follows that:

$$\lambda_M = \frac{\mu_M}{\sigma_M^2} \qquad \text{(A.11)}$$

Substituting the right-hand side of equation (A.11) for λ_M in equation (A.7), and rearranging terms, yields the familiar CAPM equation for expected excess returns:

$$\vec{\mu} = \vec{\beta}\mu_M \qquad \text{(A.12)}$$

where

$$\vec{\beta} = \frac{\boldsymbol{\Psi}\vec{x}_M}{\sigma_M^2} \qquad \text{(A.13)}$$

Valuation under the CAPM

Since CAPM is a one-period model, the value of each security *j* can be written as:

$$v_j = \frac{E[\tilde{y}_j]}{1+r_f+\beta_j\mu_M} \qquad \text{(A.14)}$$

where

v_j = the total market value of security *j*
$\tilde{y}_j$ = the random exogenous end-of-period total value of security *j*
r_f = the risk-free rate

Let

$\tilde{y}_M$ = the random end-of-period value of the market as a whole
v_M = the value of the market as whole

Be definition:

$$\tilde{y}_M = \sum_{j=1}^{n} \tilde{y}_j \tag{A.15}$$
$$v_M = \sum_{j=1}^{n} v_i \tag{A.16}$$
$$x_j = \frac{v_j}{v_M} \tag{A.17}$$

The realized total return on security *j* is:

$$\tilde{r}_j = \frac{\tilde{y}_j}{v_j} - 1 \tag{A.18}$$

Let $\tilde{\bar{y}}$ denote the vector of random end-of-period total security values. The distribution of $\tilde{\bar{y}}$ constitutes the real economy. I denote the variance-covariance matrix of $\tilde{\bar{y}}$ as $\boldsymbol{\Omega}$. From the definition of $\boldsymbol{\Omega}$ and equation (A.15), it follows that the *jq* element of the variance-covariance matrix of returns, $\boldsymbol{\Psi}$, can be written as:

$$\boldsymbol{\Psi}_{jq} = \frac{\boldsymbol{\Omega}_{jq}}{v_j v_q} \tag{A.19}$$

So the formula for β_j can be rewritten as follows:

$$\beta_j = \frac{\sum_{i=1}^{n} \Psi_{ij} x_i}{\sum_{i=1}^{n}\sum_{k=1}^{n} \Psi_{ik} x_i x_k} = \frac{{}^{1}\!/_{v_j}}{{}^{1}\!/_{v_M}} \frac{\sum_{i=1}^{n} \Omega_{ij}}{\sum_{i=1}^{n}\sum_{k=1}^{n} \Omega_{ik}} = \frac{\gamma_j}{x_j} \tag{A.20}$$

where

$$\gamma_j = \frac{\sum_{q=1}^{n} \Omega_{jq}}{\sum_{q=1}^{n}\sum_{s=1}^{n} \Omega_{qs}} \tag{A.21}$$

Substituting the final term in equation in equation (A.18) for β_j in equation (A.12), rearranging terms and simplifying yields the following equation for the total value of security *j*:

$$v_j = \frac{E[\tilde{y}_j] - v_M \gamma_j \mu_M}{1 + r_f} \tag{A.22}$$

The value of the market as a whole is:

$$v_M = \frac{E[\tilde{y}_M]}{1 + \mu_M + r_f} \tag{A.23}$$

Substituting the right-hand side of equation (A.23) for v_M in equation (A.22) yields:

$$v_j = \frac{E[\tilde{y}_j] - E[\tilde{y}_M]\gamma_j \frac{\mu_M}{1 + \mu_M + r_f}}{1 + r_f} \tag{A.24}$$

Solving the CAPM

Equation (A.24) stated the values of the risky securities in terms of the underlying economic variables ($\tilde{\vec{y}}$), the market risk premium (μ_M), and the risk-free rate (r_f). From these values, I can derive all of the other variables in the CAPM using the earlier equations. Since I take the risk-free rate as given, I still need to solve for the market risk premium.

The market risk premium depends on the market risk aversion. To see exactly how, first consider what equation (A.21) implies about σ_M^2:

$$\sigma_M^2 = \sum_{i=1}^n \sum_{j=1}^n x_{Mi} x_{Mj} \Psi_{ij} = \sum_{i=1}^n \sum_{j=1}^n \frac{\Omega_{ij}}{v_M^2} = \frac{Var[\tilde{y}_M]}{v_M^2} \qquad \text{(A.25)}$$

Substituting the right-hand side of equation (A.23) for v_M in equation (A.25) yields:

$$\sigma_M^2 = \frac{\left(1+r_f+\mu_M\right)^2}{\kappa} \qquad \text{(A.26)}$$

where

$$\kappa = \frac{E[\tilde{y}_M]^2}{Var[\tilde{y}_M]} \qquad \text{(A.27)}$$

Substituting the right-had side of equation (A.10) for σ_M^2 in equation (A.26) and rearranging terms yields a quadratic equation for μ_M:

$$\mu_M^2 + \left(2\left(1+r_f\right) - \frac{\kappa}{\lambda_M}\right)\mu_M + \left(1+r_f\right)^2 = 0 \qquad \text{(A.28)}$$

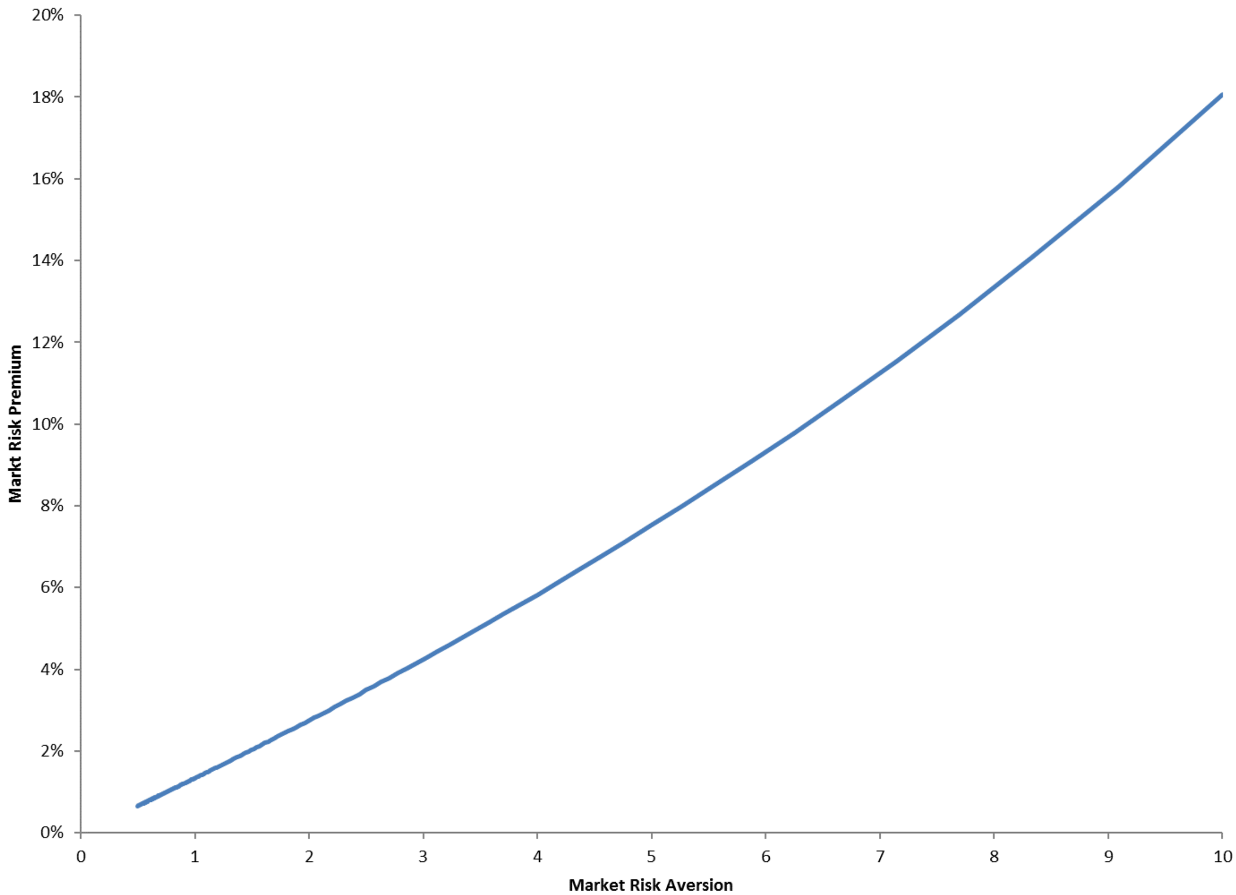
The relevant solution is:

$$\mu_M = \frac{\kappa\lambda_M - 2\left(1+r_f\right) - \sqrt{\left(2\left(1+r_f\right) - \frac{\kappa}{\lambda_M}\right)^2 - 4\left(1+r_f\right)^2}}{2} \qquad \text{(A.29)}$$

This is in effect the solution to the CAPM.

Exhibit A.1 is a plot showing the relationship between market risk aversion (λ_M) and the market risk premium (μ_M), taking the risk-free rate (r_f) and the parameters of the distribution of total market end-of-period value (κ) as given. The relationship is positive so that the higher the higher the level of market risk aversion, the higher the market risk premium.

Exhibit A.1: Relationship between Market Risk Tolerance and the Market Risk Premium under the CAPM



Appendix B: Formal Presentation of Popularity Asset Pricing Model (PAPM)

Setup of the PAPM

Let

p = the number of popularity characteristics

$\boldsymbol{C}$ = $n \times p$ matrix of characteristics of the securities

$\vec{\phi}_i$ = p-element vector of investor *i*'s attitudes toward the characteristics (The elements can be positive or negative.)

Investor *i*'s problem is:

$$\max_{\vec{x}_i} \vec{\mu}'\vec{x}_i + \vec{\phi}_i{}'\boldsymbol{C}'\vec{x}_i - \frac{\lambda_i}{2}\vec{x}_i{}'\boldsymbol{\Psi}\vec{x}_i \quad \text{(B.1)}$$

From the first-order condition, we have:

$$\vec{\mu} = \lambda_i \boldsymbol{\Psi}\vec{x}_i - \boldsymbol{C}\vec{\phi}_i \quad \text{(B.2)}$$

The solution is:

$$\vec{x}_i = \frac{1}{\lambda_i}\boldsymbol{\Psi}^{-1}\left(\vec{\mu} + \boldsymbol{C}\vec{\phi}_i\right) \quad \text{(B.3)}$$

Aggregating equation (B.3) across investors, we have:

$$\vec{x}_M = \frac{1}{\lambda_M}\boldsymbol{\Psi}^{-1}(\vec{\mu} + \boldsymbol{C}\vec{\boldsymbol{\pi}}) \quad \text{(B.4)}$$

where

$$\vec{\pi} = \lambda_M \sum_{i=1}^{m} \frac{w_i}{\lambda_i}\vec{\phi}_i \quad \text{(B.5)}$$

For reasons that will become apparent below, I call $\vec{\pi}$ the vector *of popularity premiums*.

From equations (B.3) and (B.4), I derive an equation for the portfolio decision of each investor relative to the market portfolio:

$$\vec{x}_i = \frac{\lambda_M}{\lambda_i}\vec{x}_M + \frac{1}{\lambda_i}\boldsymbol{\Psi}^{-1}\boldsymbol{C}\left(\vec{\phi}_i - \vec{\pi}\right) \quad \text{(B.6)}$$

Solving equation (B.4) for $\vec{\mu}$ yields:

$$\vec{\mu} = \lambda_M \boldsymbol{\Psi}\vec{x}_M - \boldsymbol{C}\vec{\pi} \quad \text{(B.7)}$$

Multiplying equation (B.7) through by $\vec{x}_M'$ yields:

$$\mu_M = \lambda_M \sigma_M^2 - \vec{c}_M{}'\vec{\pi} \quad \text{(B.8)}$$

where $\vec{c}_M = \boldsymbol{C}'\vec{x}_M$. This is the vector of characteristics of the market portfolio.

From equation (B.8), it follows that:

$$\lambda_M = \frac{\mu_M + \vec{c}_M{}'\vec{\pi}}{\sigma_M^2} \tag{B.9}$$

Substituting the right-hand side of equation (B.9) for λ_M in equation (B.7), and rearranging terms, yields the generalization of the CAPM equation for expected excess returns:

$$\vec{\mu} = \vec{\beta}\mu_M + \left(\vec{\beta}\vec{c}_M' - \boldsymbol{C}\right)\vec{\pi} \tag{B.10}$$

Equation (B.10) looks like multifactor asset pricing model, but with the popularity premiums rather than risk premiums. Let

$$\delta_{jk} = \beta_j c_{Mj} - C_{jk} \tag{B.11}$$

so we can write:

$$\mu_j = \beta_j \mu_M + \sum_{k=1}^{p} \delta_{jk}\pi_j \tag{B.12}$$

I call δ_{jk} the security *j*'s *popularity loading* on a characteristic *k*. It is positive if its security *j*'s exposure to characteristic *k* is less than that of the beta-adjusted market portfolio and negative if the reverse is true. In this way, a popularity loading of a security is positive for a given characteristic if the security is unpopular with respect to the characteristic and negative if it is popular.

As a special case, it is possible that the net attitude towards characteristics is zero ($\vec{\pi}$=0) so that the CAPM equation for expected excess returns still prevails and the market portfolio is mean-variance efficient. But even in that case, each investor still tilts his portfolio towards the characteristics that he likes and away from the ones that he dislikes as described by equation (B.6).

Valuation in the PAPM

Just as the equation for expected excess returns in the CAPM can be used to derive a one-period valuation formula (equation A.20), in similar fashion, equation (B.10) can be used to derive a valuation formula in NET2.

To do this, first I write (B.10) for a single security as follows:

$$\mu_j = \beta_j(\mu_M + \vec{c}_M'\vec{\pi}) - \vec{c}_j'\vec{\pi} \tag{B.13}$$

where $\vec{c}_j$ is the vector formed from the j^{th} row of $\boldsymbol{C}$.

Equations (A.17) and (A.20) hold under the PAPM just as they do for the CAPM. From them, we have:

$$\beta_j = \frac{\gamma_j}{x_j} = \frac{\gamma_j v_M}{v_j} \qquad \text{(B.14)}$$

where γ_j is as defined in equation (A.21).

The value of security *j* can be written as:

$$v_i = \frac{E[\tilde{y}_i]}{1+r_f+\mu_i} \qquad \text{(B.15)}$$

Substituting the right-hand side of equation (B.13) for μ_j in equation (B.15), rearranging term, and solving for v_j yields the valuation equation presented as equation (22) in the main text:

$$v_j = \frac{E[\tilde{y}_j]-\gamma_j v_M(\mu_M+\vec{c}_M'\vec{\pi})}{1+r_f-\vec{c}_j'{}'\vec{\pi}} \qquad \text{(B.16)}$$

<u>Solving the PAPM</u>

Unlike the CAPM, the PAPM has no closed form solution. Instead, to solve it, we need to solve a system of nonlinear equations.

Let $\vec{f}(.)$ denote the *n*-element vector-valued function that we are seeking to set to $\vec{0}$ by finding the value of the vector of security values, $\vec{v}$, that does so. That is, we seek the solution to:

$$\vec{f}(\vec{v}) = \vec{0} \qquad \text{(B.17)}$$

Once the solution is found, the all of the values of the variables of the model can be derived from the values of $\vec{v}$ and the above equations in this appendix.

The values of $\vec{f}(\vec{v})$ are determined by making the follows set of calculations:

$$\vec{x}_{M1} = \frac{\vec{v}}{\sum_{j=1}^{n} v_j} \qquad \text{(B.18)}$$

$$\Psi^{-1} = diag(\vec{v})\Omega^{-1}diag(\vec{v}) \qquad \text{(B.19)}$$

$$\mu_j = \frac{E[\tilde{y}_j]}{v_j} - (1+r_f), \quad \text{for } j = 1,2\ldots,n \qquad \text{(B.20)}$$

$$\vec{x}_{M2} = \frac{1}{\lambda_M}\,\Psi^{-1}(\vec{\mu} + \boldsymbol{C\vec{\pi}}) \qquad \text{(B.21)}$$

$$\vec{f}(\vec{v}) = \vec{x}_{M1} - \vec{x}_{M2} \qquad \text{(B.22)}$$

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