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Abstract

This paper analyzes a class of singular control problems for which value functions are not necessarily smooth. Necessary and sufficient conditions for the well-known smooth fit principle, along with the regularity of the value functions, are given. Explicit solutions for the optimal policy and for the value functions are provided. In particular, when payoff functions satisfy the usual Inada conditions, the boundaries between action and no-action regions are smooth and strictly monotonic as postulated and exploited in the existing literature (Dixit and Pindyck (1994); Davis, Dempster, Sethi, and Vermes (1987); Kobila (1993); Abel and Eberly (1997); Øksendal (2000); Scheinkman and Zariphopoulou (2001); Merhi and Zervos (2007); Alvarez (2006)). Illustrative examples for both smooth and non-smooth cases are discussed, to highlight the pitfall of solving singular control problems with *a priori* smoothness assumptions.

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1 Introduction

Consider the following problem in reversible investment/capacity planning that arises naturally in resource extraction and power generation. Facing the risk of market uncertainty, companies extract resources (such as oil or gas) and choose the capacity level in response to the random fluctuation of market price for the resources, subject to some capacity constraints, as well as the associated costs for capacity expansion and contraction. The goal of the company is to maximize its long-term profit, subject to these constraints and the rate of resource extraction.

This kind of capacity planning with price uncertainty and partial (or no) reversibility originated from the economics literature and has since attracted the interest of the applied mathematics community. (See Dixit and Pindyck (1994); Brekke and Øksendal (1994); Davis, Dempster, Sethi, and Vermes (1987); Kobila (1993); Abel and Eberly (1997); Baldursson and Karatzas (1997); Øksendal (2000); Scheinkman and Zariphopoulou (2001); Wang (2003); Chiarolla and Haussmann (2005); Guo and Pham (2005) and the references therein.) Mathematical analysis of such control problems has evolved considerably from the initial heuristics to the more sophisticated and standard stochastic control approach, and from the very special case study to general payoff functions. (See Harrison and Taksar (1983); Karatzas (1985); Karatzas and Shreve (1985); El Karoui and Karatzas (1988, 1989); Ma (1992); Davis and Zervos (1994, 1998); Boetius and Kohlmann (1998); Alvarez (2000, 2001); Bank (2005); Boetius (2005)). Most recently, Merhi and Zervos (2007) analyzed this problem in great generality and provided explicit solutions for the *special case* where the payoff is of Cobb-Douglas type. Their method is to directly solve the HJB equations, assuming certain regularity conditions for both the value function and the boundaries between the action and no-action regions. Guo and Tomecek (2007) later established sufficient conditions for the smoothness of the value function by connecting the singular control problem with a collection of optimal switching problems.

However, for many singular control problems in reversible investment and in areas such as queuing and wireless communications (Martins, Shreve, and Soner (1996); Assaf (1997); Harrison and Van Mieghem (1997); Ata, Harrison, and Shepp (2005)), there is no regularity for *either the value function or the boundaries*. Therefore, two important mathematical issues remain: 1) necessary conditions for regularity properties; and 2) characterization for the value function and for the action and no-action regions when these regularity conditions fail. Understanding these issues is especially important in cases where only numerical solutions are available, and for which the assumption on the degree of the smoothness is *wrong* (see also discussions in Section 5.2).

This paper addresses these two issues via the study of a class of singular control problems. *Both necessary and sufficient conditions* on the differentiability of the value function and on the smooth fit principle are established. Moreover, these conditions lead to a derivative-based characterization of the investment, disinvestment and continuation regions even for non-smooth value functions. In fact, *when the payoff function is not smooth*, this paper is the first to rigorously characterize the action and no-action regions, and to explicitly construct both the optimal policy and the value function. To be consistent with the literature in (ir)reversible investment, the running payoff function in this paper depends on the resource extraction rate and the market price in the form of $H(Y)X^{\lambda}$. It is worth noting that $H(\cdot)$ is *any concave function* of the capacity, and may be *neither monotonic nor differentiable.* This includes the special cases investigated by Guo and Pham (2005); Merhi and Zervos (2007); Guo and Tomecek (2007). In particular, when H satisifies the well-known Inada conditions (i.e., continuously differentiable, strictly increasing, strictly concave, with $H(0) = 0$, $H'(0^+) = \infty$, $H'(\infty) = 0$), our results show that the boundaries between regions are indeed continuous and strictly increasing as postulated and exploited in previous works: Dixit and Pindyck (1994); Davis, Dempster, Sethi, and Vermes (1987); Kobila (1993); Abel and Eberly (1997); Øksendal (2000); Scheinkman and Zariphopoulou (2001); Merhi and Zervos (2007); Alvarez (2006). Also note that our method can be applied to more general (diffusion) processes for the price dynamics, other than the geometric Brownian motion assumed for explicitness in this paper. Finally, the construction between the functional form of the boundaries and the payoff function itself is also novel, as the value function and the boundaries may be neither smooth nor strictly monotonic as in the existing literature.

The most relevant and recent work to this paper is Alvarez (2006), which provides a great deal of economic insight into the problem. However, Alvarez (2006) only handles payoff functions satisfying the Inada conditions. In contrast, our solution is *independent of* the regularity of the payoff and value functions.

Outline. The control problem is formally stated, with its value function and optimal policy described in Section 2; details of the derivation are in Section 3. The main result of this paper regarding the regularity of the value function is in Section 4. Examples are provided in Section 5, including cases for which the value function is not differentiable, the optimal controlled process not continuous, the boundaries of the action regions not smooth, and the interior of the continuation region not simply connected.

2 Mathematical Problem and Solution

2.1 Problem

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and assume a given bounded interval $[a, b] \subset (-\infty, \infty)$. Consider the following problem:

Problem A.

$$V_H(x, y) := \sup_{(\xi^+, \xi^-) \in \mathcal{A}''_y} J_H(x, y; \xi^+, \xi^-),$$

with the payoff function J_H given by

$$J_H(x, y; \xi^+, \xi^-) := \mathbb{E}\left[\int_0^\infty e^{-\rho t}[H(Y_t)(X_t^x)^\lambda - C_0 Y_t - C_1 \textstyle\int_0^t Y_s ds]dt \right.$$
$$\left. - \int_0^\infty e^{-\rho t}(K_1 + K_2(X_t^x)^\lambda)d\xi_t^+ - \int_0^\infty e^{-\rho t}(K_0 - K_2(X_t^x)^\lambda)d\xi_t^-\right],$$

subject to

$$\begin{aligned}
&Y_t := y + \xi_t^+ - \xi_t^- \quad \in [a, b], \quad y \in [a, b], \\
&dX_t^x := \mu X_t^x dt + \sqrt{2}\sigma X_t^x dW_t, \quad X_0 := x > 0, \\
&H : [a, b] \to \mathbb{R} \text{ concave, and continuous at } a \text{ and } b, \\
&C_0, C_1, K_2 \in \mathbb{R}, \quad \text{(and to prevent arbitrage)} \quad K_1 + K_0 > 0.
\end{aligned}$$

The supremum is taken over a set of admissible strategies

$$\begin{aligned}
\mathcal{A}_y'' := \Big\{&(\xi^+, \xi^-) : \xi^\pm \text{ are left continuous, non-decreasing processes, } \xi_0^\pm = 0; \\
&y + \xi_t^+ - \xi_t^- \in [a, b]; \\
&\mathbb{E}\left[\int_0^\infty e^{-\rho t}d\xi_t^+ + \int_0^\infty e^{-\rho t}d\xi_t^-\right] < \infty; \\
&|K_2|\mathbb{E}\left[\int_0^\infty e^{-\rho t}(X_t^x)^\lambda d\xi_t^+ + \int_0^\infty e^{-\rho t}(X_t^x)^\lambda d\xi_t^-\right] < \infty\Big\}.
\end{aligned}$$

This is a continuous time formulation of the aforementioned risk management problem. The capacity level Y is a controlled process represented by $(\xi_t^+)_{t\geq 0}$ and $(\xi_t^-)_{t\geq 0}$, which are $\mathbb{F}$-adapted, non-decreasing càglàd processes and respectively stand for the cumulative capacity expansion and reduction by time t; the market price X is modeled by a geometric Brownian motion; the rate of resource extraction is a modeled by the function $H(Y)$; K_0 is the cost of capacity reduction with $K_0 < 0$ representing a partial recovery of the initial investment; K_1 is the cost of capacity contraction; C_0 is the running cost; and C_1 is the cumulating cost. The goal of the company is to maximize its long-term profit with a payoff function that depends on both the resource extraction rate and the market price, with a form of $H(Y)X^\lambda$.

Clearly, when $\lambda \in (m, n)$, where $m < 0 < n$ are the roots of $\sigma^2\lambda^2 + (\mu - \sigma^2)\lambda - \rho = 0$, $V_H(x, y) < \infty$ and well defined. And the control problem can be reduced to an equivalent yet simpler singular control problem (the detailed proof for the equivalence can be found in the Appendix).

Fundamental problem.

$$V(x, y) := \sup_{(\xi^+,\xi^-)\in\mathcal{A}_y'} J(x, y; \xi^+, \xi^-), \tag{1}$$

with

$$J(x, y; \xi^+, \xi^-) := \mathbb{E}\left[\int_0^\infty e^{-\rho t}H(Y_t)X_t^x - \int_0^\infty e^{-\rho t}K_1 d\xi_t^+ - \int_0^\infty e^{-\rho t}K_0 d\xi_t^-\right],$$

subject to

$$
\begin{aligned}
&Y_t := y + \xi_t^+ - \xi_t^-, \quad y \in [a,b],\\
&dX_t^x := \mu X_t^x dt + \sqrt{2}\sigma X_t^x dW_t, \quad X_0 := x > 0,\\
&H : [a,b] \to \mathbb{R} \text{ is concave with } H(y) = \int_a^y h(z)dz,\\
&K_1 + K_0 > 0, \mu < \rho, \quad \text{and (without loss of generality)} \quad K_1 > 0.
\end{aligned}
$$

The supremum is taken over all strategies $(\xi^+, \xi^-) \in \mathcal{A}'_y$, where

$$
\begin{aligned}
\mathcal{A}'_y := \big\{ &(\xi^+, \xi^-) : \xi^\pm \text{ are left continuous, non-decreasing processes}, \quad \xi_0^\pm = 0;\\
&y + \xi_t^+ - \xi_t^- \in [a,b];\\
&\mathbb{E}\left[\int_0^\infty e^{-\rho t} d\xi_t^+ + \int_0^\infty e^{-\rho t} d\xi_t^-\right] < \infty. \big\}
\end{aligned}
$$

In fact, this equivalent control problem is more standard in the existing literature for both singular control and (ir)reversible investment. Thus, throughout the paper, our focus is on the fundamental problem with $K_1 > 0$ [1] and with h appropriately left- or right- continuous[2].

Remark 2.1. *A word of caution: A solution $(\hat{\xi}^+, \hat{\xi}^-) \in \mathcal{A}'_y$ that is optimal for V is also optimal for V_H if and only if $(\hat{\xi}^+, \hat{\xi}^-) \in \mathcal{A}''_y$. However, even if $(\hat{\xi}^+, \hat{\xi}^-) \notin \mathcal{A}''_y$, an ϵ-optimal policy for V_H is the truncated policy $((\hat{\xi}^+_{t\wedge T})_{t\geq 0}, (\hat{\xi}^-_{t\wedge T})_{t\geq 0})$, where T is sufficiently large.*

2.2 Preliminaries

Throughout the paper, we define $m < 0 < 1 < n$ to be the roots of $\sigma^2 x^2 + (\mu - \sigma^2)x - \rho = 0$, so that

$$
m, n = \frac{-(\mu - \sigma^2) \pm \sqrt{(\mu - \sigma^2)^2 + 4\sigma^2 \rho}}{2\sigma^2}.
$$

We also observe the identity $\rho = -\sigma^2 mn$ and define the useful quantity $\eta > 0$:

$$
\eta := \frac{1}{\rho - \mu} = \frac{-mn}{(n-1)(1-m)\rho} = \frac{1}{\sigma^2 (n-1)(1-m)}, \tag{2}
$$

Next, let $R(x,y) := J(x,y;0,0)$ be the no-action expected payoff, then

$$
R(x,y) := \mathbb{E}\left[\int_0^\infty e^{-\rho t} H(y) X_t^x dt\right] = \eta H(y) x, \tag{3}
$$

$$
r(x,y) := R_y(x,y) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} h(y) X_t^x dt\right] = \eta h(y) x. \tag{4}
$$

[1] The assumption of $K_1 > 0$ is without loss of generality. Indeed, if $K_1 \leq 0$, then one considers the control problem on $[0, b-a]$ for $b - Y_t$ instead of Y_t.

[2] h is clearly non-increasing from the concavity of H, so one can choose its left or right continuous versions without changing H or the value function of the control problem V. Moreover, if H is differentiable at y, h can be chosen to be continuous at y.

Moreover, $|J(x,y;\xi^+,\xi^-)| < \infty$ for all $(\xi^+,\xi^-) \in \mathcal{A}'_y$ from the boundedness of H. In fact, we have

Proposition 2.2. *(Finiteness of Value Function) $V(x,y) \leq \eta M x + b - a$, where $M = \sup_{y\in[a,b]} |H(y)| < \infty$.*

Proof. Let $x > 0$ and $y \in [a,b]$ be given. Since $\rho > \mu$ we have

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}[H(Y_t)X_t^x]dt\right] \leq \mathbb{E}\left[\int_0^\infty e^{-\rho t}[MX_t^x]dt\right] \leq \eta M x.$$

Note that for any given $(\xi^+,\xi^-) \in \mathcal{A}'_y$, $a - y \leq \xi_t^+ - \xi_t^- \leq b - y$. From integration by parts, for any $t > 0$,

$$-\int_{[0,T)} e^{-\rho t} d\xi_t^+ \leq -\int_{[0,T)} e^{-\rho t} d\xi_t^- + (y-a). \tag{5}$$

Which, together with $K_1 + K_0 > 0$ and $K_1 > 0$, implies

$$\mathbb{E}\left[-K_1\int_0^\infty e^{-\rho t} d\xi_t^+ - K_0\int_0^\infty e^{-\rho t} d\xi_t^-\right] \leq (y-a) - (K_1+K_0)\mathbb{E}\left[\int_0^\infty e^{-\rho t} d\xi_t^-\right] \leq b - a.$$

Since these bounds are independent of the control, we have

$$V(x,y) \leq \eta M x + b - a < \infty.$$

□

2.3 Solution: Optimal Singular Control and Value Function

The solution for problem (1) can be summarized as follows.

Theorem 2.3. [Value function]

$$V(x,y) = \eta H(a)x + \int_a^y v_1(x,z)dz + \int_y^b v_0(x,z)dz, \tag{6}$$

where v_0 and v_1 are solutions to the following optimal switching problems

$$v_k(x,z) := \sup_{\substack{\alpha\in\mathcal{B} \\ \kappa_0 = k}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left[h(z)X_t^x\right] I_t dt - \sum_{n=1}^\infty e^{-\rho\tau_n} K_{\kappa_n}\right]. \tag{7}$$

Here, $\alpha = (\tau_n,\kappa_n)_{n\geq 0}$ is an admissible switching control so that almost surely $\tau_0 = 0$, $\tau_{n+1} > \tau_n$ for $n \geq 1$, $\tau_n \to \infty$, and for all $n \geq 0$, $\kappa_n \in \{0,1\}$, with $\kappa_n = \kappa_0$ for n even and $\kappa_n = 1 - \kappa_0$ for n odd. $\mathcal{B}$ is the subset of admissible switching controls $\alpha = (\tau_n,\kappa_n)_{n\geq 0}$ such that $\mathbb{E}\left[\sum_{n=1}^\infty e^{-\rho\tau_n}\right] < \infty\}$, and I_t is the regime indicator function for any given $\alpha \in \mathcal{B}$ so that $I_t = \sum_{n=0}^\infty \kappa_n 1_{\{\tau_n < t \leq \tau_{n+1}\}}$.

Moreover, $v_k(x,y)$ can be solved explicitly based on K_0:
Case I *($K_0 \geq 0$):*

1. For each $z \in (a,b)$ such that $h(z) = 0$: $v_0(x,z) = v_1(x,z) = 0$.

2. For each $z \in (a,b)$ such that $h(z) > 0$:

$$\begin{cases} v_0(x,z) &= \begin{cases} A(z)x^n, & x < G(z), \\ \eta h(z)x - K_1, & x \geq G(z), \end{cases} \\ v_1(x,z) &= \eta h(z)x, \end{cases}$$

where $G(z) = \nu h(z)^{-1}$, and $A(z) = \frac{K_1}{(n-1)}(\frac{h(z)}{\nu})^n$, with $\nu = K_1\sigma^2 n(1-m)$.

3. For each $z \in (a,b)$ such that $h(z) < 0$:

$$\begin{cases} v_0(x,z) &= 0, \\ v_1(x,z) &= \begin{cases} B(z)x^n + \eta h(z)x, & x < F(z), \\ -K_0, & x \geq F(z), \end{cases} \end{cases}$$

where $F(z) = -\frac{\kappa}{h(z)}$, and $B(z) = \frac{K_0}{(n-1)}\kappa^{-n}(-\frac{h(z)}{\kappa})^n$, with $\kappa = K_0\sigma^2 n(1-m)$.

Case II *($K_0 < 0$):*

1. For each $z \in (a,b)$ such that $h(z) \leq 0$: $v_0(x,z) = 0, v_1(x,z) = -K_0$.

2. For each $z \in (a,b)$ such that $h(z) > 0$:

$$v_0(x,z) = \begin{cases} A(z)x^n, & x < G(z), \\ B(z)x^m + \eta h(z)x - K_1, & x \geq G(z), \end{cases} \tag{8}$$

$$v_1(x,z) = \begin{cases} A(z)x^n - K_0, & x \leq F(z), \\ B(z)x^m + \eta h(z)x, & x > F(z). \end{cases} \tag{9}$$

Here

$$A(z) = \frac{h(z)^n}{(n-m)\nu^n}\left(\frac{\nu}{\sigma^2(n-1)} + mK_1\right) = \frac{h(z)^n}{(n-m)\kappa^n}\left(\frac{\kappa}{\sigma^2(n-1)} - mK_0\right); \tag{10}$$

$$B(z) = \frac{-h(z)^m}{(n-m)\nu^m}\left(\frac{\nu}{\sigma^2(1-m)} - nK_1\right) = \frac{-h(z)^m}{(n-m)\kappa^m}\left(\frac{\kappa}{\sigma^2(1-m)} + nK_0\right). \tag{11}$$

The functions F and G are non-decreasing with

$$F(z) = \frac{\kappa}{h(z)} \quad \text{and} \quad G(z) = \frac{\nu}{h(z)}, \tag{12}$$

where $\kappa < \nu$ are the unique solutions to

$$\frac{1}{1-m}\left[\nu^{1-m} - \kappa^{1-m}\right] = -\frac{\rho}{m}\left[K_1\nu^{-m} + K_0\kappa^{-m}\right], \tag{13}$$

$$\frac{1}{n-1}\left[\nu^{1-n} - \kappa^{1-n}\right] = \frac{\rho}{n}\left[K_1\nu^{-n} + K_0\kappa^{-n}\right]. \tag{14}$$

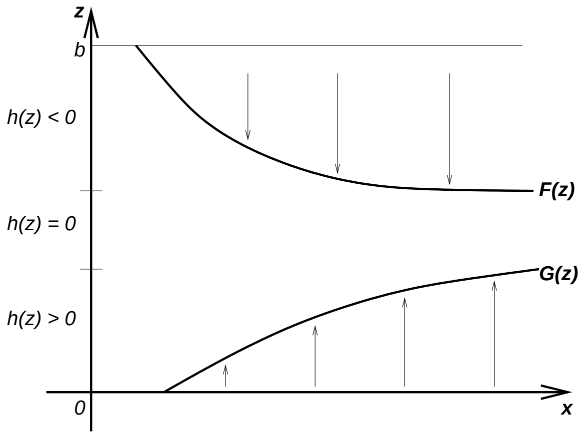

Figure 1: Illustration: Case I–when boundaries are smooth

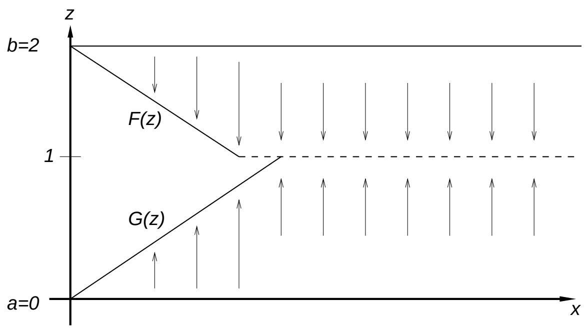

Figure 2: Illustration: Case I–when boundaries are NOT smooth: $h(z) = c/z$ for $z < 1$ and $h(z) = -d/(2-z)$ for $z > 1$ with $K_0/d < K_1/c$.

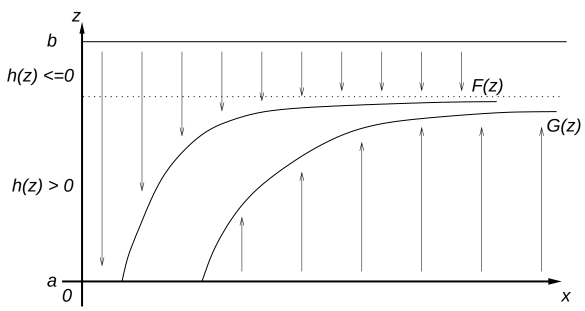

Figure 3: Ilustration of Case II–when boundaries are smooth

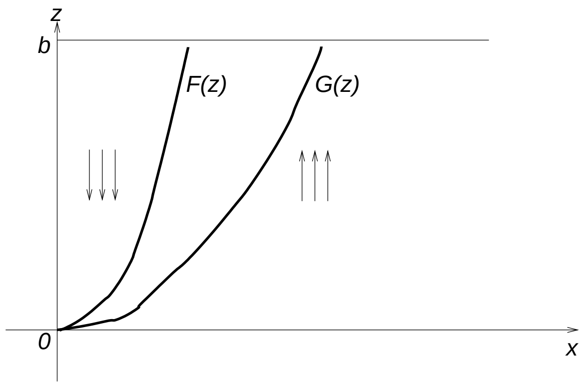

Figure 4: Example from Guo-Tomecek (2006) of Case II–when boundaries are smooth

Theorem 2.4. **[Optimal control]** *The optimal singular control* $(\hat{\xi}^+, \hat{\xi}^-) \in \mathcal{A}'_y$ *exits. For each* $z \in (a, b)$*, the optimal control is described in terms of* $F(z)$ *and* $G(z)$ *from Theorem 2.3 such that*

- *(Case I,* $K_0 \geq 0$*): For* z *such that* $h(z) > 0$*, it is optimal to invest in the project past level* z *when* $X_t^x \in [G(z), \infty)$*, and never disinvest. When* $h(z) < 0$*, it is optimal to disinvest below level* z *when* $X_t^x \in [F(z), \infty)$*, and it is never optimal to invest. When* $h(z) = 0$*, it is optimal to neither invest nor disinvest (i.e.* $F(z) = \infty = G(z)$*).*
- *(Case II,* $K_0 < 0$*): For* z *such that* $h(z) > 0$*, it is optimal to invest in the project past level* z *when* $X_t^x \in [G(z), \infty)$*, and to disinvest below level* z *when* $X_t^x \in (0, F(z)]$*. For* z *such that* $h(z) \leq 0$*, it is always optimal to disinvest.*

Theorem 2.5. **[Optimally controlled process]** *The resulting optimal control process* $\hat{Y}_t$ *is give by:*

Case I*: (up to indistinguishability) for* $t > 0$*,*

- *If* $h(y^+) > 0$ *then* $\hat{Y}_t = \max\{G^{\rightarrow}(M_t), y\}$*,*
- *If* $h(y+) = 0$ *or* $h(y-) = 0$ *then* $\hat{Y}_t = y$*,*
- *If* $h(y-) < 0$ *then* $\hat{Y}_t = \min\{F^{\rightarrow}(M_t), y\}$*.*

Here $M_t = \max\{X_s^x : s \in [0, t]\}$*, and* $F^{\rightarrow}$ *and* $G^{\rightarrow}$ *are respectively the left-continuous inverses of* F *(non-increasing) and* G *(non-decreasing).*

Case II*: (up to indistinguishability) for* $t > 0$*,*

$$\hat{Y}_t = \begin{cases} G^{\rightarrow}(M_t^0) \vee y, & \text{on } \{t \leq S_1\}, \\ F^{\leftarrow}(m_t^n) \wedge \hat{Y}_{S_n}, & \text{on } \{S_n < t \leq T_n\}, \\ G^{\rightarrow}(M_t^n) \vee \hat{Y}_{T_n}, & \text{on } \{T_n < t \leq S_{n+1}\}, \end{cases} \tag{15}$$

and $\lim_{n\to\infty} S_n = \infty = \lim_{n\to\infty} T_n$ *almost surely.*

Here $F^{\leftarrow}(x)$ and $G^{\rightarrow}(x)$ are respectively the right continuous inverse of F and the left-continuous inverse of G. Moreover, the stopping times (S_n) and (T_n) are given by

$$S_1 = \inf\{t > 0 : (X_t^x, \hat{Y}_t) \in \mathcal{S}_0\}, \qquad T_1 = \inf\{t > S_1 : (X_t^x, \hat{Y}_t) \in \mathcal{S}_1\},$$
$$S_n = \inf\{t > T_{n-1} : (X_t^x, \hat{Y}_t) \in \mathcal{S}_0\}, \qquad T_n = \inf\{t > S_n : (X_t^x, \hat{Y}_t) \in \mathcal{S}_1\}.$$

Lastly, the processes M_t^n, m_t^n are defined by $M_t^0 = \max\{X_t^x : 0 \leq s \leq t\}$, and

$$m_t^n = \min\{X_t^x : S_n \leq s \leq t\} 1_{\{S_n \leq t\}}, \qquad M_t^n = \max\{X_t^x : T_n \leq s \leq t\} 1_{\{T_n \leq t\}}.$$

Theorem 2.6. [Region characterization] *Under the optimal singular control $(\hat{\xi}^+, \hat{\xi}^-) \in \mathcal{A}'_y$, define the corresponding investment ($\mathcal{S}_1$), disinvestment ($\mathcal{S}_0$), and continuation ($\mathcal{C}$) regions by*

$$\begin{cases} \mathcal{S}_0 & := \begin{cases} \{(x,z) \in (0,\infty) \times [a,b] : x \geq \lim_{w \uparrow z} F(w)\}, & \text{if } K_0 \geq 0 \text{ (Case I)}, \\ \{(x,z) \in (0,\infty) \times [a,b] : x \leq \lim_{w \uparrow z} F(w)\}, & \text{if } K_0 < 0 \text{ (Case II)}, \end{cases} \\ \mathcal{S}_1 & := \{(x,z) \in (0,\infty) \times [a,b] : x \geq \lim_{w \downarrow z} G(w)\}, \\ \mathcal{C} & := (0,\infty) \times [a,b] \setminus (\mathcal{S}_0 \cup \mathcal{S}_1). \end{cases} \tag{16}$$

Then, the action and continuation regions can be characterized as

$$\begin{cases} \mathcal{S}_0 & = \{(x,y) \in (0,\infty) \times [a,b] : V_y(x,y) = -K_0\}, \\ \mathcal{S}_1 & = \{(x,y) \in (0,\infty) \times [a,b] : V_y(x,y) = K_1\}, \\ \mathcal{C} & = \{(x,y) \in (0,\infty) \times [a,b] : V_{y^-}(x,y) > -K_0, V_{y^+}(x,y) < K_1\}. \end{cases} \tag{17}$$

3 Derivation of the Solution

The derivation goes as follows: first, a collection of corresponding optimal switching problems is established and solved; then, the consistency of the optimal switching controls is proved; finally, the existence of the corresponding integrable optimal singular control $(\hat{\xi}^+, \hat{\xi}^-) \in \mathcal{A}'_y$ is established and the corresponding value function are derived. This method is built on the general correspondence between singular controls and switching controls outlined in Guo and Tomecek (2007).

3.1 Setup and Solution of the Optimal Switching Problems

3.1.1 Corresponding Optimal Switching Problems

In this section, we shall solve the switching problem (7),

$$v_k(x,z) := \sup_{\substack{\alpha \in \mathcal{B} \\ \kappa_0 = k}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left[h(z) X_t^x\right] I_t dt - \sum_{n=1}^\infty e^{-\rho \tau_n} K_{\kappa_n}\right].$$

First, according to Pham (2005, Theorem 1.4.1), and Ly Vath and Pham (2006, Lemma 3.2)), in addition to X being a geometric Brownian motion, we see easily

Proposition 3.1. *For fixed $z \in [a,b]$ and $k \in \{0,1\}$, $v_k(x,z)$ is C^1 in x. Moreover, for every $x > 0$, $\left|\frac{\partial}{\partial x} v_k(x,z)\right| \leq \eta |h(z)|$.*

Next, by modifying the argument in Ly Vath and Pham (2006, Theorem 3.1) for $h \geq 0$ to the case of $h < 0$, we obtain

Proposition 3.2. *v_0 and v_1 are the unique viscosity solutions with linear growth condition to the following system of variational inequalities:*

$$\min\{-\mathcal{L}v_0(x,z),\ v_0(x,z) - v_1(x,z) + K_1\} = 0, \tag{18}$$

$$\min\{-\mathcal{L}v_1(x,z) - h(x,z),\ v_1(x,z) - v_0(x,z) + K_0\} = 0, \tag{19}$$

with boundary conditions $v_0(0^+,z) = 0$ and $v_1(0^+,z) = \max\{-K_0, 0\}$. Here $\mathcal{L}$ is the generator of the diffusion X^x, killed at rate ρ, given by $\mathcal{L}u(x,z) = \sigma^2 u_{xx}(x,z) + \mu u_x(x,z) - \rho u(x,z)$.

3.1.2 Solution of the Optimal Switching Problems

Based on Ly Vath and Pham (2006, Theorem 4.2), we see that

Case I: $K_0 \geq 0$**.** For each $z \in (a,b)$, the switching regions are described in terms of $F(z)$ and $G(z)$, which take values in $(0,\infty]$.

First, for each $z \in (a,b)$ such that $h(z) = 0$, it is never optimal to switch, since $K_0 \geq 0$ and $K_1 > 0$ and so we take $F(z) = \infty = G(z)$. For this case, $v_0(x,z) = 0 = v_1(x,z)$.

Secondly, for z such that $h(z) > 0$, $G(z) < \infty$ and it is optimal to switch from regime 0 to regime 1 (to invest in the project at level z) when $X_t^x \in [G(z),\infty)$. Since $K_0 \geq 0$, it is never optimal to switch from regime 1 to regime 0 (i.e. $F(z) = \infty$). Furthermore, we have

$$v_0(x,z) = \begin{cases} A(z)x^n, & x < G(z), \\ \eta h(z)x - K_1, & x \geq G(z), \end{cases}$$

$$v_1(x,z) = \eta h(z)x,$$

Since v_0 is C^1 at $G(z)$, we get

$$\begin{cases} A(z)G(z)^n & = \eta h(z)G(z) - K_1, \\ nA(z)G(z)^{n-1} & = \eta h(z). \end{cases}$$

That is,

$$\begin{cases} G(z) & = \nu h(z)^{-1}, \\ A(z) & = \frac{K_1}{(n-1)} G(z)^{-n} = \frac{K_1}{(n-1)} \nu^{-n} h(z)^n, \end{cases}$$

where $\nu = K_1 \sigma^2 n(1-m)$.

Finally, when $h(z) < 0$, it is optimal to switch from regime 1 to regime 0 (disinvest at level z) when $X_t^x \in [F(z),\infty)$. Since $K_1 > 0$, it is never optimal to switch from regime 0 to regime 1 (i.e. $G(z) = \infty$). The derivation of the value function proceeds analogously to the derivation for the case of $h(z) > 0$.

Case II: $K_0 < 0$. First of all, for each $z \in (a,b)$ such that $h(z) \le 0$, it is always optimal to disinvest because $K_0 < 0$. That is, $F(z) = \infty = G(z)$. In this case, clearly $v_0(x,z) = 0$ and $v_1(x,z) = -K_0$.

Next, for each $z \in (a,b)$ such that $h(z) > 0$, it is optimal to switch from regime 0 to regime 1 (to invest in the project at level z) when $X_t^x \in [G(z), \infty)$, and to switch from regime 1 to regime 0 (disinvest at level z) when $X_t^x \in (0, F(z)]$, where $0 < F(z) < G(z) < \infty$.

Moreover, v_0 and v_1 are given by

$$v_0(x,z) = \begin{cases} A(z)x^n, & x < G(z), \\ B(z)x^m + \eta x h(z) - K_1, & x \ge G(z), \end{cases}$$

$$v_1(x,z) = \begin{cases} A(z)x^n - K_0, & x \le F(z), \\ B(z)x^m + \eta x h(z), & x > F(z). \end{cases}$$

Smoothness of $v(x,z)$ at $x = G(z)$ and $x = F(z)$ from Proposition 3.1 leads to

$$\begin{cases} A(z)G(z)^n &= B(z)G(z)^m + \eta G(z)h(z) - K_1, \\ nA(z)G(z)^{n-1} &= mB(z)G(z)^{m-1} + \eta h(z), \\ A(z)F(z)^n &= B(z)F(z)^m + \eta F(z)h(z) + K_0, \\ nA(z)F(z)^{n-1} &= mB(z)F(z)^{m-1} + \eta h(z). \end{cases} \tag{20}$$

Eliminating $A(z)$ and $B(z)$ from (20) yields

$$\begin{cases} K_1 G(z)^{-m} + K_0 F(z)^{-m} &= \frac{-m}{(1-m)\rho} h(z)(G(z)^{1-m} - F(z)^{1-m}), \\ K_1 G(z)^{-n} + K_0 F(z)^{-n} &= \frac{n}{(n-1)\rho} h(z)(G(z)^{1-n} - F(z)^{1-n}). \end{cases} \tag{21}$$

Since the viscosity solutions to the variational inequalities are unique and C^1 according to Proposition 3.2, for every z there is a unique solution $F(z) < G(z)$ to (21). Let $\kappa(z) = F(z)h(z)$, $\nu(z) = G(z)h(z)$, then the following system of equations for $\kappa(z)$ and $\nu(z)$ is guaranteed to have a unique solution for each z:

$$\begin{cases} K_1 \nu(z)^{-m} + K_0 \kappa(z)^{-m} &= \frac{-m}{(1-m)\rho}(\nu(z)^{1-m} - \kappa(z)^{1-m}), \\ K_1 \nu(z)^{-n} + K_0 \kappa(z)^{-n} &= \frac{n}{(n-1)\rho}(\nu(z)^{1-n} - \kappa(z)^{1-n}). \end{cases}$$

Moreover, these equations depend on z only through $\nu(z)$ and $\kappa(z)$, implying that there exist unique constants κ, ν such that $\kappa(z) \equiv \kappa$ and $\nu(z) \equiv \nu$ for all z. Hence $F(z) = \kappa h(z)^{-1}$, $G(z) = \nu h(z)^{-1}$, with $\kappa < \nu$ being the unique solutions to

$$\begin{cases} \frac{1}{1-m}\left[\nu^{1-m} - \kappa^{1-m}\right] &= -\frac{\rho}{m}\left[K_1 \nu^{-m} + K_0 \kappa^{-m}\right], \\ \frac{1}{n-1}\left[\nu^{1-n} - \kappa^{1-n}\right] &= \frac{\rho}{n}\left[K_1 \nu^{-n} + K_0 \kappa^{-n}\right]. \end{cases}$$

Given $F(z)$ and $G(z)$, $A(z)$ and $B(z)$ are solved from Eq. (20),

$$\begin{cases} B(z) &= -\frac{G(z)^{-m}}{n-m}\left(\frac{G(z)h(z)}{\sigma^2(1-m)} - nK_1\right) = -\frac{F(z)^{-m}}{n-m}\left(\frac{F(z)h(z)}{\sigma^2(1-m)} + nK_0\right), \\ A(z) &= \frac{G(z)^{-n}}{n-m}\left(\frac{G(z)h(z)}{\sigma^2(n-1)} + mK_1\right) = \frac{F(z)^{-n}}{n-m}\left(\frac{F(z)h(z)}{\sigma^2(n-1)} - mK_0\right). \end{cases}$$

3.2 Derivation of the Optimal Switching Controls

In this section we shall first describe the optimal switching control $\hat{\alpha}(z) = (\hat{\tau}_n(z), \hat{\kappa}_n(z))_{n\geq 0}$ for all $z \in (a, b)$. We then define a collection of these switching controls and prove that this collection satisfies the consistency property from Guo and Tomecek (2007), which implies that it corresponds to an admissible singular control. Note that since the regime values $\hat{\kappa}_n$ are alternating, it suffices to define the switching times $\hat{\tau}_n$.

3.2.1 Optimal Switching Controls

Given the solution to the optimal switching problems, it is clear that the optimal switching control for any level $z \in (a, b)$ is given by the following:

Case I: For $z \in (a, b)$ and $x > 0$, let F and G be as given in Theorem 2.3 for Case I. The switching control $\hat{\alpha}_k(x, z) = (\hat{\tau}_n(x, z), \hat{\kappa}_n(z))_{n\geq 0}$, starting from $\hat{\tau}_0(x, z) = 0$ and $\hat{\kappa}_0(z) = k$ is given by, for $n \geq 1$

- If $k = 0$, $\hat{\tau}_1(x, z) = \inf\{t > 0 : X_t^x \in [G(z), \infty)\}$ and for $n \geq 2$, $\hat{\tau}_n(z) = \infty$,
- If $k = 1$, $\hat{\tau}_1(x, z) = \inf\{t > 0 : X_t^x \in [F(z), \infty)\}$ and for $n \geq 2$, $\hat{\tau}_n(z) = \infty$.

Case II: For $z \in (a, b)$ and $x > 0$, F and G as given in Theorem 2.3 for case II. The switching control $\hat{\alpha}_k(x, z) = (\hat{\tau}_n(x, z), \hat{\kappa}_n(z))_{n\geq 0}$, starting from $\hat{\tau}_0(x, z) = 0$ and $\hat{\kappa}_0(z) = k$ is given by, for $n \geq 1$

- If $\hat{\kappa}_{n-1}(z) = 0$, $\hat{\tau}_n(x, z) = \inf\{t > \tau_{n-1} : X_t^x \in [G(z), \infty)\}$, $\hat{\kappa}_n(z) = 1$.
- If $\hat{\kappa}_{n-1}(z) = 1$, $\hat{\tau}_n(x, z) = \inf\{t > \tau_{n-1} : X_t^x \in (0, F(z)]\}$, $\hat{\kappa}_n(z) = 0$,

3.2.2 Consistency of the Switching Controls

Now, define the collection of admissible switching controls $(\hat{\alpha}(x, z))_{z\in(a,b)}$ so that $\hat{\alpha}(x, z) = \hat{\alpha}_0(x, z)$ for $z > y$ and $\hat{\alpha}(x, z) = \hat{\alpha}_1(x, z)$ for $z \leq y$. Then,

Proposition 3.3. *The collection of switching controls $(\hat{\alpha}(x, z))_{z\in(a,b)}$ is consistent.*

To prove the consistency, the following monotonicity property of F and G are essential: F is non-increasing and G is non-decreasing in Case I, and F is non-decreasing and G is non-increasing in Case II.

To start, for each $z \in (a, b)$, denote $\hat{I}_t(x, z)$ to be the regime indicator function of the optimal switching control $\hat{\alpha}(x, z)$. That is, $\hat{I}_t(x, z) = \sum_{n=0}^{\infty} \hat{\kappa}_n(z) 1_{\{\hat{\tau}_n(x,z)<t\leq\hat{\tau}_{n+1}(x,z)\}}$.

Then the consistency follows from the following lemmas.

Lemma 3.4. *For every $x > 0$ and $t > 0$, $\hat{I}_t(x, \cdot)$ is non-increasing.*

Proof. For simplicity, we omit the dependence on x from the notation.

- Case I: Fix $x > 0$ and $t > 0$. Let $w < z$ be given and suppose that $\hat{I}_t(z) = 1$. On the event that $t \leq \hat{\tau}_1(z)$ we have $w < z \leq y$ and hence $F(w) \geq F(z)$ since F is non-increasing. So by definition $\hat{\tau}_1(w) \geq \hat{\tau}_1(z) \geq t$. Thus, $\hat{I}_t(w) = 1$ for $w \leq y$.

  Now on the event that $t > \hat{\tau}_1(z)$, $\hat{I}_t(z) = 1$ implies that for some $s < t$, $X_s^x \in [G(z), \infty)$, i.e., $\sup\{s \leq t : X_s^x\} \geq G(z)$. However, since G is non-decreasing, $G(z) \geq G(w)$. Hence $\sup\{s \leq t : X_s^x\} \geq G(w)$ and $\hat{I}_t(w) = 1$.

  Since $\hat{I}_t(z) = 1$ implies that $\hat{I}_t(w) = 1$ for any $w < z$, $\hat{I}_t(x, \cdot)$ is non-increasing.

- Case II: Fix $x > 0$ and $t > 0$. Let $w < z$ be given and suppose that $\hat{I}_t(z) = 1$. On the event that $t \leq \hat{\tau}_1(z)$ we have $w < z \leq y$ and hence $F(w) \leq F(z)$. So by definition $\hat{\tau}_1(w) \geq \hat{\tau}_1(z) \geq t$. Thus, $\hat{I}_t(w) = 1$ for $w \leq y$.

  Now on the event that $t > \hat{\tau}_1(z)$, $\hat{I}_t(z) = 1$ implies that for some $s < t$, $X_s^x \in [G(z), \infty)$ and also that X^x must have been in the set $[G(z), \infty)$ more recently than in $[0, F(z)]$, i.e.,

  $$\sup\{s \leq t : X_s^x \in [G(z), \infty)\} > \sup\{s \leq t : X_s^x \in (0, F(z)]\}.$$

  However, since $[G(z), \infty) \subset [G(w), \infty)$ and $(0, F(w)] \subset (0, F(z)]$ for $w < z$, this implies,

  $$\begin{aligned} \sup\{s \leq t : X_s^x \in [G(w), \infty)\} &\geq \sup\{s \leq t : X_s^x \in [G(z), \infty)\} \\ &> \sup\{s \leq t : X_s^x \in (0, F(z)]\} \\ &\geq \sup\{s \leq t : X_s^x \in (0, F(w)]\}. \end{aligned}$$

  Hence X^x was in $[G(w), \infty)$ more recently than in $(0, F(w)]$, meaning $\hat{I}_t(w) = 1$.

  Since $\hat{I}_t(z) = 1$ implies that $\hat{I}_t(w) = 1$ for any $w < z$, $\hat{I}_t(x, \cdot)$ is non-increasing.

□

Lemma 3.5. *For every $x > 0$, $t > 0$, $\int_a^b (\hat{I}_t^+(x, z) + \hat{I}_t^-(x, z))dz < \infty$, almost surely.*

Proof.

- Case I is easy by recalling that $\hat{I}_t^+(x, z) + \hat{I}_t^-(x, z)$ represents the number of switches at level z up to time t. Since there is at most one switch at each level z, $\hat{I}_t^+(x, z) + \hat{I}_t^-(x, z) \leq 1$. Hence $\int_a^b (\hat{I}_t^+(x, z) + \hat{I}_t^-(x, z))dz \leq b - a < \infty$.

- Case II: Since $[a, b]$ is bounded, it suffices to show that for all (x, t), $\hat{I}_t^+(x, z) + \hat{I}_t^-(x, z)$ is almost surely bounded in z. Let $x > 0$ and $t > 0$ be given. Recall that $\hat{I}_t^+(x, z) + \hat{I}_t^-(x, z)$ represents the number of switches at level z up to time t. When $h(z) \leq 0$, there is exactly one switch. When $h(z) > 0$, $0 < F(z) < G(z) < \infty$, $G(z) = \nu h(z)^{-1}$ and $F(z) = \kappa h(z)^{-1}$. Note that after the first switch, each subsequent switch requires that X^x move from $(0, F(z)]$ to $[G(z), \infty)$ or vice versa.

Alternatively, $\log(X^x)$ must move from $(-\infty, \log(F(z))]$ to $[\log(G(z)), \infty)$, travelling a minimum distance of $\log(G(z)) - \log(F(z)) = \log(\nu) - \log(\kappa) > 0$ for each switch. In particular, this quantity is independent of z.

Since $\log(X^x)$ is a Brownian motion with drift, its sample paths are almost surely uniformly continuous on $[0, t]$. Thus, for almost all $\omega \in \Omega$, there exists some $\delta(\omega) > 0$ such that for any $x > 0$ and all $s, r \in [0, t]$, with $|s - r| < \delta(\omega)$,

$$|\log(X_s^x(\omega)) - \log(X_r^x(\omega))| < \log(\nu) - \log(\kappa) = \log(G(z)) - \log(F(z)).$$

Hence, for any level $z \in [a, b]$, there is at least $\delta(\omega)$ amount of time in between any two switches (after the first one). Hence there can be at most $1 + \frac{t}{\delta(\omega)}$ switches at level z in $[0, t]$. Thus

$$\hat{I}_t^+(x, z) + \hat{I}_t^-(x, z) \leq 1 + \frac{t}{\delta} < \infty, \quad \text{almost surely.}$$

□

3.3 Derivation of the Optimal Singular Control

Clearly, the following proposition holds, from Merhi and Zervos (2007).

Proposition 3.6. *For any $y \in [a, b]$ and any pair (ξ^+, ξ^-) of left-continuous, non-decreasing processes, with $\xi_0^\pm = 0$ and $y + \xi_t^+ - \xi_t^- \in [a, b]$ for all t, either*

A. $(\xi^+, \xi^-) \in \mathcal{A}'_y$, or

B. there exists an $\mathbb{F}$-adapted process Z such that $U_\cdot \leq Z_\cdot$ almost surely, $\mathbb{E}[|Z_T|] < \infty$ for all $T \geq 0$, and $\limsup_{T\to\infty} \mathbb{E}[Z_T] = -\infty$, where

$$U_T(y, \xi^+, \xi^-) := \int_0^T e^{-\rho t}[H(Y_t)X_t^x]dt - K_1 \int_{[0,T)} e^{-\rho t} d\xi_t^+ - K_0 \int_{[0,T)} e^{-\rho t} d\xi_t^-.$$

Therefore, after verifying the Standing Assumptions $\mathbb{A}1$, $\mathbb{A}2$ and $\mathbb{A}3$, one can invoke Guo and Tomecek (2007, Proposition 2.13, Theorem 3.13, Theorem 3.10) and conclude that there exists a corresponding integrable singular control $(\hat{\xi}^+, \hat{\xi}^-) \in \mathcal{A}'_y$, and that the value function $V(x, y)$ is given by (6). We define $\hat{Y}_t = y + \hat{\xi}_t^+ - \hat{\xi}_t^-$.

To prove Theorem 2.5, we first establish some Lemmas. First, from Guo and Tomecek (2007, Proposition 2.13) and Lemma 3.4,

Lemma 3.7. *Given $(x, y) \in (0, \infty) \times [a, b]$, the optimally controlled process $\hat{Y}$ is indistinguishable from* $\sup\{z \in (a, b) : \hat{I}_t(x, z) = 1\} = \inf\{z \in (a, b) : \hat{I}_t(x, z) = 0\}$.

Lemma 3.8. *Let $S \leq T$ be non-negative random variables. Then with probability one,*

- *$\hat{Y}$ is non-decreasing on $(S, T]$ for $(X^x, \hat{Y}) \in (\mathcal{S}_0)^c$ on (S, T);*

- *$\hat{Y}$ is non-increasing on $(S,T]$ for $(X^x,\hat{Y}) \in (\mathcal{S}_1)^c$ on (S,T);*
- *$\hat{Y}$ is constant on $(S,T]$ for $(X^x,\hat{Y}) \in \mathcal{C}$ on (S,T).*

Consequently, with probability one, $(X^x_t,\hat{Y}_t) \in \overline{\mathcal{C}}$ for all $t > 0$ and $d\hat{Y}_t$ is supported on $\partial\mathcal{C}$.

Proof. We shall prove only the first claim. (The second one follows by a similar argument, and the last one is immediate from the definition of $\mathcal{C}$ and the first two.) Take any $x > 0$. If $(X^x,\hat{Y}) \in (\mathcal{S}_0)^c$ on (S,T), then in light of Lemma 3.7 and the fact that h has at most countably many discontinuities, clearly it suffices to show that for any $z \in (a,b)$ such that h is continuous at z, $\hat{I}_t(x,z)$ is almost surely non-decreasing on $(S,T]$.

Given $z \in (a,b)$ where h is continuous. Fix $t > 0$ and consider the event that $t \in (S,T)$ and $\hat{I}_t(x,z) = 1$. On this event $\hat{Y}_t \geq z$ almost surely. Furthermore, for any $s \in [t,T)$, $(X^x_s,\hat{Y}_s) \in \mathcal{C}$, and hence $X^x_s > F(\hat{Y}^-_s) \geq F(z^-) = F(z)$, since z is a continuity point of h. This implies that there is no switching to regime 0 at level z, and hence with probability one, $\hat{I}_s(x,z) = 1$ for all $s \in [t,T)$. By the left continuity of $\hat{I}$, this implies $\hat{I}_T(x,z) = 1$ as well. Since $\hat{I}_t(x,z) \in \{0,1\}$, this implies that $\hat{I}_t(x,z)$ is indeed non-decreasing on $(S,T]$.

□

Proof. of Theorem 2.5. We shall prove Cases I and II separately.

Case I: The left-continuous inverses of F (non-increasing) and G (non-decreasing) are given by

$$F^{\rightarrow}(x) = \inf\{z \in (a,b) : F(z) < x\} = \sup\{z \in (a,b) : F(z) \geq x\},$$
$$G^{\rightarrow}(x) = \inf\{z \in (a,b) : G(z) \geq x\} = \sup\{z \in (a,b) : G(z) < x\},$$

with $\inf\emptyset = b$ and $\sup\emptyset = a$.

Recall the optimal switching controls for Case I. Suppose $0 < h(y^+)$ then $0 < h(y)$ since h is non-increasing and thus $F(z) = \infty$ and $G(z) < \infty$. Let $t > 0$ be fixed and observe that $\hat{I}_t(z) \equiv 1$ for all $z \leq y$ and for $z > y$, $\hat{I}_t(z) = 1_{\{\hat{\tau}_1(z)<t\}}$. So $\hat{I}_t(z) = 1$ if and only if $z \leq y$ or $t > \hat{\tau}_1(z)$. Almost surely, $t > \hat{\tau}_1(z)$ is equivalent to $M_t > G(z)$. Hence

$$\begin{aligned}\hat{Y}_t &= \sup\{z \in (a,b) : I_t(z) = 1\} = y \vee \sup\{z \in (a,b) : t > \hat{\tau}_1(z)\}\\ &= y \vee \sup\{z \in (a,b) : G(z) < M_t\} = \max\{G^{\rightarrow}(M_t), y\}.\end{aligned}$$

Now, $\hat{Y}_t$ and since M is increasing, $\max\{G^{\rightarrow}(M_t), y\}$ is also left-continuous, thus, they are indistinguishable.

A similar argument proves the result for $h(y-) < 0$.

Suppose $h(y+) = 0$ or $h(y-) = 0$. Then for all $z > y$, $h(z) \leq 0$ and hence it is never optimal to switch to regime 1. Since $\hat{I}_t(0) = 0$, this is true for all t and $\hat{I}_t(z) \equiv 0$. Similarly, for all $z \leq y$, $h(z) \leq 0$ and so $\hat{I}_t(z) \equiv 1$. Thus $\hat{Y}_t = y$ for all t.

Case II: The right continuous inverse of F and the left-continuous inverse of G, both of which are non-decreasing, are given by

$$\begin{aligned}F^{\leftarrow}(x) &= \inf\{z \in (a,b) : F(z) > x\} = \sup\{z \in (a,b) : F(z) \le x\},\\ G^{\rightarrow}(x) &= \inf\{z \in (a,b) : G(z) \ge x\} = \sup\{z \in (a,b) : G(z) < x\},\end{aligned}$$

with $\inf \emptyset = b$ and $\sup \emptyset = a$.

First we show that $\lim_{n\to\infty} S_n = \infty = \lim_{n\to\infty} T_n$ almost surely. Let $\tilde{S}_n = \sup\{t < T_n : (X_t^x, \hat{Y}_t) \in \mathcal{S}_0\}$ be the last exit time of the process $(X^x, \hat{Y})$ from $\mathcal{S}_0$ before T_n. Then $S_n \le \tilde{S}_n \le T_n$, and $(X_t^x, \hat{Y}_t) \in \mathcal{C}$ on $(\tilde{S}_n, T_n)$. By Lemma 3.8, $\hat{Y}$ is constant on $(\tilde{S}_n, T_n]$. Thus, in between $\tilde{S}_n$ and T_n, the process $(X_t^x, \hat{Y}_{T_n})$, must travel between $\mathcal{S}_0$ and $\mathcal{S}_1$. This means that between $\tilde{S}_n$ and T_n, $\log(X^x)$ must travel between $\log(F(\hat{Y}_{T_n}^-)$ and $\log(G(\hat{Y}_{T_n}^+)$.

Meanwhile, we have

$$\begin{aligned}\log(G(\hat{Y}_{T_n}^+)) - \log(F(\hat{Y}_{T_n}^-)) &= \log(\nu) - \log(h(\hat{Y}_{T_n}^+)) - \log(\kappa) + \log(h(\hat{Y}_{T_n}^-))\\ &\ge \log(\nu) - \log(\kappa) > 0.\end{aligned}$$

Since this quantity is positive, and independent of n, and $\log(X^x)$ is a Brownian motion, there exists a positive random variable $\epsilon > 0$ such that $\epsilon \le T_n - \tilde{S}_n \le T_n - S_n \le S_{n+1} - S_n$. Hence $\lim_{n\to\infty} S_n = \infty$ almost surely. Since $T_n \ge S_n$ for all n, $\lim_{n\to\infty} T_n = \infty$ almost surely as well.

Next, fix $t > 0$ and note that that almost surely $t \in (T_n, S_{n+1}]$ or $t \in (S_n, T_n]$, for some n, where $T_0 = 0$. We consider the case that $t \in (T_n, S_{n+1}]$ for some $n \ge 0$. The proof for the case $t \in (S_n, T_n]$ is similar.

Note that $(X^x, \hat{Y}) \in (\mathcal{S}_0)^c$ on $(\tilde{S}_n, S_{n+1})$, and hence by Lemma 3.8, $\hat{I}_s(x, z)$ is non-decreasing on $[T_n, S_{n+1}] \subset (\tilde{S}_n, S_{n+1}]$ for all $z \in (a, b)$ such that h is continuous at z.

Thus, on the event that $t \in (T_n, S_{n+1}]$ we know that $\hat{I}_{T_n}(z) = 1$ for all $z < \hat{Y}_{T_n}$ and $\hat{I}_{T_n}(z) = 0$ for all $z > \hat{Y}_t$. Since $\hat{I}$ is non-decreasing on $[T_n, S_{n+1}]$, this means that $\hat{I}_t(x, z) = 1$ if and only if $z < \hat{Y}_{T_n}$ or if $X_s^x \ge G(z)$ for some $s \in [T_n, t)$. The latter condition is almost surely equivalent to $G(z) < M_t^n$. Thus, by Lemma 3.7, on the event that $t \in (T_n, S_{n+1}]$, we almost surely have

$$\begin{aligned}\hat{Y}_t &= \sup\{z \in (a,b) : I_t(z) = 1\} = \hat{Y}_{T_n} \vee \sup\{z \in (a,b) : G(z) < M_t^n\}\\ &= G^{\rightarrow}(M_t^n) \vee \hat{Y}_{T_n}.\end{aligned}$$

A similar argument shows that on the event that $t \in (S_n, T_n]$, we almost surely have $\hat{Y}_t = F^{\leftarrow}(m_t^n) \wedge \hat{Y}_{S_n}$. Hence, we have proved that for each t, the statement in (15) holds almost surely. Moreover, since M^n is increasing and $G^{\rightarrow}$ is left continuous, $G^{\rightarrow}(M_t^n)$ is left-continuous in t. Similarly, since m^n is decreasing and $F^{\leftarrow}$ is right continuous, $F^{\leftarrow}(m_t^n)$ is left-continuous in t. Thus, the right hand side of (15) is left-continuous in t, and hence indistinguishable from $\hat{Y}$.

□

4 Regularity, Smooth Fit and Region Characterization

In this section, we shall establish necessary and sufficient conditions for the smooth fit principle by exploiting both the structure of the payoff function and the explicit solution of the value function. This analysis leads to the proof of Theorem 2.6 on region characterization.

4.1 Regularity and Smooth Fit

Theorem 4.1. **[Sufficient Conditions]** *$V(x,y)$ is C^1 in x for all $(x,y) \in (0,\infty)\times[a,b]$, and*

$$\frac{\partial}{\partial x}V(x,y) = \eta H(a) + \int_a^y \frac{\partial}{\partial x}v_1(x,z)dz + \int_y^b \frac{\partial}{\partial x}v_0(x,z)dz.$$

Moreover, if H is C^1 on an open interval $\mathcal{J} \subset [a,b]$, then $V(x,y)$ is C^1 in y on $(0,\infty)\times\mathcal{J}$; that is, $V(x,y)$ is $C^{1,1}$ on $(0,\infty)\times\mathcal{J}$.

Proof. First, by the representation of $V(x,y)$ in Eq. (6), it suffices to check that for a fixed $y \in [a,b]$, $u'(x) = \int_a^y \frac{\partial}{\partial x}v_1(x,z)dz$ for all $x > 0$, where $u(x) = \int_a^y v_1(x,z)dz$.

Note that $\int_a^y |v_1(x,z)|dz < \infty$, and $|\frac{\partial}{\partial x}v_1(x,z)|$ is locally bounded by a constant factor of $h(z)$ by Proposition 3.1. Moreover, for every $\delta > 0$ such that $x - \delta > 0$, there exists a constant C such that

$$\int_a^y \int_{-\delta}^{\delta} \Big|\frac{\partial}{\partial x}v_1(x+\theta,z)\Big| d\theta dz \le \int_a^y \int_{-\delta}^{\delta} Ch(z)d\theta dz = 2\delta C[H(y)-H(a)] < \infty.$$

Hence, by the Dominated Convergence Theorem, v is continuous; and by Durrett (1996, Theorem A.9.1), $u'(x) = \int_a^y \frac{\partial}{\partial x}v_1(x,z)dz$ for all $x > 0$.

Furthermore, suppose that $H(y)$ is C^1 in an open interval $\mathcal{J} \subset [a,b]$. Then for $x > 0$, and $y \in \mathcal{J}$,

$$\begin{aligned}\lim_{z\to y}\mathbb{E}\left[\int_0^\infty |e^{-\rho t}h(z)(X_t^x) - e^{-\rho t}h(y)X_t^x|dt\right] &= \lim_{z\to y}\mathbb{E}\left[\int_0^\infty e^{-\rho t}X_t^x dt\right]|h(z)-h(y)| \\ &= \eta x \lim_{z\to y}|h(z)-h(y)| = 0\end{aligned}$$

So by Proposition 3.15 in Guo and Tomecek (2007), $v_k(x,\cdot)$ is continuous at y and hence $V(x,y)$ is C^1 in y for all $(x,y) \in (0,\infty)\times\mathcal{J}$. □

To study the necessary conditions for the continuous differentiability of the value function on y, we start by defining $d(x,y) = V_{y^+}(x,y) - V_{y^-}(x,y)$.

First, by Guo and Tomecek (2007) and by recalling that $h(y)$ is non-increasing and hence $\mathbb{E}\left[\int_0^\tau e^{-\rho t}h(z)X_t^x dt\right]$ is non-increasing in z for any stopping time τ, we have

Lemma 4.2. *For $x > 0$, $v_1(x,\cdot) - v_0(x,\cdot)$ is decreasing. Therefore, $d(x,\cdot)$ has only countably many discontinuities.*

This lemma, coupled with the variational inequalities in (18) and (19), leads to

Proposition 4.3. *$V(x,y)$ is both left and right differentiable in y, with V_{y^+} and V_{y^-} decreasing in y and $-K_0 \le V_{y^+}(x,y) \le V_{y^-}(x,y) \le K_1$. Thus, $d(x,y) \le 0$.*

Note that the above results on regularity are based on the general properties of the payoff function H and on the relation between the value function $V(x,y)$ of singular control problem (1) with the value functions $v_k(x,z)$ of the corresponding optimal switching problems.

In the following, we exploit the explicit solutions of $v_k(x,y)$ to establish further regularity properties of $V(x,y)$ with respect to y.

Proposition 4.4. *The left and right derivatives $V_{y^-}(x,y)$ and $V_{y^+}(x,y)$ are C^1 in x. That is, $d(x,y)$ is C^1 in x.*

Proof. We provide the proof for $V_{y^+}(x,y)$ in Case II with $h(y^+) > 0$, and other cases can be verified by similar arguments. Clearly, it suffices to verify that $V_{y^+}(x,y)$ is continuous and differentiable (with zero derivative) at $x = F(y^+)$ and $x = G(y^+)$.

In Case II, F and G are non-decreasing, and so taking limits of the difference between v_0 and v_1 in (9) and (8) gives

$$V_{y^+}(x,y) = \begin{cases} -K_0, & x \le F(y^+), \\ B(y^+)x^m - A(y^+)x^n + \eta h(y^+)x, & F(y^+) < x \le G(y^+), \\ K_1, & x > G(y^+), \end{cases} \tag{22}$$

$$V_{y^-}(x,y) = \begin{cases} -K_0, & x < F(y^-), \\ B(y^-)x^m - A(y^-)x^n + \eta h(y^-)x, & F(y^-) \le x < G(y^-), \\ K_1, & x \ge G(y^-). \end{cases} \tag{23}$$

By the continuity of v_1 and v_0 in (8), we have

$$\begin{aligned} \lim_{x \downarrow G(y^+)} V_{y^+}(x,y) &= K_1 \\ \lim_{x \uparrow G(y^+)} V_{y^+}(x,y) &= B(y^+)G(y^+)^m - A(y^+)G(y^+)^n + \eta h(y^+)G(y^+) \\ &= \lim_{z \downarrow y} [B(z)G(z)^m - A(z)G(z)^n + \eta h(z)G(z)] \\ &= \lim_{z \downarrow y} [v_1(G(z),z) - v_0(G(z),z)] = K_1. \end{aligned}$$

Hence $V_{y^+}(x,y)$ is continuous at $G(y^+)$.

Moreover, by the continuous differentiability of v_1 and v_0 in (8) and (9), we have

$$\lim_{h\downarrow 0}\frac{V_{y^+}(G(y^+)+h,y)-V_{y^+}(G(y^+),y)}{h}=\lim_{h\downarrow 0}\frac{K_1-K_1}{h}=0$$

$$\begin{aligned}
&\lim_{h\uparrow 0}\frac{V_{y^+}(G(y^+)+h,y)-V_{y^+}(G(y^+),y)}{h}\\
&\quad=mB(y^+)G(y^+)^{m-1}-nA(y^+)G(y^+)^{n-1}+\eta h(y^+)\\
&\quad=\lim_{z\downarrow y}[mB(z)G(z)^{m-1}-nA(z)G(z)^{n-1}+\eta h(z)]\\
&\quad=\lim_{z\downarrow y}\frac{\partial}{\partial x}[v_1(G(z),z)-v_0(G(z),z)]=0.
\end{aligned}$$

Hence $V_{y^+}(x,y)$ is C^1 at $G(y^+)$, (and similarly, at $F(y^+)$). □

Theorem 4.5. [Necessary and Sufficient Conditions for Smooth Fit] *$V(x,y)$ is continuously differentiable in x for all $(x,y)\in(0,\infty)\times[a,b]$. $V(x,y)$ is differentiable in y at the point (x,y) if and only if*

$$(x,y)\in\{(x,y)\in(0,\infty)\times(a,b): H \text{ is differentiable at } y\}\cup\mathcal{S}_0\cup\mathcal{S}_1,$$

where $\mathcal{S}_0$ and $\mathcal{S}_1$ are given in Eq. (16). Alternatively, it is not differentiable in y at the point (x,y) if and only if

$$(x,y)\in\{(x,y)\in(0,\infty)\times(a,b): H \text{ is not differentiable at } y\}\cap\mathcal{C}.$$

This theorem follows naturally from the following Lemma and the Proposition.

Lemma 4.6. *If h is continuous at y, then for all $x>0$, $d(x,y)=0$.*

Proposition 4.7. *If h is not continuous at y, then in Case I, $d(x,y)=0$ for $x\geq\min\{F(y^-),G(y^+)\}$ and $d(x,y)<0$ for $x<\min\{F(y^-),G(y^+)\}$. In Case II, $d(x,y)=0$ for $x\leq F(y^-)$ and $x\geq G(y^+)$ and $d(x,y)<0$ for $x\in(F(y^-),G(y^+))$.*

Proof. (of Proposition 4.7). Suppose that there exists $y\in(a,b)$ where h is not continuous. We shall prove the result in Case II when $h(y^+)>0$, and other cases can be verified by similar arguments.

First, since h is non-increasing, $\lim_{z\downarrow y}h(z)<\lim_{z\uparrow y}h(z)$. This also implies that the switching boundaries $F(z)=\kappa h(z)^{-1}$ and $G(z)=\nu h(z)^{-1}$ are discontinuous at y. Clearly, by (22) and (23), $d(x,y)=0$ for $x<F(y^-)$ and for $x>G(y^+)$. By the continuity of d, this is also true of $x=F(y^-)$ and $x=G(y^+)$.

Next, without loss of generality, assume that h and hence G is right continuous. Then, pick x such that $G(z^-)\leq x<G(y)=G(y^+)$. Since $x<G(y)$, then $v_1(x,y)-v_0(x,y)<K_1$ from the HJB Eqs. (18) and (19)). Furthermore, by Lemma 4.2, $v_1(x,z)-v_0(x,z)\leq v_1(x,y)-v_0(x,y)<K_1$ for all $z>y$. Hence

$$V_{y^+}(x,y)=\lim_{z\downarrow y}v_1(x,z)-v_0(x,z)\leq v_1(x,y)-v_0(x,y)<K_1, \text{ and}$$

$$V_{y^-}(x,y)=\lim_{z\uparrow y}v_1(x,z)-v_0(x,z)=K_1,$$

where the last equality follows from the fact that $x \geq G(z)$ for all $z < y$. Thus, $d(x, y) < 0$ for all $x \in [G(y^-), G(y^+))$. A similar argument proves that in addition to the above, $d(x, y) < 0$ for all $x \in (F(y^-), F(y^+)]$.

Finally, let $x_0 \in (F(y^+), G(y^-))$ be given. We know that $d(F(y^-), y) = 0 = d(G(y^+), y)$, $d(x, y) \leq 0$ and that d is C^1 in x. Suppose $d(x_0, y) = 0$, implying that x_0 is a local maximum, and hence $d_x(x_0, y) = 0$. Furthermore, by the mean value theorem, there must be two points $x_1 \in (F(y^-), x_0)$ and $x_2 \in (x_0, G(y^+))$ such that $d_x(x_1, y) = 0 = d_x(x_2, y)$. In fact, since $d(x, y) < 0$ for all $x \in (F(y^-), F(y^+)]$ and $x \in [G(y^-), G(y^+))$, we must have $x_1 \in (F(y^+), x_0)$ and $x_2 \in (x_0, G(y^-))$.

Let $f(x) = x^{-(m-1)} d_x(x, y)$. Since $x_0, x_1, x_2 > 0$ and $0 = d_x(x_0, y) = d_x(x_1, y) = d_x(x_2, y)$, we must also have $0 = f(x_0) = f(x_1) = f(x_2)$. However, by (22) and (23), for $x \in (F(y^+), G(y^-))$, $d(x, y) = \Delta B(y)x^m - \Delta A(y)x^n + \eta \Delta h(y)x$, with $\Delta B(y) = B(y^+) - B(y^-)$, $\Delta A(y) = A(y^+) - A(y^-)$ and $\Delta h(y) = h(y^+) - h(y^-)$. So by differentiating, we have that for $x \in (F(y^+), G(y^-))$,$f(x) = x^{-(m-1)} d_x(x, y) = m\Delta B(y) - n\Delta A(y)x^{n-m} + \eta \Delta h(y)x^{1-m}$.

Now, f is C^1 on $(F(y^+), G(y^-))$, hence by the mean value theorem again, there must be two points, $\hat{x}_1 \in (x_1, x_0) \subset (F(y^+), G(y^-))$ and $\hat{x}_2 \in (x_0, x_2) \subset (F(y^+), G(y^-))$ such that $f_x(\hat{x}_1) = 0 = f_x(\hat{x}_2)$. Thus f_x must have at least two positive roots. Differentiating again, we have, for $x \in (F(y^+), G(y^-))$,

$$\begin{aligned} f_x(x) &= -n(n-m)\Delta A(y)x^{n-m-1} + (1-m)\eta \Delta h(y)x^{-m} \\ &= x^{-m}\left((1-m)\eta \Delta h(y) - n(n-m)\Delta A(y)x^{n-1}\right). \end{aligned}$$

Thus, $f_x(x)$ can have at most one positive root, contradiction. Thus $d(x_0, y) < 0$. Since $x_0 \in (F(y^+), G(y^-))$ was arbitrary, $d(x, y) < 0$ for all $x \in (F(y^+), G(y^-))$. □

Finally, we can explicitly compute V_{xy} and V_{yx} from the derivatives of $v_k(x, y)$.

Theorem 4.8. *If $V_y(x, \hat{y})$ exists in a neighborhood of $\hat{x}$, then V_{xy} and V_{yx} exist at $(\hat{x}, \hat{y})$, with $V_{xy}(\hat{x}, \hat{y}) = V_{yx}(\hat{x}, \hat{y}) = \frac{\partial}{\partial x}[v_1(\hat{x}, \hat{y}) - v_0(\hat{x}, \hat{y})]$.*

Proof. The existence of V_{yx} exists at $(\hat{x}, \hat{y})$ is clear with $V_{yx}(\hat{x}, \hat{y}) = \frac{\partial}{\partial x}[v_1(\hat{x}, \hat{y}) - v_0(\hat{x}, \hat{y})]$. Moreover, By Theorem 4.5, the existence of $V_y(\hat{x}, y)$ for all y in a neighborhood of $\hat{y}$ means that either $\hat{y}$ is a continuity point of h, or $(\hat{x}, \hat{y})$ is in the interior of $\mathcal{S}_0 \cup \mathcal{S}_1$.

If $\hat{y}$ is a continuity point of h, by the representation of V_x in Theorem 4.1, it is sufficient to show that $u_1(y) := \frac{\partial}{\partial x} v_1(x, y)$ and $u_0(y) := \frac{\partial}{\partial x} v_0(x, y)$ are continuous at $\hat{y}$.

We prove that $u_0(y)$ is continuous at $\hat{y}$ for Case II. (Similar arguments apply to other cases.) In this case, v_0 is C^1 in x and from (8),

$$u_0(y) = \frac{\partial}{\partial x} v_0(x, z) = \begin{cases} nA(z)x^{n-1}, & x < G(z), \\ mB(z)x^{m-1} + \eta h(z), & x \geq G(z), \end{cases}$$

where $nA(z)G(z)^{n-1} = mB(z)G(z)^{m-1} + \eta h(z)$. Since h is continuous at $\hat{y}$, the continuity of A, B and G follows by their representation in Theorem 2.3, hence the continuity of $u_0(y)$ at $\hat{y}$ from its expression.

If $(\hat{x}, \hat{y})$ is in the interior of $\mathcal{S}_1$. Then, the explicit forms in Theorem 2.3 imply that for all (x, y) in a neighborhood of $(\hat{x}, \hat{y})$, we have $\frac{\partial}{\partial x} v_0(x, y) = \frac{\partial}{\partial x} v_1(x, y)$, and the limits in y from both the left and the right exist. Thus, by the representation in Theorem 4.1, the left and right derivatives of $V_x(\hat{x}, \hat{y})$ exist and are given by

$$V_{xy^+}(\hat{x}, \hat{y}) = \lim_{y \downarrow \hat{y}} \frac{\partial}{\partial x} v_1(\hat{x}, y) - \lim_{y \downarrow \hat{y}} \frac{\partial}{\partial x} v_0(\hat{x}, y) = \lim_{y \downarrow \hat{y}} \left(\frac{\partial}{\partial x} v_1(\hat{x}, y) - \frac{\partial}{\partial x} v_0(\hat{x}, y) \right) = 0,$$
$$V_{xy^-}(\hat{x}, \hat{y}) = \lim_{y \uparrow \hat{y}} \frac{\partial}{\partial x} v_1(\hat{x}, y) - \lim_{y \uparrow \hat{y}} \frac{\partial}{\partial x} v_0(\hat{x}, y) = \lim_{y \uparrow \hat{y}} \left(\frac{\partial}{\partial x} v_1(\hat{x}, y) - \frac{\partial}{\partial x} v_0(\hat{x}, y) \right) = 0.$$

Thus, V_{xy} exists, since the left- and right- derivatives are equal. Furthermore, it is easy to verify that for $(\hat{x}, \hat{y})$ in the interior of $\mathcal{S}_1$, $V_{yx}(\hat{x}, \hat{y}) = \frac{\partial}{\partial x}[v_1(\hat{x}, \hat{y}) - v_0(\hat{x}, \hat{y})] = 0$. A similar argument applies to $(\hat{x}, \hat{y})$ in the interior of $\mathcal{S}_0$, thereby proving the claim.

□

Corollary 4.9. *If H is C^1 on an open interval $\mathcal{J} \subset [a, b]$ then V_{yx} and V_{xy} exist and are continuous with $V_{xy}(x, y) = V_{yx}(x, y) = \frac{\partial}{\partial x}[v_1(x, y) - v_0(x, y)]$ on $(0, \infty) \times \mathcal{J}$.*

4.2 Derivation of Theorem 2.6 on Region Characterization

Proof. of Theorem 2.6.

Recall that V_{y^-} and V_{y^+} exist by Proposition 4.3 and that $-K_0 \leq V_{y^+} \leq V_{y^-} \leq K_1$.

Thus, $V_y(x, y) = -K_0$ if and only if $V_{y-}(x, y) = -K_0$, and from the expression for V_{y^-} in (23), we have that $V_{y^-}(x, y) = -K_0$ for $x < F(y^-)$ (in Case II). However, by the continuity of V_{y^-} in Proposition 4.4, we get $V_{y^-}(x, y) = -K_0$ if and only if $x \leq F(y^-)$, which is true if and only if $(x, y) \in \mathcal{S}_0$ by Eq. (16). Thus, $V_y(x, y) = -K_0$ if and only if $(x, y) \in \mathcal{S}_0$.

The same argument applied to $V_{y^+}(x, y) = K_1$ shows that $V_y(x, y) = K_1$ if and only if $(x, y) \in \mathcal{S}_1$. Lastly, the claim for $\mathcal{C}$ follows since it is the complement of $\mathcal{S}_0 \cup \mathcal{S}_1$.

A similar argument also applies in Case I. □

5 Examples and Discussions

By now, it is clear from our analysis that without sufficient smoothness of the payoff function, the value function may be non-differentiable and the boundaries may be non-smooth or not strictly monotonic. Moreover, when the payoff function H is not continuously differentiable, the interior of $\mathcal{C}$ may not be simply connected. Note, however, the regions $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{C}$ are mutually disjoint and simply connected by the monotonicity of F and G.

We elaborate on these points with some concrete examples.

5.1 Examples

Taking parameters κ, ν, h as defined in the main results in Section 2.3, we fix here $[a, b] = [0, 2]$, $K_0 < 0$ and $0 < \beta < 1$. Recall that since $K_0 < 0$, $F(z) = \kappa h(z)^{-1}$ and $G(z) = \nu h(z)^{-1}$.

Example 5.1. **[Value function NOT C^1, boundaries are C^1 but NOT strictly increasing: because H is not strictly concave]**

$$H(z) = \begin{cases} z, & z \leq 1, \\ \arctan(z-1) + 1, & z > 1. \end{cases}$$

$$h(z) = \begin{cases} 1, & z \leq 1, \\ \frac{1}{1+(z-1)^2}, & z > 1. \end{cases}$$

See Figure 5.

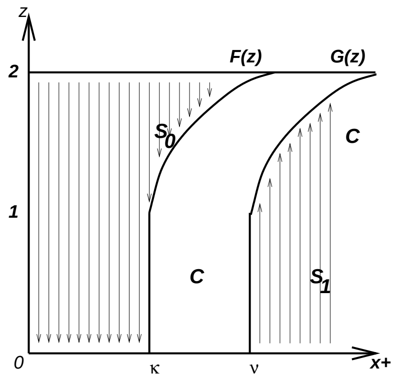

Figure 5: Value function is C^1, F, G are C^1, but NOT strictly increasing: because H is NOT strictly concave.

Example 5.2. **[Value function C^1, F, G are only C^0: because H is not C^2]**

$$H(z) = \begin{cases} z, & z \leq 1, \\ \frac{z^\beta - 1}{\beta} + 1, & z > 1, \end{cases}$$

$$h(z) = \begin{cases} 1, & z \leq 1, \\ z^{\beta-1}, & z > 1. \end{cases}$$

See Figure 6.

Example 5.3. **[Value function NOT C^1, F, G NOT continuous: because H is not C^1.]**

$$H(z) = \begin{cases} z, & z \leq 1, \\ \frac{2\kappa}{(\kappa+\nu)} \frac{z^\beta - 1}{\beta} + 1, & z > 1, \end{cases}$$

$$h(z) = \begin{cases} 1, & z \leq 1, \\ \frac{2\kappa}{\kappa+\nu} z^{\beta-1}, & z > 1. \end{cases}$$

See Figure 7.

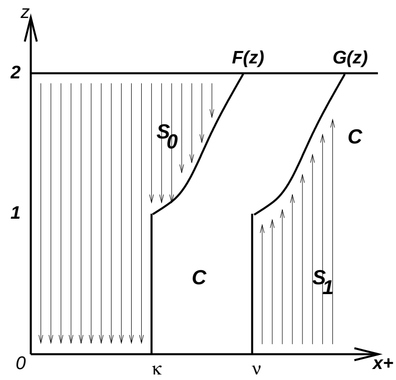

Figure 6: Value function C^1, F, G are only C^0: because H is not C^2.

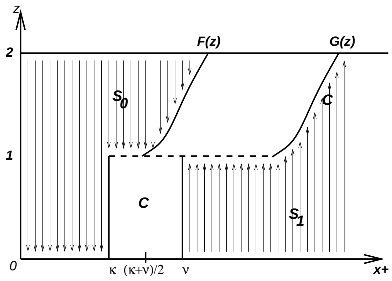

Figure 7: Value function NOT C^1, F, G NOT continuous: because H is not C^1.

Example 5.4. [Interior of continuation region NOT connected]

$$H(z) = \begin{cases} z, & z \leq 1, \\ \frac{\kappa}{2\nu} \frac{z^\beta - 1}{\beta} + 1, & z > 1, \end{cases}$$

$$h(z) = \begin{cases} 1, & z \leq 1, \\ \frac{\kappa}{2\nu} z^{\beta - 1}, & z > 1. \end{cases}$$

See Figure 8.

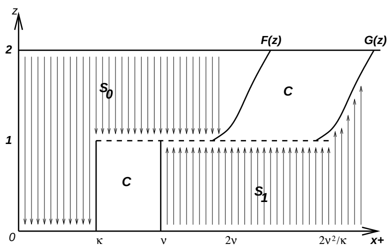

Figure 8: Interior of continuation region NOT connected

Note that Examples 5.2 - 5.4 all have payoff functions of the form

$$H(z) = \begin{cases} z, & z \leq 1, \\ \phi \frac{z^\beta - 1}{\beta} + 1, & z > 1, \end{cases}$$

for some constant ϕ. To ensure the concavity of H, we must have $\phi \in [0,1]$. When $\phi = 1$, we recover Example 5.1 and Figure 6. For $\frac{\kappa}{\nu} < \phi < 1$, the regions are described by Figure 7. Lastly, for $0 < \phi \leq \frac{\kappa}{\nu}$, the interior of the continuation region is not connected as in Figure 8.

5.2 Discussion

The above examples demonstrate how the regularity assumptions typically assumed by the traditional HJB approach may fail. According to that approach, the value function $V(x, y)$ would satisfy some (quasi)-Variational Inequalities so that

$$\max\{\sigma^2 x^2 V_{xx}(x,y) + bxV_x(x,y) - rV(x,y) + H(x,y), V_y(x,y) - K_1, -V_y(x,y) - K_0\} = 0.$$

In general, while searching for a solution, one would assume *a priori* smoothness for the value function and the boundary. For example, in Alvarez (2006) and Merhi and Zervos (2007), V is derived from the class of $C^{2,1}$. However, Example 5.2 shows that although the HJB variational inequality may still hold, one should search for a solution in a larger class of functions, such as in $C^{1,0}$.

Furthermore, Example 5.3 shows that in general, one may not have the smoothness of the boundary, as the boundaries F and G are not necessarily continuous or not even strictly increasing. Indeed, in this example, F and G are inversely proportional to h, which may be neither.

Finally, we compare our results and method with those in Alvarez (2006).

Example 5.5. [General case of Alvarez (2006) for GBM] *Let $x > 0$ and $y \in [a, b]$, with $K_0 < 0$ and $h > 0$ on $[a, b]$. Then F and G are non-decreasing and given by Eq. (12). Define*

$$y_0(x) = G^{\leftarrow}(x) \wedge b = \sup\{z : G(z) \leq x\} = \sup\{z : h(z) \leq (x/\nu)^{-1}\} \wedge b,$$
$$y_1(x) = F^{\rightarrow}(x) \vee a = \inf\{z : F(z) \geq x\} = \inf\{z : h(z) \geq (x/\kappa)^{-1}\} \vee a.$$

Then $y_0(x) \leq y_1(x)$, and

- *$x \leq F(z)$ for $z > y_1(x)$;*
- *$F(z) < x < G(z)$ for $y_0(x) < z < y_1(x)$;*
- *$G(z) \leq x$ for $z < y_0(x)$*

When, in addition, H satisfies the Inada conditions, this example generalizes those in Alvarez (2006) when X is a geometric Brownian motion. Compared to the very special form appearing in Alvarez (2006), our results show that, in order to compute the value function, integration of $v_k(x, z)$ is necessary, which we reduce to three possible cases, depending on whether (x, y) is in $\mathcal{S}_0$, $\mathcal{S}_1$ or $\mathcal{C}$.

1. $(x,y) \in \mathcal{S}_0$: *Then* $y_1 \le y$ *and*

$$V(x,y) = \eta H(y_1)x + x^m \int_a^{y_1} B(z)dz + x^n \int_{y_1}^b A(z)dz - K_0(y - y_1).$$

2. $(x,y) \in \mathcal{C}$: *Then* $y_0 < y < y_1$ *and*

$$V(x,y) = \eta H(y)x + x^m \int_a^{y} B(z)dz + x^n \int_{y}^b A(z)dz.$$

3. $(x,y) \in \mathcal{S}_1$: *Then* $y \le y_0$ *and*

$$V(x,y) = \eta H(y_0)x + x^m \int_a^{y_0} B(z)dz + x^n \int_{y_0}^b A(z)dz - K_1(y_0 - y),$$

where A *and* B *are given by Eqs. (10)-(11).*

A Appendix: Equivalence of Problem A and Problem (1)

Proposition A.1. *For all* $(\xi^+, \xi^-) \in \mathcal{A}''_y$, , *we have*

$$J_H(x,y;\xi^+,\xi^-) = Ky + x^\lambda[K_2(y-a) + \eta H(a)] + \tilde{J}_{\tilde{H}}(x^\lambda, y; \xi^+, \xi^-)$$

where $K = \frac{-C_0}{\rho} + \frac{-C_1}{\rho^2}$, $\eta = \frac{1}{\rho - \tilde{\mu}}$ *and*

$$\tilde{J}_{\tilde{H}}(x,y;\xi^+,\xi^-) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\tilde{H}(Y_t)\tilde{X}_t^x dt - \tilde{K}_1 \int_0^\infty e^{-\rho t} d\xi_t^+ - \tilde{K}_0 \int_0^\infty e^{-\rho t} d\xi_t^-\right]$$

where

$$\begin{aligned}
&\tilde{K}_0 = K_0 + K, \quad \tilde{K}_1 = K_1 - K,\\
&\tilde{H}(y) = H(y) - H(a) - K_2(\rho - \tilde{\mu})(y-a) = \int_a^y [H'(z) - K_2(\rho - \tilde{\mu})]dz,\\
&d\tilde{X}_t^x = \tilde{\mu}\tilde{X}_t^x dt + \sqrt{2}\tilde{\sigma}\tilde{X}_t^x d\tilde{W}_t, \quad \tilde{X}_0^x := x > 0,\\
&\tilde{\mu} = \sigma^2\lambda^2 + (\mu - \sigma^2)\lambda < \rho, \quad \tilde{\sigma} = \sigma|\lambda|,\\
&\tilde{W}_t = sign(\lambda) W_t.
\end{aligned}$$

Proof. Let $(\xi^+, \xi^-) \in \mathcal{A}''_y$ be given. To obtain $\tilde{K}_1$ and $\tilde{K}_0$, observe that

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} C_1 (\textstyle\int_0^t Y_s ds) dt\right] = \mathbb{E}\left[\int_0^\infty \int_s^\infty e^{-\rho t} C_1 Y_s dt ds\right] = \mathbb{E}\left[\int_0^\infty e^{-\rho s} \frac{C_1}{\rho} Y_s ds\right].$$

Thus, from the integration by parts formula,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}[-C_0 - C_1/\rho]Y_t dt\right] = Ky + K\mathbb{E}\left[\int_0^\infty e^{-\rho t} d\xi_t^+\right] - K\mathbb{E}\left[\int_0^\infty e^{-\rho t} d\xi_t^-\right],$$

where $K = \frac{-C_0}{\rho} + \frac{-C_1}{\rho^2}$.

Next, for any $\lambda \in (m, n)$, let $\tilde{x} = x^\lambda$, $\tilde{X}_t^{\tilde{x}} = (X_t^x)^\lambda$. Ito's formula shows

$$d\tilde{X}_t^{\tilde{x}} = \tilde{\mu}\tilde{X}_t^{\tilde{x}} dt + \sqrt{2}\tilde{\sigma}\tilde{X}_t^{\tilde{x}} d\tilde{W}_t, \quad \tilde{X}_0^{\tilde{x}} := \tilde{x} > 0,$$

where $\tilde{\mu} = \sigma^2\lambda^2 + (\mu - \sigma^2)\lambda$, $\tilde{\sigma} = \sigma|\lambda|$ and $\tilde{W}_t = \operatorname{sign}(\lambda) W_t$.

To get $\tilde{H}(y)$, let $Z_t = e^{-\rho t}\tilde{X}_t^{\tilde{x}}$. Then $\mathbb{E}[\int_0^\infty Z_s ds] < \infty$, and by Ito's formula, we have

$$dZ_t = -(\rho - \tilde{\mu})Z_t dt + \sqrt{2}\sigma Z_t dW_t, \quad Z_0 = \tilde{x} > 0.$$

Note that $(\rho - \tilde{\mu}) > 0$ for $\lambda \in (m, n)$. Thus $\lim_{t\to 0} Z_t = 0$ almost surely and in L^1.

Now, from the integration by parts formula in Protter (2004, p. 68) and noting that X and Y are left continuous,

$$\begin{aligned}\int_0^t Z_{s-} dY_s &= Z_t Y_t - \int_0^t Y_{s-} dZ_s - [Z, Y]_t \\ &= Z_t Y_t + (\rho - \tilde{\mu})\int_0^t Y_s Z_s ds - \sqrt{2}\sigma \int_0^t Y_s Z_s dW_s - \tilde{x}y,\end{aligned}$$

because $[Z, Y]_t = Z_0 Y_0 = \tilde{x}y$ for all t from the finite variation of Y.

Moreover, from Protter (2004, p. 63), the process $\int_0^t Y_s Z_s dW_s$ is a martingale, so for all $t > 0$,

$$\mathbb{E}\left[\int_0^t Z_s dY_s\right] = \mathbb{E}\left[Z_t Y_t + (\rho - \tilde{\mu})\int_0^t Y_s Z_s ds\right] - \tilde{x}y.$$

And since $Z_t \to 0$ in L^1 and $Y_t \in [a, b]$ is bounded, $\lim_{t\to\infty} \mathbb{E}[Z_t Y_t] = 0$.

Furthermore, since $(\xi^+, \xi^-) \in \mathcal{A}_y''$ and Z_t is integrable, the dominated convergence theorem gives

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty Z_s dY_s\right] &= \lim_{t\to\infty} \mathbb{E}\left[\int_0^t e^{-\rho s}(X_s^x)^\lambda dY_s\right] = \lim_{t\to\infty} \mathbb{E}\left[Z_t Y_t + (\rho - \tilde{\mu})\int_0^t Y_s Z_s ds\right] - \tilde{x}y \\ &= \mathbb{E}\left[(\rho - \tilde{\mu})\int_0^\infty Y_s Z_s ds\right] - \tilde{x}y.\end{aligned}$$

Hence,

$$\mathbb{E}\left[-K_2 \int_0^\infty e^{-\rho t}\tilde{X}_t^{\tilde{x}} d\xi_t^+ + K_2 \int_0^\infty e^{-\rho t}\tilde{X}_t^{\tilde{x}} d\xi_t^-\right] = K_2 x^\lambda y - \mathbb{E}\left[K_2(\rho - \tilde{\mu})\int_0^\infty e^{-\rho t} Y_t \tilde{X}_t^x dt\right].$$

Lastly, for $\eta = \frac{1}{\rho - \tilde{\mu}}$, $\mathbb{E}\left[\int_0^\infty e^{-\rho t}[H(a) + K_2(\rho - \tilde{\mu})a]\tilde{X}_t^{\tilde{x}} dt\right] = x^\lambda[\eta H(a) + K_2 a]$. The conclusion is immediate by putting the above calculations together. □

Now, the equivalence of the control problems follows from the above equivalence of the payoffs, with an easy application of the Dominated Convergence Theorem.

Theorem A.2. *The value function V_H defined in Problem A is given by*

$$V_H(x,y) = Ky + x^\lambda[K_2(y-a) + \eta H(a)] + V(x^\lambda, y),$$

where $K = \frac{-C_0}{\rho} + \frac{-C_1}{\rho^2}$, $\eta = \frac{1}{\rho-\mu}$ *and*

$$V(x,y) = \sup_{(\xi^+,\xi^-)\in\mathcal{A}'_y} \tilde{J}_{\tilde{H}}(x,y;\xi^+,\xi^-). \tag{24}$$

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