A Two-Player Price Impact Game

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Single Player Problem

Bank, Soner, V. ('17)

For a given predictable $\xi \in L^2(\mathbb{P} \otimes dt)$ and given $x \in \mathbb{R}$, $\sigma > 0$, $\lambda > 0$, $\gamma > 0$, find an absolutely continuous, adapted process $X = x + \int_0^{\cdot} \alpha_t dt$ with $\alpha \in L^2(\mathbb{P} \otimes dt)$ which minimizes

$$\mathbb{E}\left[\int_0^T (X_t - \xi_t)^2 \sigma \, dt + \lambda \int_0^T \alpha_t^2 \, dt + \gamma \int_0^T \alpha_t \cdot (X_t - x) \, dt\right]$$

subject to $X_T = \Xi_T$ for some given $\Xi_T \in L^2(\mathscr{F}_{T-}, \mathbb{P})$.

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References:

Kohlmann/Tang ('02), Rogers/Singh ('10), Frei/Westray ('13), Horst/Naujokat ('14), Almgren/Li ('14), Cartea/Jaimungal ('15), Bank/Soner/V. ('17), . . .

Single Player Problem: Solution

Theorem (Bank, Soner, V. ('17))

Under suitable assumptions the optimal control $\hat{\alpha}$ with strategy $\hat{X} = x + \int_0^{\cdot} \hat{\alpha}_t dt$ is given by

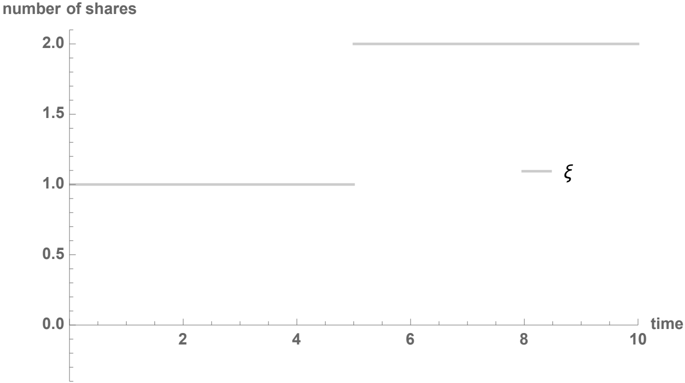
$$\hat{\alpha}_t = \tilde{c}_t \cdot \left(\hat{\xi}_t - \hat{X}_t\right)$$

with deterministic function $\tilde{c}_t > 0$ satisfying $\lim_{t \uparrow T} \tilde{c}_t = +\infty$ and

$$\hat{\xi}_t = \tilde{w}_t^1 \cdot \mathbb{E}\left[\Xi_T \,|\, \mathscr{F}_t\right] + \tilde{w}_t^2 \cdot \mathbb{E}\left[\int_t^T \xi_s \cdot \tilde{K}(t,s)\, ds \,\middle|\, \mathscr{F}_t\right]$$

with deterministic nonnegative weights $\tilde{w}_\cdot^1 + \tilde{w}_\cdot^2 = 1$, $\lim_{t \uparrow T} \tilde{w}_t^1 = 1$, $\lim_{t \uparrow T} \tilde{w}_t^2 = 0$ and deterministic kernel $\tilde{K}$.

Illustration: Stock-Buying Schedule



Single player: $x = 0$, $\xi_t = 1 \cdot 1_{\{0 \leq t < 5\}} + 2 \cdot 1_{\{5 \leq t \leq 10\}}$, $\Xi_T = 2$.

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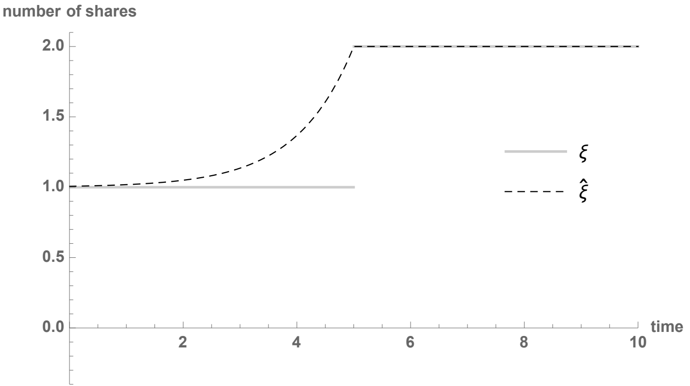

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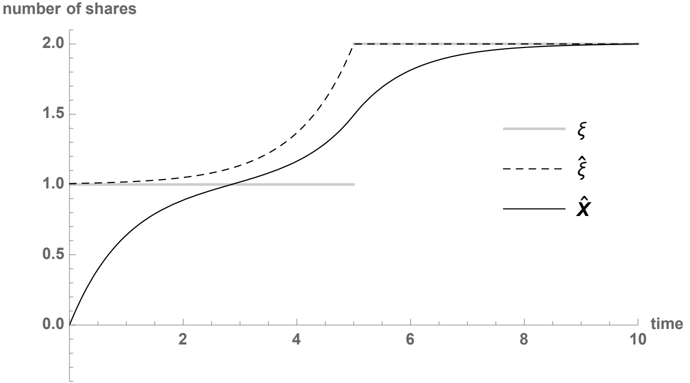

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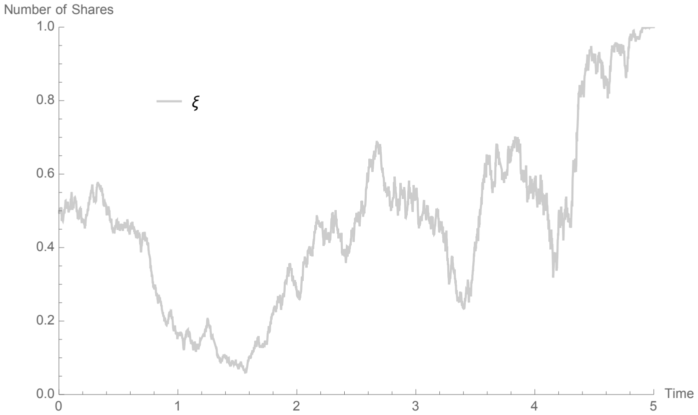
Illustration: Running after the delta



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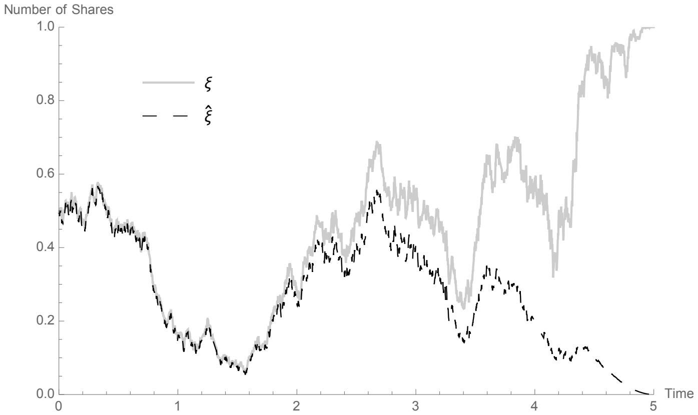

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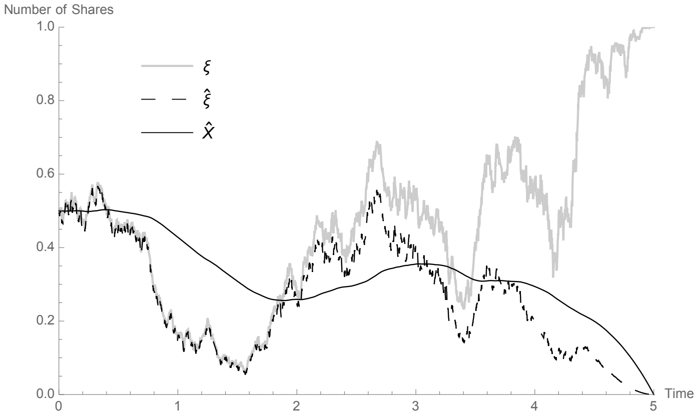

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Two-Player Problem

Player 1: $X^1_\cdot = x^1 + \int_0^\cdot \alpha^1_t dt$ and targets ξ^1, Ξ^1_T

$$J^1(\alpha^1 \quad) \triangleq \mathbb{E}\left[\int_0^T (X^1_t - \xi^1_t)^2 \sigma dt + \lambda \int_0^T \alpha^1_t \cdot \left(\alpha^1_t \quad \right) dt \right] \to \min_{\substack{\alpha^1 \\ \text{s.t. } X^1_0 = x^1,\, X^1_T = \Xi^1_T}}$$

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Player 2: $X^2_\cdot = \qquad \int_0^\cdot \alpha^2_t dt$

$$J^2(\alpha^1,\alpha^2) \triangleq \mathbb{E}\left[\int_0^T (X^2_t \qquad)^2 \sigma dt + \lambda \int_0^T \alpha^2_t \cdot \left(\alpha^1_t + \alpha^2_t\right) dt + \gamma \int_0^T \alpha^2_t \cdot \left(X^1_t - x^1\right) dt\right] \to \min_{\substack{\alpha^2 \\ \text{s.t. } X^2_0 = 0,\, X^2_T = 0}}$$

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Two strategic agents are competing for liquidity!

Nash equilibrium

A pair of admissible strategies $(\hat{\alpha}^1, \hat{\alpha}^2)$ is called a **Nash equilibrium** for the game if for all admissible strategies α^1, α^2 we have

$$J^1(\hat{\alpha}^1, \hat{\alpha}^2) \leq J^1(\alpha^1, \hat{\alpha}^2) \quad \text{and} \quad J^2(\hat{\alpha}^1, \hat{\alpha}^2) \leq J^2(\hat{\alpha}^1, \alpha^2),$$

that is, neither player has an incentive to deviate from $(\hat{\alpha}^1, \hat{\alpha}^2)$.

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Note: We will only consider an **open loop** Nash equilibrium!

Related Literature: Optimal Portfolio Liquidation

Player 1 ("distressed trader"): $x^1 > 0$, $\Xi^1_T = 0, \xi^1 \equiv 0$
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Explicit results:

1. **Carlin, Lobo, Viswanathan ('07)**: risk-neutral ($\sigma = 0$), deterministic open loop

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Numerical analysis:

4. **Carmona & Yang ('08)**: adopt optimization problem of Carlin et al., stochastic closed loop strategies, noise traders, also allow longer time horizon $\tilde{T} > T$ for predator (two stage model)

Nash equilibrium in optimal liquidation

Qualitative property of Nash equilibrium:

predatory trading *vs.* liquidity provision

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  - preying in plastic market ($\gamma \gg \lambda$)
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Qualitative property of Nash equilibrium:

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  - liquidity provision in elastic market ($\lambda \gg \gamma$)
- Schied & Zhang: predatory trading or liquidity provision occurs

Two-Player Problem: Solution

Theorem

Under suitable assumptions there exists a unique open loop Nash equilibrium $(\hat{\alpha}^1, \hat{\alpha}^2)$ with **Player 1's** control $\hat{\alpha}^1$ given by

$$\hat{\alpha}_t^1 = c_t \cdot \left(\hat{\xi}_t^1 - w_t^5 \cdot \hat{X}_t^2 - \hat{X}_t^1 \right)$$

with deterministic function $c_t > 0$ satisfying $\lim_{t \uparrow T} c_t = +\infty$ and

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with deterministic nonnegative weights $w_\cdot^1 + w_\cdot^2 + w_\cdot^3 + w_\cdot^4 = 1$, $w_\cdot^5 \in [-1, 1]$, $\lim_{t \uparrow T} w_t^{3,4,5} = 0$, $\lim_{t \uparrow T} w_t^{1,2} = 1/2$, and deterministic kernels K^1, K^2.

Two-Player Problem: Solution

Theorem (cont.)

And, similarly, with **Player 2's** control $\hat{\alpha}^2$ given by

$$\hat{\alpha}_t^2 = c_t \cdot \left(\hat{\xi}_t^2 - w_t^5 \cdot \hat{X}_t^1 - \hat{X}_t^2 \right)$$

and

$$\begin{aligned} \hat{\xi}_t^2 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathscr{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^2 - \Xi_T^1 \mid \mathscr{F}_t] \\ + w_t^3 \cdot \mathbb{E}\left[\int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t,s)\, ds \,\middle|\, \mathscr{F}_t \right] \\ + w_t^4 \cdot \mathbb{E}\left[\int_t^T (\xi_s^2 - \xi_s^1) \cdot K^2(t,s)\, ds \,\middle|\, \mathscr{F}_t \right]. \end{aligned}$$

Optimal Portfolio Liquidation Revisited

Schied & Zhang ('17)

Player 1 ("distressed trader"): $x^1 = 1$, $\Xi^1_T = 0$, $\xi^1_t \equiv 0$.
Player 2 ("predator"): $x^2 = 0$, $\Xi^2_T = 0$, $\xi^2_t \equiv 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

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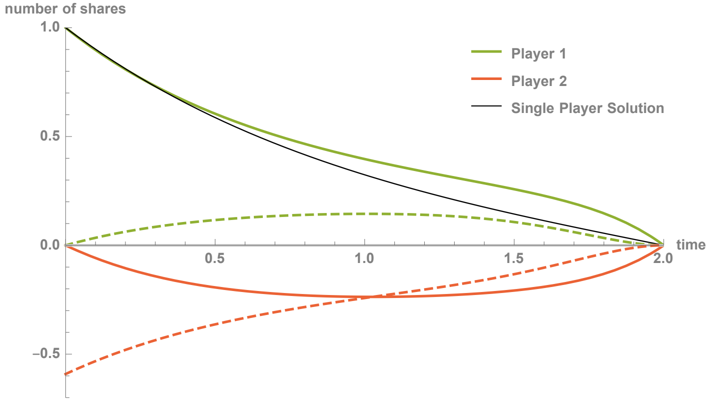

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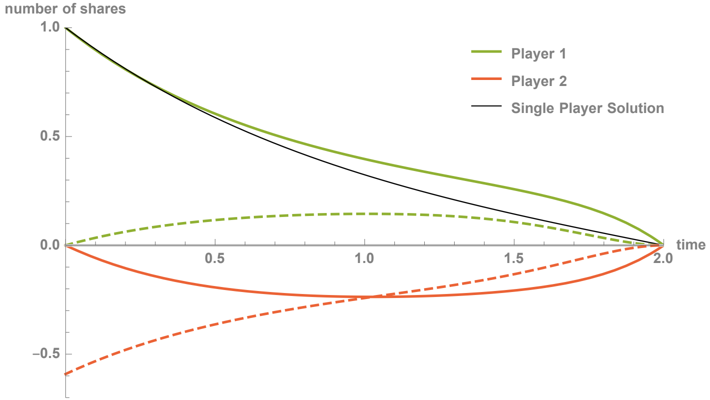

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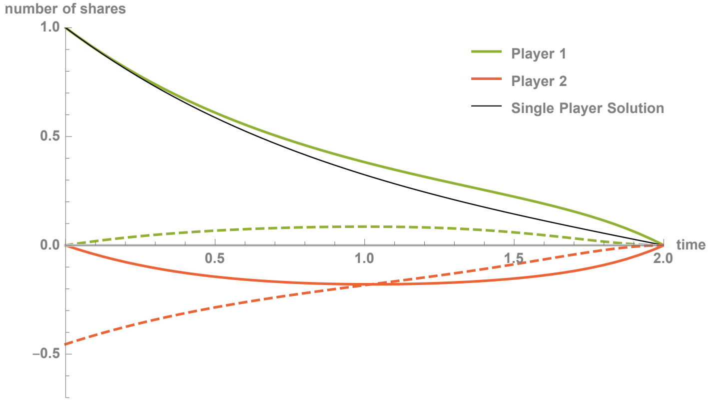

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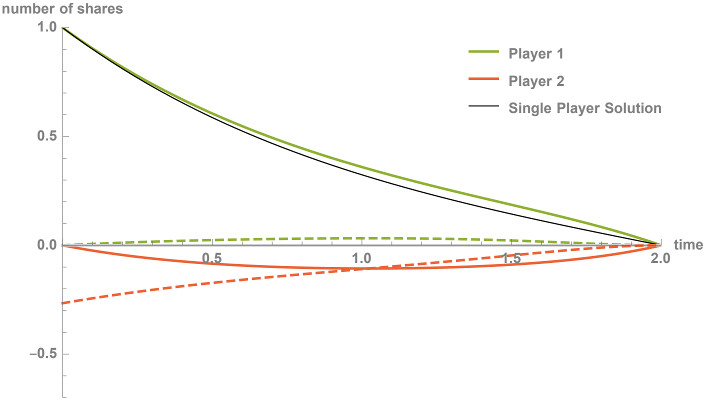

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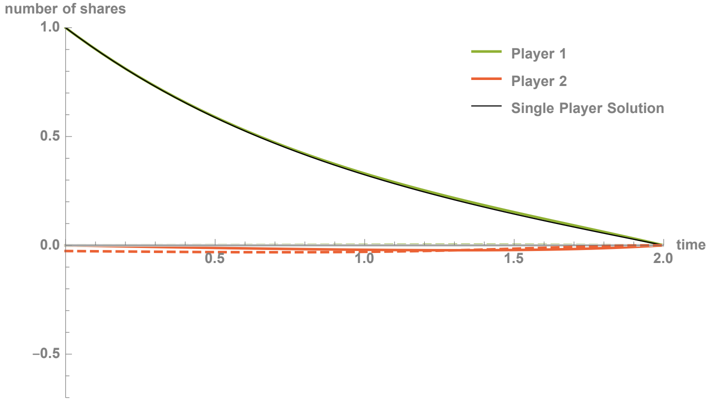

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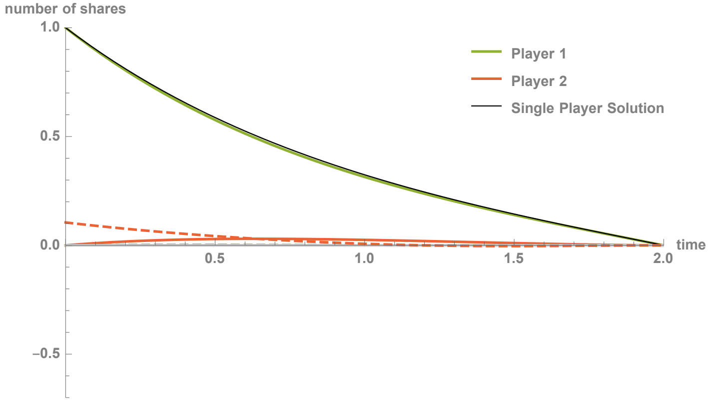

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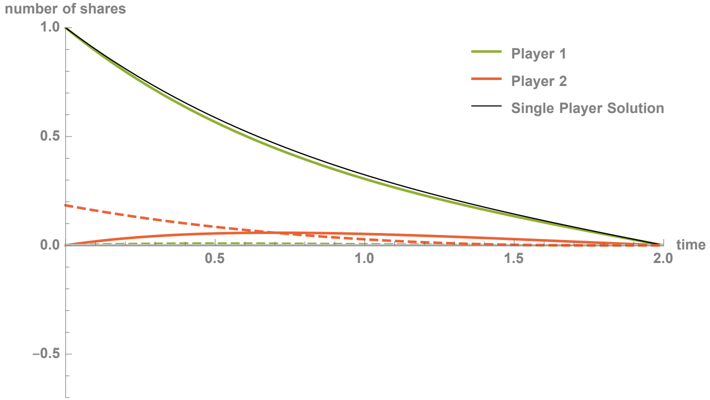

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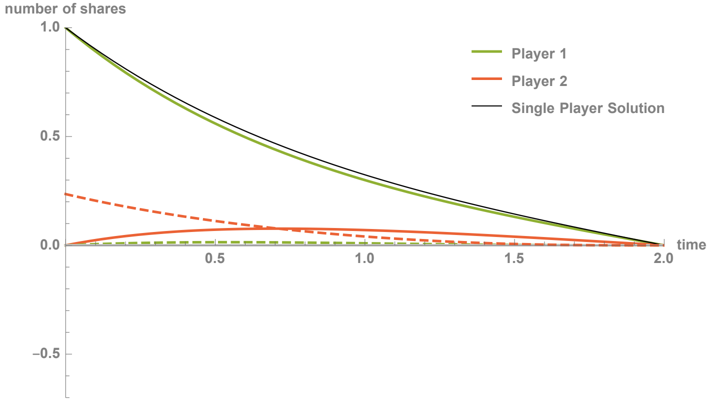

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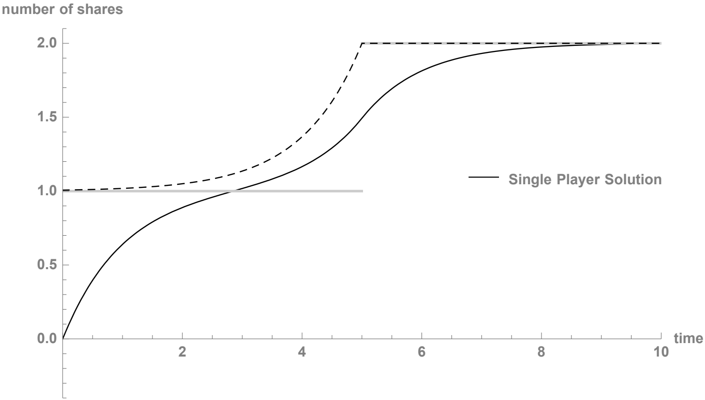

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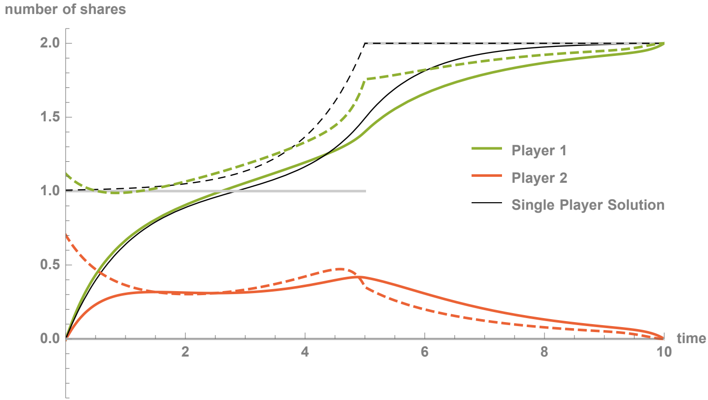

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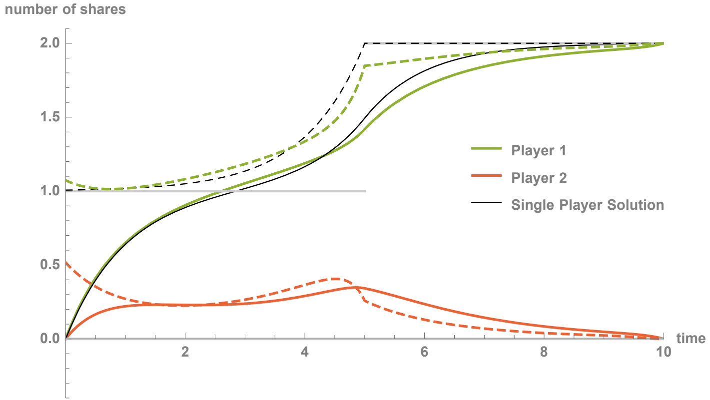
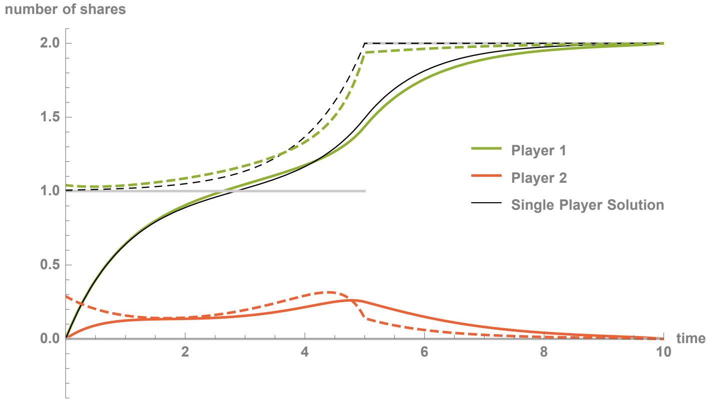
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Stock-Buying Schedule



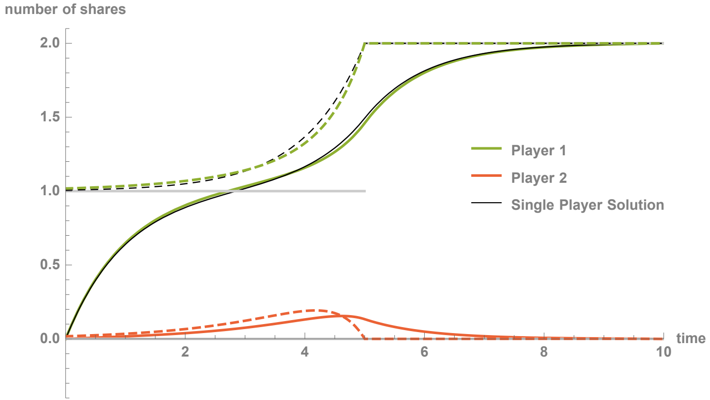
Player 1: $x^1 = 0$, $\xi_t^1 = 1 \cdot 1_{\{0 \le t < 5\}} + 2 \cdot 1_{\{5 \le t \le 10\}}$, $\Xi_T^1 = 2$.
Player 2 ("predator"): $x^2 = 0$, $\xi_t^2 \equiv 0$, $\Xi_T^2 = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1.5$.

Stock-Buying Schedule



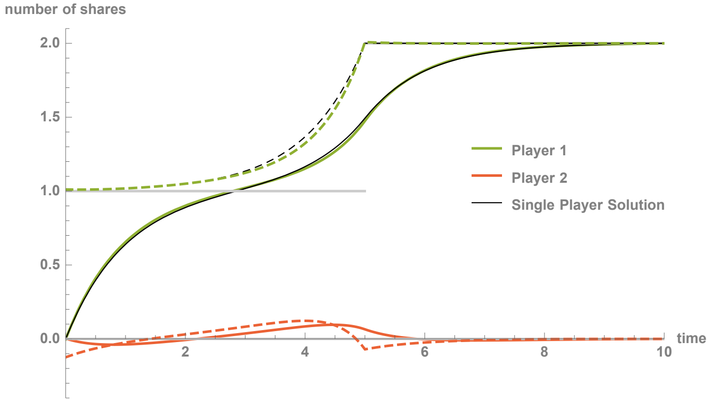
Player 1: $x^1 = 0$, $\xi_t^1 = 1 \cdot 1_{\{0 \le t < 5\}} + 2 \cdot 1_{\{5 \le t \le 10\}}$, $\Xi_T^1 = 2$.
Player 2 ("predator"): $x^2 = 0$, $\xi_t^2 \equiv 0$, $\Xi_T^2 = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1$.

Stock-Buying Schedule



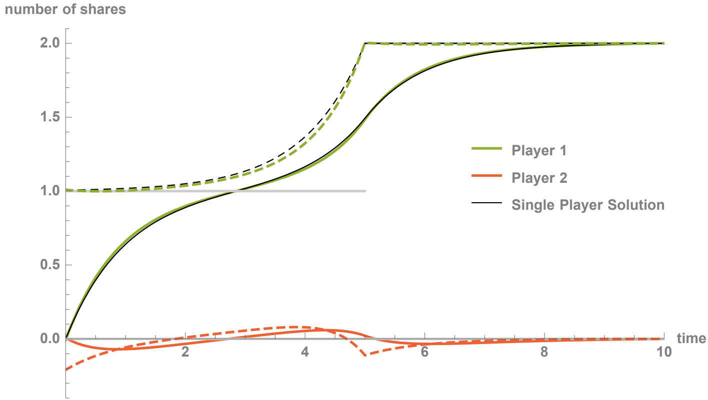
Player 1: $x^1 = 0$, $\xi_t^1 = 1 \cdot 1_{\{0 \le t < 5\}} + 2 \cdot 1_{\{5 \le t \le 10\}}$, $\Xi_T^1 = 2$.
Player 2 ("predator"): $x^2 = 0$, $\xi_t^2 \equiv 0$, $\Xi_T^2 = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.5$.

Stock-Buying Schedule



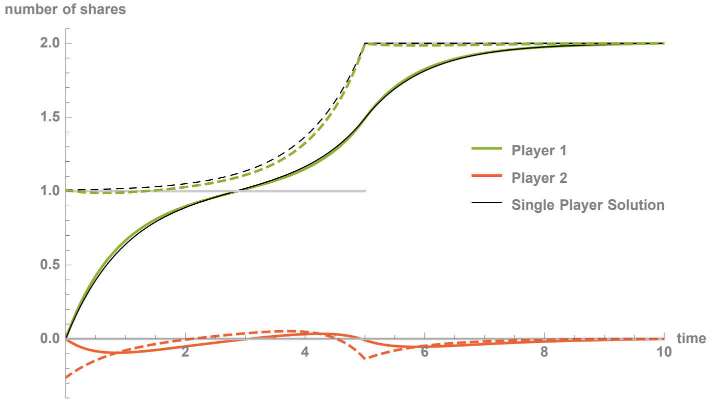
Player 1: $x^1 = 0$, $\xi_t^1 = 1 \cdot 1_{\{0 \le t < 5\}} + 2 \cdot 1_{\{5 \le t \le 10\}}$, $\Xi_T^1 = 2$.
Player 2 ("predator"): $x^2 = 0$, $\xi_t^2 \equiv 0$, $\Xi_T^2 = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.25$.

Stock-Buying Schedule



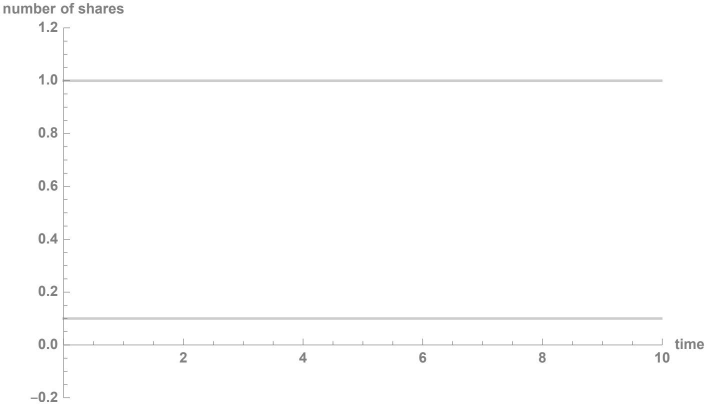
Player 1: $x^1 = 0$, $\xi_t^1 = 1 \cdot 1_{\{0 \le t < 5\}} + 2 \cdot 1_{\{5 \le t \le 10\}}$, $\Xi_T^1 = 2$.
Player 2 ("predator"): $x^2 = 0$, $\xi_t^2 \equiv 0$, $\Xi_T^2 = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.1$.

Stock-Buying Schedule



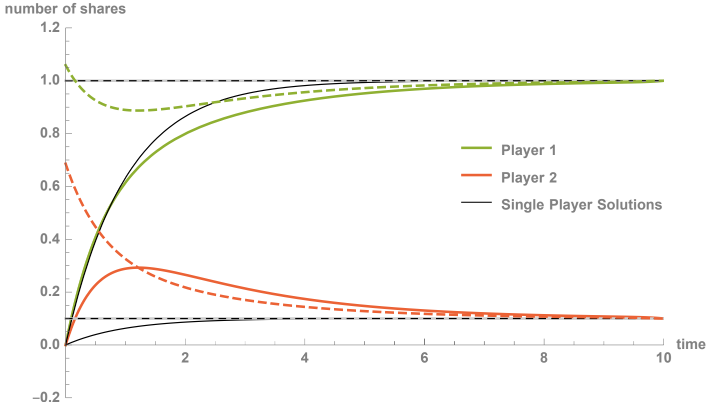
Player 1: $x^1 = 0$, $\xi_t^1 = 1 \cdot 1_{\{0 \le t < 5\}} + 2 \cdot 1_{\{5 \le t \le 10\}}$, $\Xi_T^1 = 2$.
Player 2 ("predator"): $x^2 = 0$, $\xi_t^2 \equiv 0$, $\Xi_T^2 = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0$.

Constant Inventory Targets



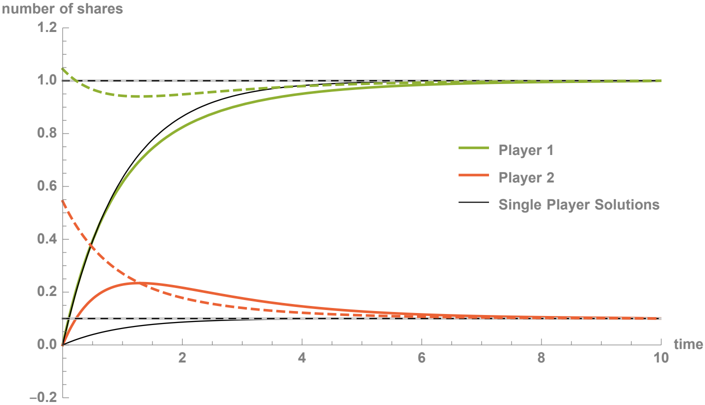
Player 1: $x^1 = 0$, $\xi^1_t \equiv 1$, $\Xi^1_T = 1$.
Player 2: $x^2 = 0$, $\xi^2_t \equiv 0.1$, $\Xi^2_T = 0.1$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Constant Inventory Targets



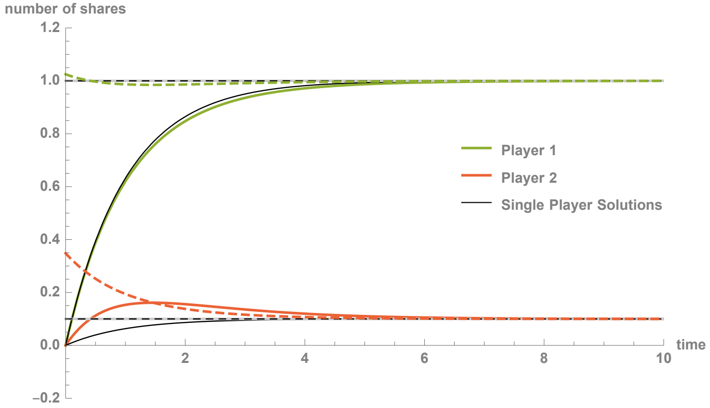
Player 1: $x^1 = 0$, $\xi_t^1 \equiv 1$, $\Xi_T^1 = 1$.
Player 2: $x^2 = 0$, $\xi_t^2 \equiv 0.1$, $\Xi_T^2 = 0.1$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Constant Inventory Targets



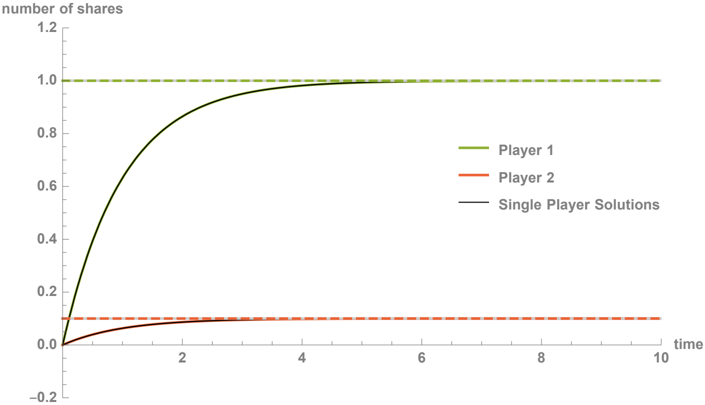
Player 1: $x^1 = 0$, $\xi^1_t \equiv 1$, $\Xi^1_T = 1$.
Player 2: $x^2 = 0$, $\xi^2_t \equiv 0.1$, $\Xi^2_T = 0.1$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1.5$.

Constant Inventory Targets



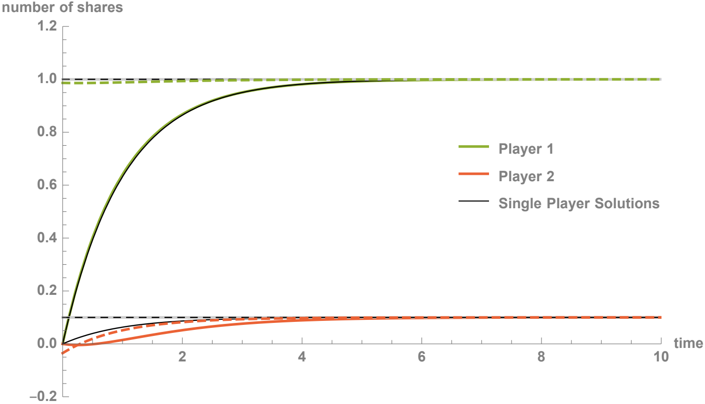
Player 1: $x^1 = 0$, $\xi_t^1 \equiv 1$, $\Xi_T^1 = 1$.
Player 2: $x^2 = 0$, $\xi_t^2 \equiv 0.1$, $\Xi_T^2 = 0.1$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1$.

Constant Inventory Targets



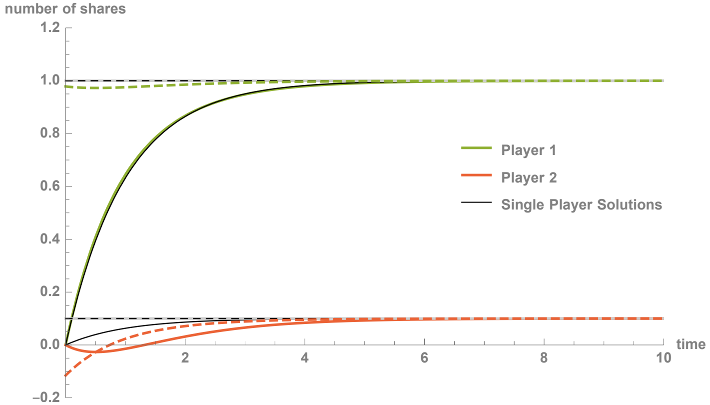
Player 1: $x^1 = 0$, $\xi_t^1 \equiv 1$, $\Xi_T^1 = 1$.
Player 2: $x^2 = 0$, $\xi_t^2 \equiv 0.1$, $\Xi_T^2 = 0.1$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.5$.

Constant Inventory Targets



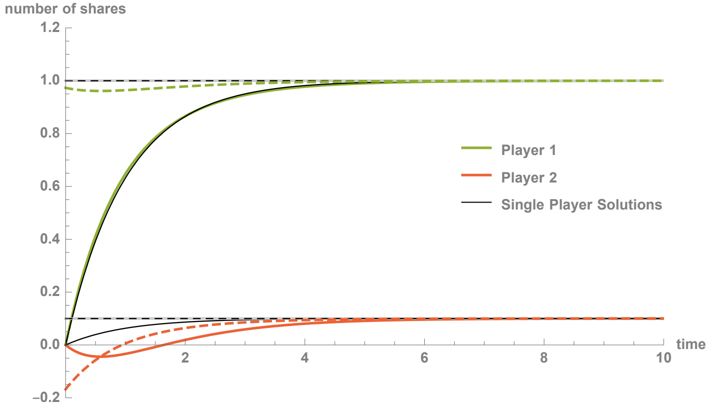
Player 1: $x^1 = 0$, $\xi_t^1 \equiv 1$, $\Xi_T^1 = 1$.
Player 2: $x^2 = 0$, $\xi_t^2 \equiv 0.1$, $\Xi_T^2 = 0.1$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.25$.

Constant Inventory Targets



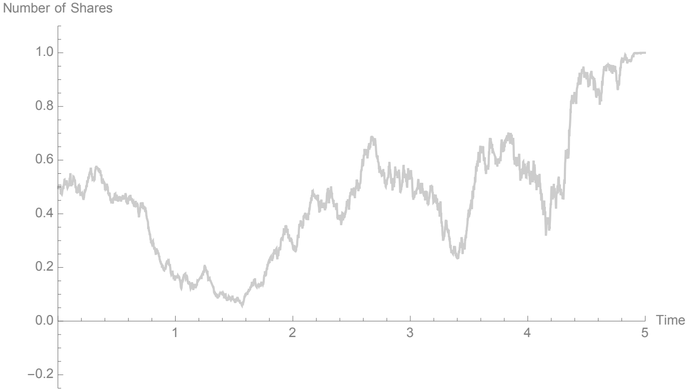
Player 1: $x^1 = 0$, $\xi_t^1 \equiv 1$, $\Xi_T^1 = 1$.
Player 2: $x^2 = 0$, $\xi_t^2 \equiv 0.1$, $\Xi_T^2 = 0.1$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.1$.

Constant Inventory Targets



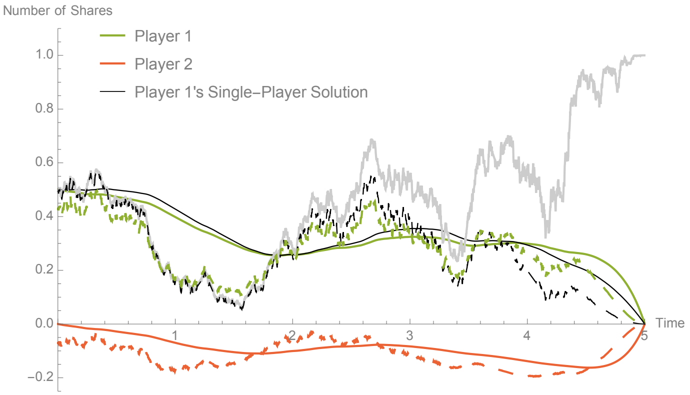
Player 1: $x^1 = 0$, $\xi_t^1 \equiv 1$, $\Xi_T^1 = 1$.
Player 2: $x^2 = 0$, $\xi_t^2 \equiv 0.1$, $\Xi_T^2 = 0.1$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0$.

Running after the delta



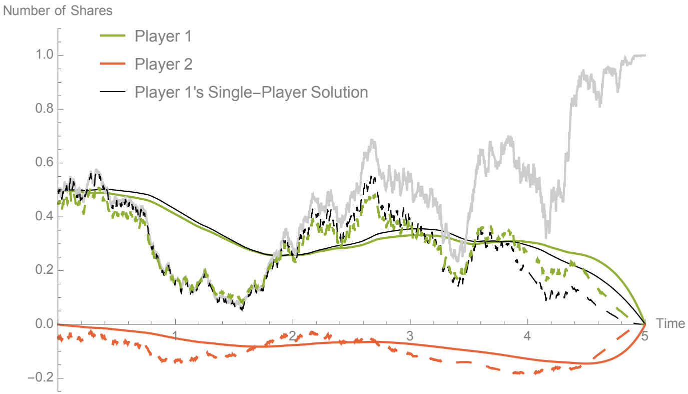
Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2 ("predator"): $x^2 = 0$, $\xi^2_t \equiv 0$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Running after the delta



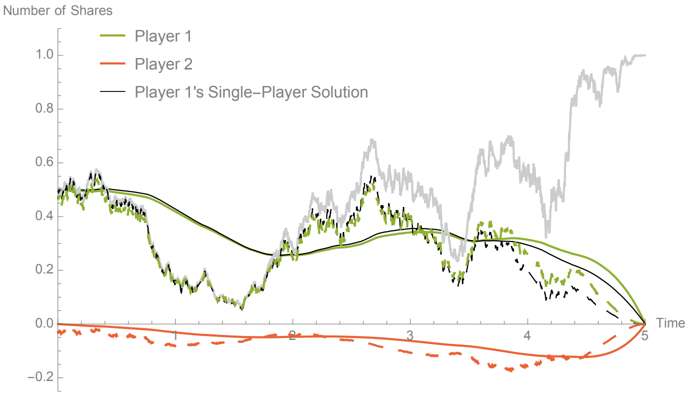
Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2 ("predator"): $x^2 = 0$, $\xi^2_t \equiv 0$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Running after the delta



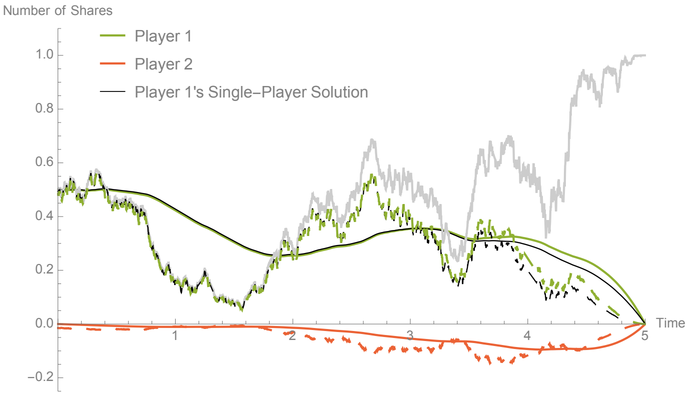
Player 1: $x^1 = 1/2$, $\xi_t^1 =$ delta-hedge, $\Xi_T^1 = 0$.
Player 2 ("predator"): $x^2 = 0$, $\xi_t^2 \equiv 0$, $\Xi_T^2 = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1.5$.

Running after the delta



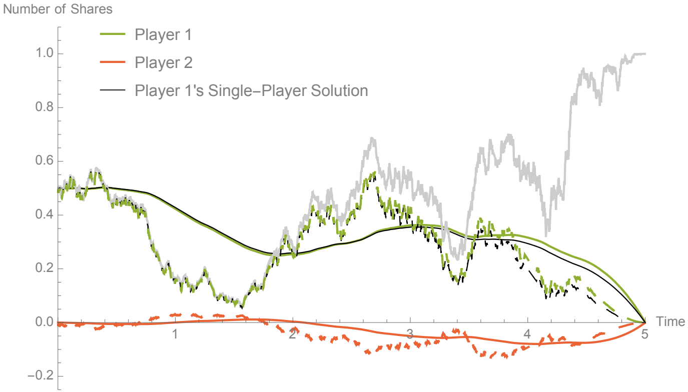
Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2 ("predator"): $x^2 = 0$, $\xi^2_t \equiv 0$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1.0$.

Running after the delta



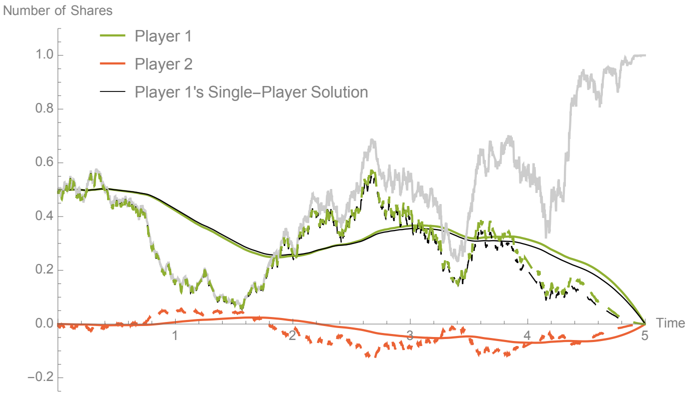
Player 1: $x^1 = 1/2$, $\xi_t^1 =$ delta-hedge, $\Xi_T^1 = 0$.
Player 2 ("predator"): $x^2 = 0$, $\xi_t^2 \equiv 0$, $\Xi_T^2 = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.5$.

Running after the delta



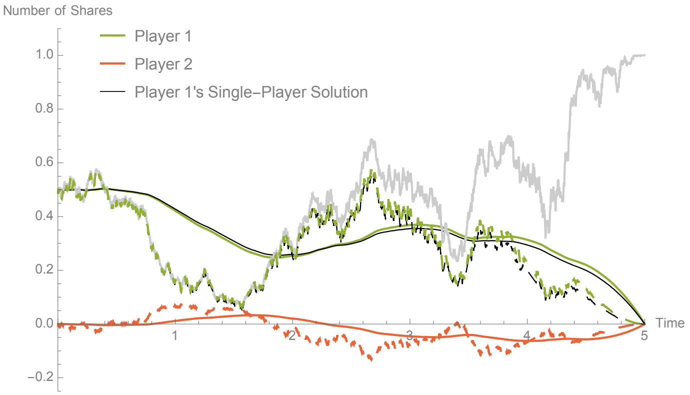
Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2 ("predator"): $x^2 = 0$, $\xi^2_t \equiv 0$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.25$.

Running after the delta



Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2 ("predator"): $x^2 = 0$, $\xi^2_t \equiv 0$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.1$.

Running after the delta

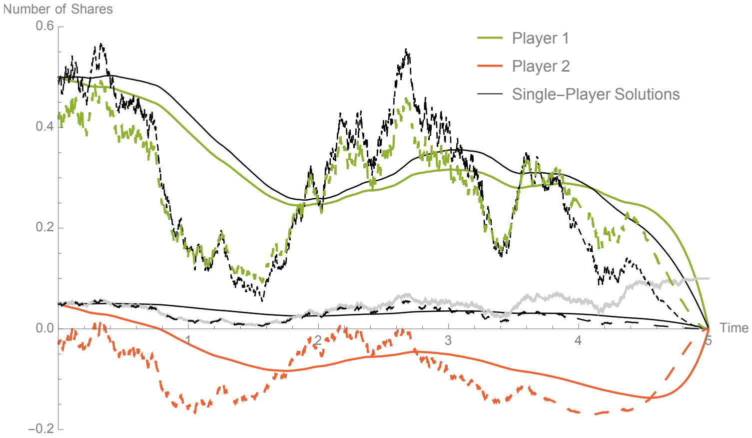


Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2 ("predator"): $x^2 = 0$, $\xi^2_t \equiv 0$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0$.

Running after the delta

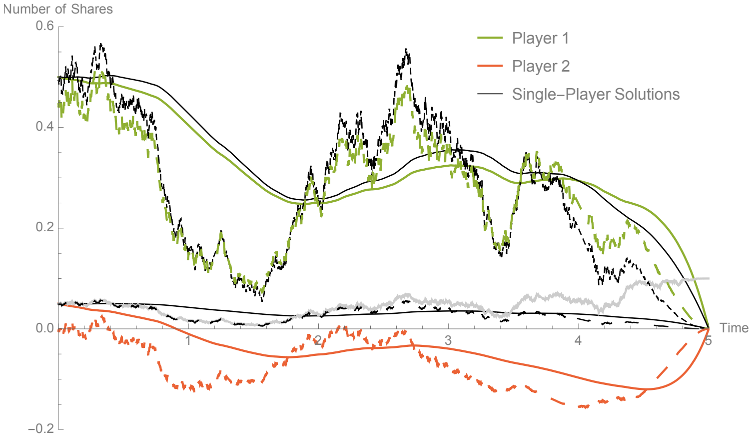
Player 1: $x^1 = 1/2$, $\xi_t^1 =$ delta-hedge, $\Xi_T^1 = 0$.
Player 2: $x^2 = 1/20$, $\xi_t^2 = 0.1 \cdot \xi_t^1$, $\Xi_T^2 = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Running after the delta



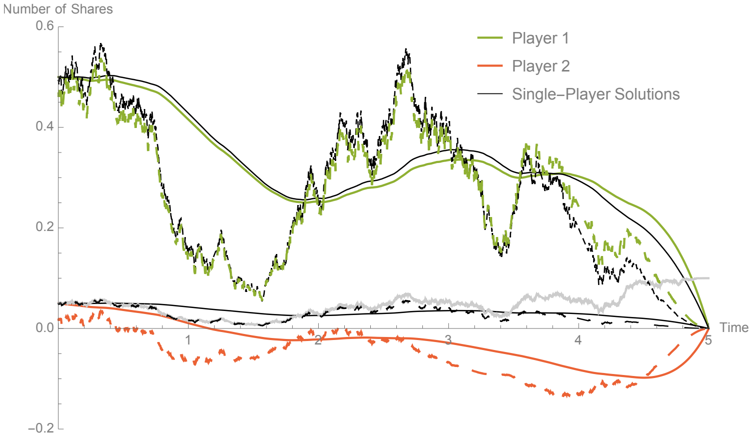
Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2: $x^2 = 1/20$, $\xi^2_t = 0.1 \cdot \xi^1_t$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Running after the delta



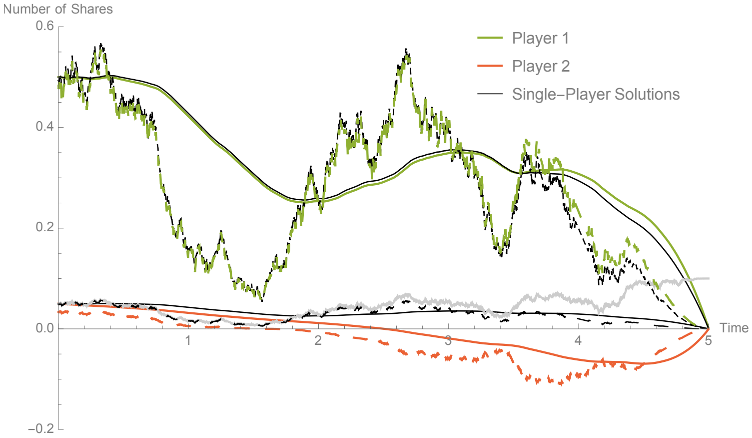
Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2: $x^2 = 1/20$, $\xi^2_t = 0.1 \cdot \xi^1_t$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1.5$.

Running after the delta



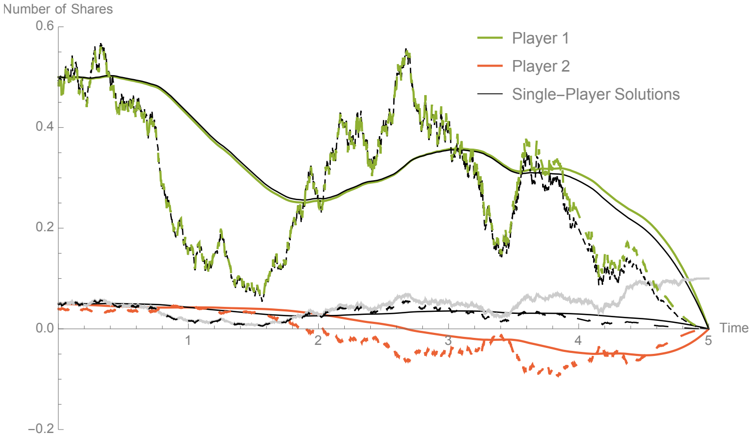
Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2: $x^2 = 1/20$, $\xi^2_t = 0.1 \cdot \xi^1_t$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1.0$.

Running after the delta



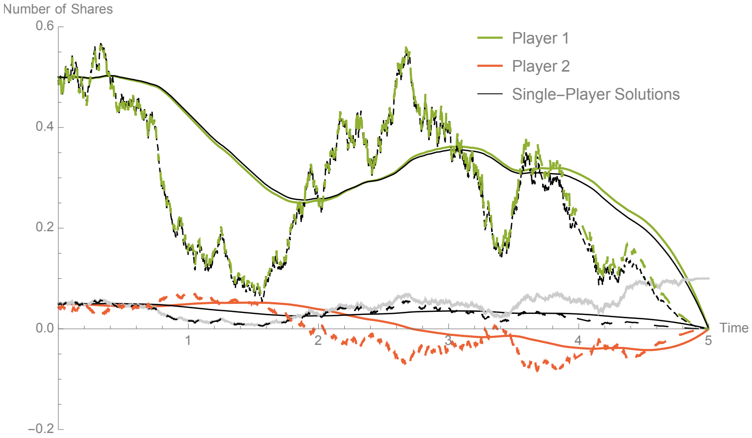
Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2: $x^2 = 1/20$, $\xi^2_t = 0.1 \cdot \xi^1_t$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.5$.

Running after the delta



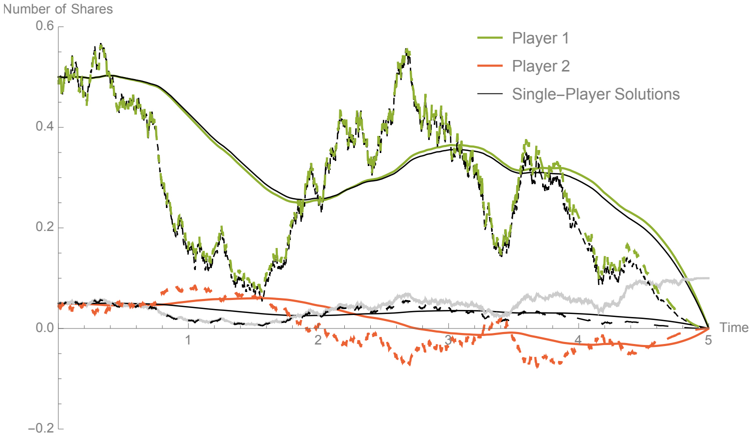
Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2: $x^2 = 1/20$, $\xi^2_t = 0.1 \cdot \xi^1_t$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.25$.

Running after the delta



Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2: $x^2 = 1/20$, $\xi^2_t = 0.1 \cdot \xi^1_t$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.1$.

Running after the delta

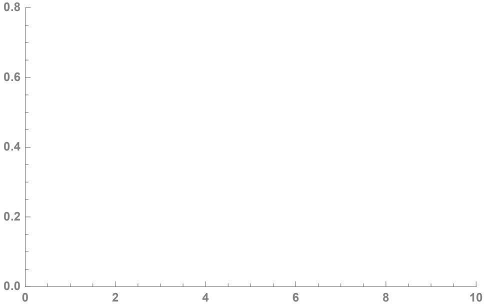


Player 1: $x^1 = 1/2$, $\xi^1_t =$ delta-hedge, $\Xi^1_T = 0$.
Player 2: $x^2 = 1/20$, $\xi^2_t = 0.1 \cdot \xi^1_t$, $\Xi^2_T = 0$.
Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0$.

Illustration: Weights

Recall:

$$\hat{\xi}^1_t = w^1_t \cdot \mathbb{E}[\Xi^1_T + \Xi^2_T \mid \mathscr{F}_t] + w^2_t \cdot \mathbb{E}[\Xi^1_T - \Xi^2_T \mid \mathscr{F}_t]$$
$$+ w^3_t \cdot \mathbb{E}\left[\int_t^T (\xi^1_s + \xi^2_s) \cdot K^1(t,s)\, ds \,\middle|\, \mathscr{F}_t \right] + w^4_t \cdot \mathbb{E}\left[\int_t^T (\xi^1_s - \xi^2_s) \cdot K^2(t,s)\, ds \,\middle|\, \mathscr{F}_t \right]$$

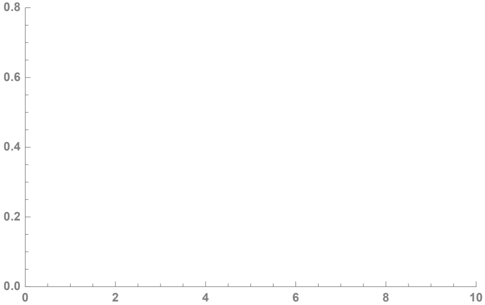


Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Illustration: Weights

Recall:

$$\hat{\xi}_t^1 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathscr{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathscr{F}_t]$$
$$+ w_t^3 \cdot \mathbb{E}\left[\int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t,s)\, ds \,\middle|\, \mathscr{F}_t\right] + w_t^4 \cdot \mathbb{E}\left[\int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t,s)\, ds \,\middle|\, \mathscr{F}_t\right]$$

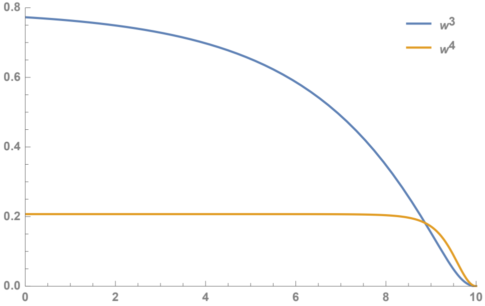


Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Illustration: Weights

Recall:

$$\hat{\xi}_t^1 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathscr{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathscr{F}_t]$$
$$+ w_t^3 \cdot \mathbb{E}\left[\int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t,s)\, ds \,\middle|\, \mathscr{F}_t\right] + w_t^4 \cdot \mathbb{E}\left[\int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t,s)\, ds \,\middle|\, \mathscr{F}_t\right]$$

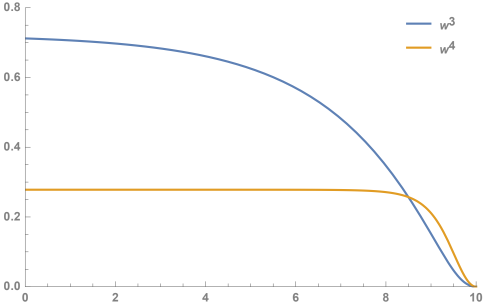


Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 2$.

Illustration: Weights

Recall:

$$\hat{\xi}_t^1 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathscr{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathscr{F}_t]$$
$$+ w_t^3 \cdot \mathbb{E}\left[\int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t,s)\, ds \,\middle|\, \mathscr{F}_t\right] + w_t^4 \cdot \mathbb{E}\left[\int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t,s)\, ds \,\middle|\, \mathscr{F}_t\right]$$

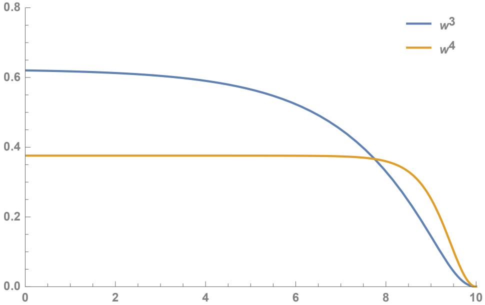


Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1.5$.

Illustration: Weights

Recall:

$$\hat{\xi}^1_t = w^1_t \cdot \mathbb{E}[\Xi^1_T + \Xi^2_T \mid \mathscr{F}_t] + w^2_t \cdot \mathbb{E}[\Xi^1_T - \Xi^2_T \mid \mathscr{F}_t]$$
$$+ w^3_t \cdot \mathbb{E}\left[\int_t^T (\xi^1_s + \xi^2_s) \cdot K^1(t,s)\, ds \,\middle|\, \mathscr{F}_t\right] + w^4_t \cdot \mathbb{E}\left[\int_t^T (\xi^1_s - \xi^2_s) \cdot K^2(t,s)\, ds \,\middle|\, \mathscr{F}_t\right]$$

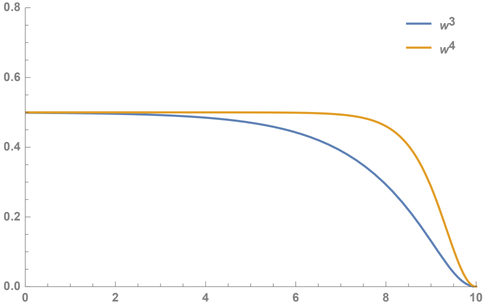


Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1$.

Illustration: Weights

Recall:

$$\hat{\xi}_t^1 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathscr{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathscr{F}_t] + w_t^3 \cdot \mathbb{E}\left[\int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t,s)\, ds \,\middle|\, \mathscr{F}_t\right] + w_t^4 \cdot \mathbb{E}\left[\int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t,s)\, ds \,\middle|\, \mathscr{F}_t\right]$$



Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.5$.

Illustration: Weights

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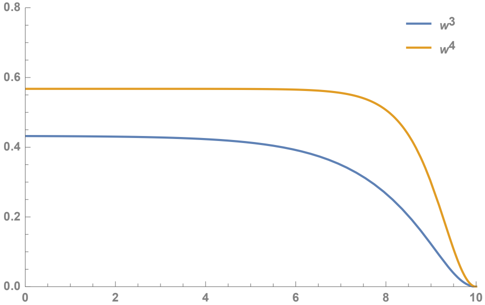

Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0.25$.

Illustration: Weights

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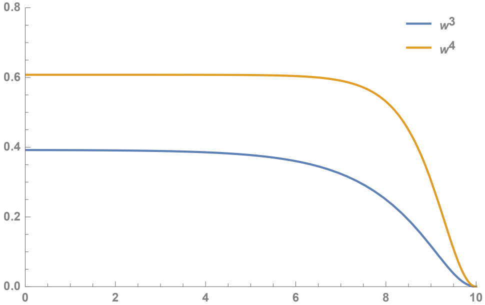

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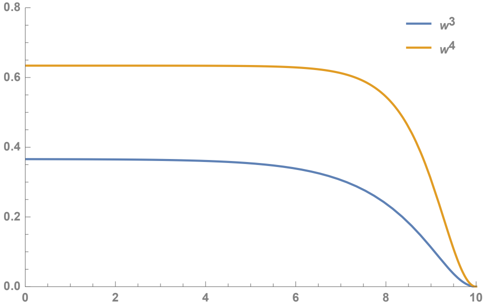

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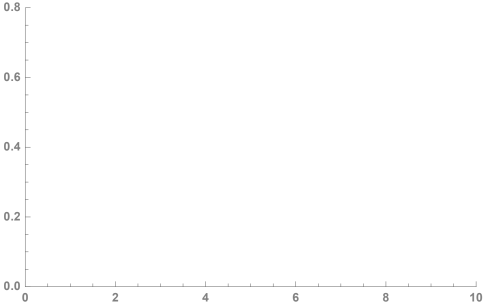

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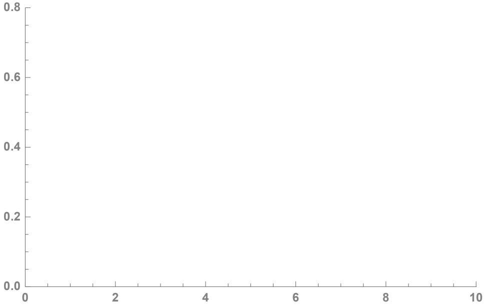

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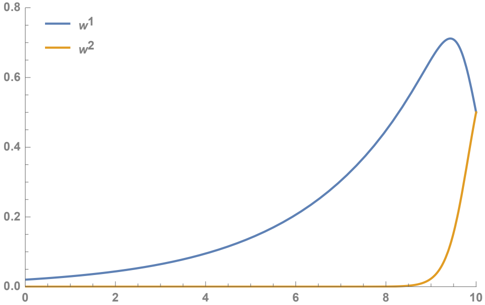

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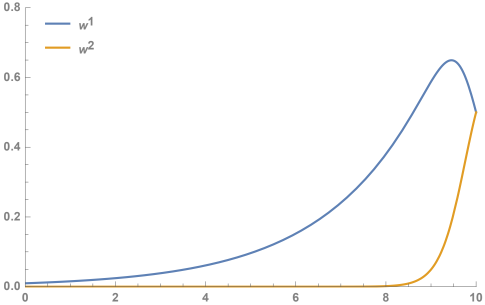

Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 1.5$.

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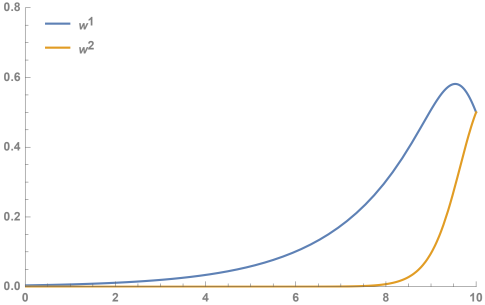

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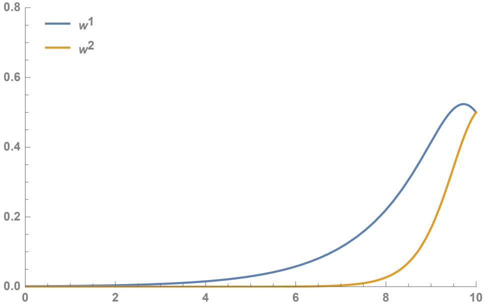

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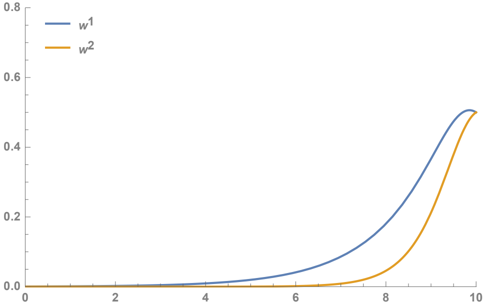

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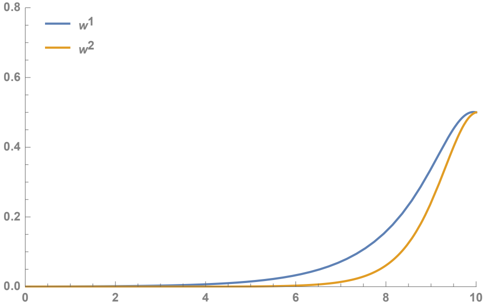

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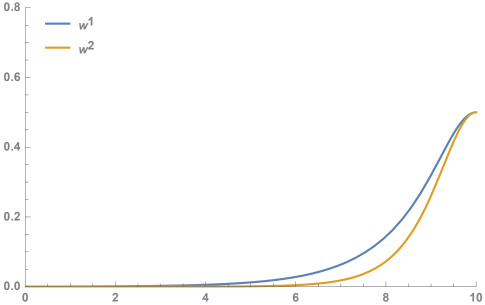

Parameters: $\sigma = 1$, $\lambda = 1$, $\gamma = 0$.

Sketch of Proof

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For fixed $\tilde{\alpha}^2 \in \mathscr{A}^2$, $\tilde{\alpha}^1 \in \mathscr{A}^1$ the mappings

$$\alpha^1 \mapsto J^1(\alpha^1, \tilde{\alpha}^2) \qquad \alpha^2 \mapsto J^2(\tilde{\alpha}^1, \alpha^2)$$

are strictly convex (over convex admissible sets $\mathscr{A}^1$, $\mathscr{A}^2$).

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Lemma

There exists at most one Nash equilibrium.

Sketch of Proof

Given two controls $\tilde{\alpha}^1 \in \mathscr{A}^1$, $\tilde{\alpha}^2 \in \mathscr{A}^2$ we can introduce the Gâteaux derivatives of the mappings:

$$\alpha^1 \mapsto J^1(\alpha^1, \tilde{\alpha}^2) \qquad \alpha^2 \mapsto J^2(\tilde{\alpha}^1, \alpha^2)$$

at $\alpha^1 \in \mathscr{A}^1$ and $\alpha^2 \in \mathscr{A}^2$, respectively, in any (suitable) directions $\beta^1, \beta^2 \in \mathscr{A}^0$:

$$\langle \nabla_1 J^1(\alpha^1, \tilde{\alpha}^2), \beta^1 \rangle \triangleq \lim_{\varepsilon \to 0} \frac{J^1(\alpha^1 + \varepsilon\beta^1, \tilde{\alpha}^2) - J^1(\alpha^1, \tilde{\alpha}^2)}{\varepsilon},$$

$$\langle \nabla_2 J^2(\tilde{\alpha}^1, \alpha^2), \beta^2 \rangle \triangleq \lim_{\varepsilon \to 0} \frac{J^2(\tilde{\alpha}^1, \alpha^2 + \varepsilon\beta^2) - J^2(\tilde{\alpha}^1, \alpha^2)}{\varepsilon}.$$

Sketch of Proof

Lemma

For $\alpha^1 \in \mathscr{A}^1, \alpha^2 \in \mathscr{A}^2$ *we have*

$$\langle \nabla_1 J^1(\alpha^1, \alpha^2), \beta^1 \rangle$$
$$= \mathbb{E}\left[\int_0^T \beta_s^1 \left(\lambda \alpha_s^1 + \frac{\lambda}{2}\alpha_s^2 + \gamma(X_s^2 - x^2) + \int_s^T (X_t^1 - \xi_t^1)\sigma dt\right) ds\right]$$

and

$$\langle \nabla_2 J^2(\alpha^1, \alpha^2), \beta^2 \rangle$$
$$= \mathbb{E}\left[\int_0^T \beta_s^2 \left(\lambda \alpha_s^2 + \frac{\lambda}{2}\alpha_s^1 + \gamma(X_s^1 - x^1) + \int_s^T (X_t^2 - \xi_t^2)\sigma dt\right) ds\right]$$

for any $\beta^1, \beta^2 \in \mathscr{A}^0$.

Sketch of Proof

Lemma

Suppose that $(\hat{X}^1, \hat{X}^2)$ *with controls* $(\hat{\alpha}^1, \hat{\alpha}^2) \in \mathscr{A}^1 \times \mathscr{A}^2$ *solves following coupled forward backward SDE system*

$$\begin{cases} dX_t^1 = \alpha_t^1 dt, \qquad X_0^1 = x^1, \\ dX_t^2 = \alpha_t^2 dt, \qquad X_0^2 = x^2, \\ d\alpha_t^1 = \frac{\sigma}{\lambda}(X_t^1 - \xi_t^1)dt - \frac{\gamma}{\lambda}\alpha_t^2 dt - \frac{1}{2}d\alpha_t^2 + dM_t^1, \qquad X_T^1 = \Xi_T^1, \\ d\alpha_t^2 = \frac{\sigma}{\lambda}(X_t^2 - \xi_t^2)dt - \frac{\gamma}{\lambda}\alpha_t^1 dt - \frac{1}{2}d\alpha_t^1 + dM_t^2, \qquad X_T^2 = \Xi_T^2, \end{cases}$$

for two suitable square integrable martingales $(M_t^1)_{0\le t<T}$ *and* $(M_t^2)_{0\le t<T}$*. Then* $(\hat{\alpha}^1, \hat{\alpha}^2)$ *is a Nash equilibrium.*

Conclusion

- study competition of two strategic agents for liquidity in a financial market
- agents interact through common aggregated temporary and permanent price impact à la Almgren & Chriss
- resulting stochastic linear quadratic differential game with terminal state constraints allows for an explicitly available open loop Nash equilibrium
- closed-from solution reveals how the equilibrium strategies of the two players take into account the other agent's trading targets
- rich set of phenomena occurring in equilibrium: coexistence of cooperation and predation (depending on the ratio between temporary and permanent price impact)

Future Research

- opens door to study N-player stochastic differential game with mean field interaction through common price impact (Game with Major and Minor Players)
- study the Mean-Field limit $N \to +\infty$

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- different information structure (agents have private filtrations)

- incorporate transient price impact à la Obizhaeva & Wang

Reference

A Two-Player Price Impact Game
Preprint available on arXiv:1911.05122.

Hedging with Temporary Price Impact
with Peter Bank, H. Mete Soner
Mathematics and Financial Economics, 2017.

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Thank you very much!