A Term Structure Model for Dividends and Interest Rates

Damir Filipović
Joint work with Sander Willems

École Polytechnique Fédérale de Lausanne
Swiss Finance Institute

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Overview

1. Introduction
2. Polynomial Framework
3. Option Pricing
4. Linear Jump-Diffusion Model
5. Calibration
6. Extensions
A new market for dividend derivatives

- How can we trade dividends?
  - Synthetic replication.
  - Dividend swaps (OTC) or dividend futures (on exchange).
  - Latest innovations: single names, options, dividend-rates hybrids, ...

- Dividend derivative pricing.

- Asset pricing: term structure of equity risk premium.

- Interest rates: hybrid products, long maturity dividend claims.
## Notional Outstanding Dividend Swaps and Futures

<table>
<thead>
<tr>
<th>Underlying Index</th>
<th>Notional Amount Outstanding (U.S. Dollars Millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equity Index Future</td>
</tr>
<tr>
<td>EURO STOXX 50</td>
<td>137,717</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>320,964</td>
</tr>
<tr>
<td>Nikkei 225</td>
<td>23,924</td>
</tr>
<tr>
<td>FTSE 100*</td>
<td>59,616</td>
</tr>
</tbody>
</table>

**Figure:** Total notional outstanding as of June 2015. Source: Mixon and Onur (2016)
Notional Outstanding Dividend Swaps and Futures

**Figure:** Notional outstanding per expiry as of June 2015. Source: Mixon and Onur (2016)
Contribution of this paper

Term-structure model for dividends and interest rates with
- Closed-form prices for dividend futures/swaps, bonds, and dividend paying stocks.
- Moment-based approximations for a broad class of exotic payoffs.
- Positive dividends and possible seasonal behaviour.
- Flexible correlation between dividends and interest rates.
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Factor process

- Filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})\), with \(\mathbb{Q}\) risk-neutral pricing measure.

- Multivariate factor process \(X_t\) on \(E \subseteq \mathbb{R}^d\)

\[
dX_t = \kappa(\theta - X_t)dt + dM_t,
\]

for \(\kappa \in \mathbb{R}^{d \times d}\), \(\theta \in \mathbb{R}^d\), and martingale \(M_t\) such that \(X_t\) is a polynomial jump-diffusion, cfr. Filipović and Larsson (2017).

- Generator \(G\) maps polynomials to polynomials:

\[
\mathcal{GPol}_n(E) \subseteq \text{Pol}_n(E), \quad \forall n \in \mathbb{N},
\]

with \(\text{Pol}_n(E)\) space of polynomials on \(E\) of degree \(n\) or less.
PJD Moment Formula

- Fix polynomial basis for $\text{Pol}_n(E)$:
  \[ H_n(x) = (h_1(x), \ldots, h_{N_n}(x))^\top, \]
  with $N_n = \dim(\text{Pol}_n(E)) \leq \binom{n + d}{n}$.
- $\mathcal{G}$ restricts to a linear operator on $\text{Pol}_n(E)$
  \[ \mathcal{G}H_n(x) = G_n H_n(x). \]
- First order linear ODE for $s \mapsto \mathbb{E}_t[H_n(X_s)]$
  \[ \mathbb{E}_t[H_n(X_T)] = H_n(X_t) + G_n \int_t^T \mathbb{E}_t[H_n(X_s)] \, ds \]
- Solving ODE gives for all $t \leq T$
  \[ \mathbb{E}_t[H_n(X_T)] = e^{G_n(T-t)}H_n(X_t). \]
Dividend Futures

- **Instantaneous dividend rate:**
  
  \[ D_t = p^\top H_1(X_t), \]
  
  for \( p \in \mathbb{R}^{d+1} \) such that \( p^\top H_1(x) \geq 0 \) for all \( x \in E \).

- **Linear dividend futures price:**
  
  \[
  D_{fut}(t, T_1, T_2) = \mathbb{E}_t \left[ \int_{T_1}^{T_2} D_s \, ds \right] = p^\top \int_{T_1}^{T_2} e^{G_1(s-t)} \, ds \, H_1(X_t).
  \]

- E.g., if \( H_1(x) = (1, x^\top)^\top \), then
  
  \[
  G_1 = \begin{bmatrix} 0 & 0 \\ \kappa \theta & -\kappa \end{bmatrix}.
  \]
Interest Rates

- Risk-neutral discount factor:
  \[ \zeta_t = \zeta_0 e^{-\int_0^t r_s \, ds}, \quad t \geq 0, \]

  where \( r_t \) denotes the short rate.

- Directly specify \( \zeta_t \):
  \[ \zeta_t := e^{-\gamma t} q^\top H_1(X_t), \]

  for \( \gamma \in \mathbb{R} \) and \( q \in \mathbb{R}^{d+1} \) such that \( \zeta_t \) is a positive and absolutely continuous process.

- Implied short rate:
  \[ r_t = \gamma - \frac{q^\top G_1 H_1(X_t)}{q^\top H_1(X_t)}. \]

Bond Prices

- Time- \( t \) price of zero-coupon bond maturing at \( T \geq t \):

\[
P(t, T) = \frac{1}{\zeta_t} \mathbb{E}_t[\zeta_T] = e^{-\gamma(T-t)} \frac{q^\top e^{G_1(T-t)H_1(X_t)}}{q^\top H_1(X_t)}.
\]

- Linear discounted bond price \( \zeta_t P(t, T) \).

- If \( \Re(\text{eig}(\kappa)) > 0 \):

\[
\lim_{T \to \infty} - \frac{\log(P(t, T))}{T - t} = \gamma.
\]
Dividend Paying Stock

- Fundamental stock price

\[ S_t^* = \frac{1}{\zeta_t} \mathbb{E}_t \left[ \int_t^\infty \zeta_s D_s \, ds \right]. \]

- If \( \Re(\text{eig}(G_2)) < \gamma \), then

\[ S_t^* = \bar{v}^\top (\gamma \text{Id} - G_2)^{-1} H_2(X_t) < \infty, \]

- Quadratic discounted fundamental stock price \( \zeta_t S_t^* \).
- Define arbitrage-free stock price as

\[ S_t = \frac{L_t}{\zeta_t} + S_t^*, \]

for some nonnegative (local) martingale \( L_t \), cfr. Buehler (2015), Jarrow et al. (2007, 2010).
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Maximum Entropy Moment Matching

- Pricing problem:
  \[
  \pi_t = \mathbb{E}_t \left[ F\left(g(X_T)\right) \right],
  \]
  with \( g \in \text{Pol}_n(E) \) and \( F: \mathbb{R} \to \mathbb{R} \).

- Goal: Approximate density of \( g(X_T) \) based on moments.

- Maximize Boltzmann-Shannon entropy:
  \[
  \max_f - \int f(x) \ln f(x) \, dx
  \]
  s.t. \( \int x^n f(x) \, dx = M_n, \quad n = 0, \ldots, N \)

- Unique solution:
  \[
  f(x) = \exp \left( - \sum_{i=0}^{N} \lambda_i x^i \right)
  \]
Option Pricing

- Swaptions:

\[
\pi_{t}^{swpt} = \frac{1}{\zeta_{t}} \mathbb{E}_{t} \left[ \left( \zeta_{T} \pi_{T}^{swap} \right)^{+} \right]
\]

\[
= \frac{1}{\zeta_{t}} \mathbb{E}_{t} \left[ \left( \zeta_{T} - \zeta_{T} P(T, T_{n}) - \delta K \sum_{k=1}^{n} \zeta_{T} P(T, T_{k}) \right)^{+} \right]
\]

- Stock options

\[
\pi_{t}^{stock} = \frac{1}{\zeta_{t}} \mathbb{E}_{t} \left[ \left( \zeta_{T} S_{T} - \zeta_{T} K \right)^{+} \right]
\]

\[
= \frac{1}{\zeta_{t}} \mathbb{E}_{t} \left[ \left( L_{T} + \zeta_{T} S_{T}^{*} - \zeta_{T} K \right)^{+} \right]
\]
Option Pricing

- Dividend options

\[ \pi_t^{\text{div}} = \mathbb{E}_t \left[ \left( \int_{T_0}^{T_1} D_s \, ds - K \right)^+ \right] \]

\[ = \mathbb{E}_t \left[ (I_{T_1} - I_{T_0} - K)^+ \right], \]

with \( I_T = \int_0^T D_s \, ds \).

- Augment factor process: \((I_t, X_t)\) is PJD.

- Compute moments \( \mathbb{E}_t \left[ (I_{T_1} - I_{T_0})^n \right] \) using law of iterated expectations.
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Linear Jump-Diffusion

- Specify martingale part $dM_t$ as

$$dX_t = \kappa(\theta - X_t)\,dt + \text{diag}(X_t^-)(\Sigma\,dB_t + dJ_t)$$

- $B_t$: standard $d$-dimensional Brownian motion, $\Sigma \in \mathbb{R}^{d \times d}$ lower triangular with $\Sigma_{ii} > 0$

- $J_t$: compensated compound Poisson process, jump intensity $\xi$ and i.i.d. jump amplitudes $e^Z - 1$, $Z \sim \mathcal{N}(\mu_J, \Sigma_J)$.

- Unique positive solution if $\kappa\theta \geq 0$ and if $\kappa_{ij} \leq 0$ for $i \neq j$.

- Allows for flexible instantaneous correlation between factors through $\Sigma$.

- Moments in closed-form (PJD).
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Model Specification

- Five factor model $X_t = (X_{0t}^I, X_{1t}^I, X_{2t}^I, X_{1t}^D, X_{2t}^D)^\top$

- Rate factors $X_t^I = (X_{0t}^I, X_{1t}^I, X_{2t}^I)^\top$:

$$dX_t^I = \begin{bmatrix} \kappa_0^I & -\kappa_0^I & 0 \\ 0 & \kappa_1^I & -\kappa_1^I \\ 0 & 0 & \kappa_2^I \end{bmatrix} \begin{bmatrix} \theta^I - X_{0t}^l \\ \theta^I - X_{1t}^l \\ \theta^I - X_{2t}^l \end{bmatrix} dt + \text{diag}(X_t^I) \begin{bmatrix} 0 & 0 \\ \Sigma_{11}^I & 0 \\ \Sigma_{21}^I & \Sigma_{22}^I \end{bmatrix} \begin{bmatrix} dB_{1t} \\ dB_{2t} \end{bmatrix},$$

with $\zeta_t = e^{-\gamma t} X_{0t}^l$, $\theta^l = 1$, and $\gamma = 4.2\%$.

- Dividend factors $X_t^D = (X_{1t}^D, X_{2t}^D)^\top$:

$$dX_t^D = \begin{bmatrix} \kappa_1^D & -\kappa_1^D \\ 0 & \kappa_2^D \end{bmatrix} \begin{bmatrix} \theta^D - X_{1t}^D \\ \theta^D - X_{2t}^D \end{bmatrix} dt + \text{diag}(X_t^D) \begin{bmatrix} 0 \\ \Sigma_{11}^D \\ \Sigma_{21}^D \end{bmatrix} \begin{bmatrix} dB_{3t} \\ dB_{4t} \end{bmatrix} + \begin{bmatrix} dB_{1t} \\ dB_{2t} \end{bmatrix},$$

with $D_t = X_{1t}^D$.

- Explicit restrictions on parameters s.t. $S_t^* < \infty$. 

Calibration
Jump-diffusive stock price bubble:

\[ dL_t = L_t - (\sigma^L dB^L_t + dJ^L_t) \]

Assume \( L_t \) independent of \( X_t \)

\[
\pi^\text{stock}_t = \frac{1}{\zeta_t} \mathbb{E}_t \left[ (L_T + \zeta_T S_T^* - \zeta_T K)^+ \right]
\]

\[
= \frac{1}{\zeta_t} \mathbb{E}_t \left[ C^M_t (L_t, \tilde{K}(X_T)) \right],
\]

where \( C^M_t (L_t, \tilde{K}(X_T)) \) is the Merton (1976) option price with spot \( L_t \) and strike \( \tilde{K}(X_T) = \zeta_T K - \zeta_T S_T^* \).

Dependence between \( L_t \) and \( X_t \) is possible as long as \((X_t, L_t)\) remains jointly a PJD (at the cost of an increased dimension).
Data

Calibration date: December 21 2015.

- **Euribor swaps (7)**
  Tenors: 1, 2, 3, 5, 7, 10, 15y.

- **Euribor swaptions (12)**
  Expiries: 3m, 6m, 1y
  Tenors: 3, 5, 10, 15y
  Moneyness: ATM

- **Eurostoxx 50 index dividend futures (10)**
  Expiry: 1-10 y

- **Eurostoxx 50 index dividend options (21)**
  Expiry: 2, 3, 4y
  Moneyness: 0.9, 0.95, 0.975, 1, 1.025, 1.05, 1.1

- **Eurostoxx 50 index options (24)**
  Expiry: 3m, 6m, 1y
  Moneyness: 0.8, 0.9, 0.95, 0.975, 1, 1.025, 1.05, 1.1
Swaps and Dividend Futures (21/12/2015)

(a) Euribor swap rates

(b) Eurostoxx 50 dividend futures
Euribor Swaptions (21/12/2015)

(a) 3 month maturity

(b) 6 month maturity

(c) 1 year maturity
Eurostoxx 50 Dividend Futures Options (21/12/2015)

(a) Dec 2016-2017

(b) Dec 2017-2018

(c) Dec 2018-2019
Index Options (21/12/2015)

(a) 3 month maturity

(b) 6 month maturity

(c) 1 year maturity
Moments and Option Prices

(a) Swaption, 3m expiry
(b) Index option, 3m expiry
(c) Dividend option, 2y expiry
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Dividend Seasonality

Figure: Monthly dividend payments by Eurostoxx 50 constituents (in index points) from October 2009 until October 2016. Source: Eurostoxx 50 DVP index, Bloomberg.
Dividend Seasonality

- Standard choice to model annual cycles:

\[ \delta(t) = \rho_0 + \rho^\top \Gamma(t), \quad \Gamma(t) = \begin{bmatrix} \sin(2\pi t) \\ \cos(2\pi t) \\ \vdots \\ \sin(2\pi K t) \\ \cos(2\pi K t) \end{bmatrix}. \]

- Remark, \( \Gamma(t) \) is the solution of a linear ODE:

\[ d\Gamma(t) = \text{blkdiag} \left( \begin{bmatrix} 0 & 2\pi \\ -2\pi & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 2\pi K \\ -2\pi K & 0 \end{bmatrix} \right) \Gamma(t) dt. \]

→ We can add \( \Gamma \) to the state vector!

- For example:

\[ dX_t = \kappa(\delta(t) - X_t) dt + dM_t \]
Dividend Swaps

- Dividend swap/forward price:

\[
D_{\text{swap}}(t, T_1, T_2) = \frac{1}{P(t, T_2)} \frac{1}{\zeta_t} \mathbb{E}_t \left[ \zeta_{T_2} \int_{T_1}^{T_2} D_s \, ds \right] \\
= D_{\text{fut}}(t, T_1, T_2) + \frac{\text{Cov}_t \left[ \zeta_{T_2}, \int_{T_1}^{T_2} D_s \, ds \right]}{P(t, T_2) \zeta_t}.
\]

- In polynomial framework:

\[
D_{\text{swap}}(t, T_1, T_2) = \frac{w(t, T_1, T_2)^\top H_2(X_t)}{q^\top e^{G_1(T_2-t)} H_1(X_t)},
\]

with \(w(t, T_1, T_2) = \int_{T_1}^{T_2} q^\top e^{G_1(T_2-s)} Q e^{G_2(s-t)} \, ds\) and \(QH_2(x) = H_1(x)H_1(x)^\top p\).
Conclusion

- Joint term-structure model for dividends and interest rates.
- Explicit prices for dividend futures/swaps, bonds, and dividend paying stock.
- Moment-based approximations for (path dependent) option prices using principle of maximum entropy.
- Future work:
  - Time-series estimation of $S_t^*$.
  - SVJ model for bubble component.
  - Single-stock framework with credit risk.

https://ssrn.com/abstract=3016310
Thank you for your attention!
References I


References II


