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Determinants of Levered Portfolio Performance

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The cumulative return to a levered strategy is determined by five elements that fit together in a simple and useful formula. A previously undocumented element is the covariance between leverage and excess return to the fully invested source portfolio underlying the strategy. In an empirical study of volatility-targeting strategies over the 84-year period 1929–2013, this covariance accounted for a reduction in return that substantially diminished the Sharpe ratio in all cases.

Hedge funds, real estate investment trusts, and many other investment vehicles routinely use leverage. Even among the most conservative and highly regulated investors, such as US public pension funds, the use of levered investment strategies is widespread and growing (see, for example, Kozlowski 2013). In the period since the global financial crisis, strategies that explicitly lever holdings of publicly traded securities, such as risk parity strategies, have emerged as candidates for these investment portfolios.[1]

Five elements determine the cumulative return to a levered strategy, and they fit together in a simple and useful formula. Looking backward, the formula we developed can be used to attribute performance. Looking forward, an investor can populate the formula with his or her forecasts of the five determinants to generate a forecast for return to the levered strategy.

A levered strategy begins with a fully invested source portfolio, such as unlevered risk parity, unlevered minimum variance, or unlevered bonds. The source portfolio is then levered according to a rule. The most common leverage rules target volatility: They estimate the current volatility of the source portfolio and then choose leverage so that the estimated volatility of the levered strategy matches the target. Because the source portfolio typically exhibits variable volatility, volatility targeting requires *dynamic* leverage, even if the volatility target is *fixed*.

Much of our intuition about levered strategies comes from single-period models. In a single-period model, the return of the levered strategy is determined by the return of the source portfolio, the leverage, and the financing cost associated with the leverage. By definition, leverage is constant; there is no trade and, hence, no trading costs—and no compounding to take into account. Because volatility targeting requires dynamic leverage, however, we need to consider multiple periods.

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A Simple Two-Period Example

Assume that the source portfolio earns a 10% arithmetic return in Period 1 and a –10% arithmetic return in Period 2. We invest \$100.00, which is worth \$110.00 at the end of Period 1 and \$99.00 at the end of Period 2, as shown in **Table 1**. The average of the arithmetic return over the two periods is zero, but the cumulative return of the source portfolio over the two periods is

$$\frac{99-100}{100}=-0.01=-1.00\%.$$

The average arithmetic return of the source portfolio must be corrected for compounding. As noted by Booth and Fama (1992) and discussed in Appendix A, the correction subtracts half the variance of arithmetic return each period; we call this correction the "variance drag." Note that the variance of the arithmetic return is

$$\frac{\left(0.1-0\right)^2+\left(-0.1-0\right)^2}{2}=0.01=1.00\%.$$

If we subtract half the variance from the arithmetic return each period, we get a total return of

$$0.1-\frac{0.01}{2}+\left(-0.1\right)-\frac{0.01}{2}=-0.01=-1.00\%,$$

which matches the actual cumulative return over the two periods.

Now, consider a levered strategy. Suppose for simplicity that leverage can be financed at the risk-free rate, which happens to be zero. We initially invest \$100.00. Suppose we target a fixed volatility of 12% per period and our estimate of the source portfolio volatility is 6% at the beginning of the first period and 4% at the beginning of the second period. Thus, we choose leverage $\lambda_1 = 12/6 = 2$ in the first period and $\lambda_2 = 12/4 = 3$ in the second period.

If we tried to compress this into a one-period analysis, we might assume the return would be similar to that of a strategy with fixed leverage $\bar{\lambda} = (\lambda_1 + \lambda_2)/2 = 2.5$ because it is the average leverage over the two periods. Thus, we might expect to earn about 2.5 times the average arithmetic return of the source portfolio, or 2.5 × 0.00% = 0.00%. If we were a little more sophisticated and were to take compounding into account, we might expect to earn 2.5 × –1.00% = –2.50%. However, both these answers are wrong, even for the case of fixed leverage, and they are particularly wrong for the case of dynamic leverage.

Consider a fixed-leverage strategy that uses leverage of 2.5 in both periods. As noted in Table 1, we hold assets of $250.00 in the source portfolio (financed by our initial $100.00 plus $150.00 in debt) at the beginning of the first period. At the end of the first period, our assets are worth $275.00 and our debt is $150.00, so the value of the levered strategy is $125.00. Even though we want to maintain fixed leverage, we need to rebalance. We hold assets of $125.00 × 2.5 = $312.50 in the source portfolio. We must borrow, increasing our debt to $312.50 – 125.00 = $187.50 to finance the position. At the end of the second period, our assets in the source portfolio are worth $312.50 × 0.9 = $281.25, so the value of the levered strategy is $281.25 – 187.50 = $93.75; our cumulative return over the two periods is (93.75 – 100)/100 = –0.0625, a loss of 6.25%. The variance of the levered strategy return is

$$\frac{(0.25-0)^2 + (-0.25-0)^2}{2} = 0.0625 = 6.25\%.$$

The correction for compounding (variance drag) is 6.25%/2 per period; over two periods, the result is –6.25%, exactly matching the realized return. Note that the variance drag is *quadratic* in leverage, so constant leverage of 2.5 increases the variance drag by a factor of 6.25.

With this more sophisticated understanding of the quadratic nature of the variance drag, we might expect the dynamically levered strategy to have a cumulative return of about –6.25%. But that answer is also wrong. As shown in Table 1, the dynamically levered strategy holds assets of $200.00 in the source portfolio at the beginning of the first period, financed by our initial $100.00 and debt of $100.00. At the end of the first period, the assets are worth a total of $220.00 and the debt is still $100.00, so the value of the levered strategy is $120.00. We rebalance to achieve the prescribed leverage. Because the levered strategy calls for leverage of $\lambda_2 = 3$, we borrow an additional $140.00, for total debt of $240.00, and we hold assets of $360.00 in the source portfolio. We incur trading costs, which for simplicity we assume to be zero. At the end of the second period, our shares of the source portfolio are worth $360.00 × 0.9 = $324.00. Because we owe $240.00, our equity is $84.00. The cumulative return to the levered strategy over the two periods is (84 – 100)/100 = –0.16 = –16.00%. Rather than breaking even, or losing

Table 1. Returns of Source Portfolio and Levered Strategies in the Two-Period Example

Period	Source Return, r^S (%)	Assets, A ($)	Debt, D ($)	Strategy Value, $A - D$ ($)
Source portfolio				
Beginning of 1		100.00	0.00	100.00
End of 1	10	110.00	0.00	110.00
End of 2	–10	99.00	0.00	99.00
Fixed-leverage strategy				
Beginning of 1		250.00	150.00	100.00
End of 1	10	275.00	150.00	125.00
End of 1′		312.50	187.50	125.00
End of 2	–10	281.25	187.50	93.75
Dynamically levered strategy				
Beginning of 1		200.00	100.00	100.00
End of 1	10	220.00	100.00	120.00
End of 1′		360.00	240.00	120.00
End of 2	–10	324.00	240.00	84.00

Note: The "End of 1" rows represent the levered strategy prior to rebalancing; the "End of 1′" rows represent the levered strategy after rebalancing.

2.5%—or losing 6.25%—as we expected from our single-period intuition, we have lost 16%.

We went wrong because we ignored the *covariance* between leverage, λ, and source portfolio return, r^S:

$$\begin{aligned}\operatorname{cov}\left(\lambda, r^S\right) &= \frac{\left(\lambda_1-\bar{\lambda}\right)\left(r_1^S-0\right)+\left(\lambda_2-\bar{\lambda}\right)\left(r_2^S-0\right)}{2}\\ &= \frac{(2-2.5)(0.1-0)+(3-2.5)(-0.1-0)}{2}\\ &= \frac{-0.05-0.05}{2}\\ &= -0.05\\ &= -5.00\%.\end{aligned}$$

The covariance term reduces return by 5.0% *each period*, producing a return of –10.0%. The arithmetic return of the dynamically levered strategy is 0.2 in the first period and –0.3 in the second period, so the variance of the return of the levered strategy is

$$\begin{aligned}\frac{\left[0.2-(-0.05)\right]^2+\left[-0.3-(-0.05)\right]^2}{2} &= \frac{(0.25)^2+(-0.25)^2}{2}\\ &= 0.0625\\ &= 6.25\%.\end{aligned}$$

The variance drag for the dynamically levered strategy is the same as for the fixed-leverage strategy:[2] half the variance, or 3.125%, each period. Combining the covariance and variance produces a return of –8.125% per period, which suggests a cumulative loss of 16.25%, close to the actual cumulative return of –16.00% over the combination of the two periods.

As this example indicates, the covariance term can make a big difference over a few periods. Nevertheless, one might be tempted to think the covariance term would wash out over time. If that were true, the covariance term might not be particularly important. Strikingly, however, we found that the covariance term makes a *substantial difference over a very long horizon*. Our empirical examples include as source portfolios risk parity (with asset classes consisting of US stocks and US Treasury bonds) and T-bonds alone, with two types of volatility targeting and two volatility targets. In all our examples, the covariance term turned out to be negative, diminishing annualized return by amounts ranging from 0.64% to 4.23% over an 84-year horizon. Consequently, the Sharpe ratios of volatility-targeting strategies were lower than those of their source portfolios and fixed-leverage benchmarks.

Background and Synopsis

In the single-period capital asset pricing model (CAPM), the market portfolio is the unique portfolio of risky assets that maximizes the Sharpe ratio. Leverage serves only as a means to travel along the efficient frontier. Both excess return and volatility scale linearly with leverage, and a mean–variance investor will lever or de-lever the market portfolio in accordance with his or her risk tolerance.

Empirically, the Sharpe ratios of certain low-volatility portfolios have been higher than that of the market portfolio (see, for example, Anderson, Bianchi, and Goldberg 2012), which suggests that levering a low-volatility source portfolio could establish an attractive risk–return trade-off. However, market frictions, such as the difference between borrowing and lending rates, and the correlations that arise in multi-period models make the relationship between the realized return of a levered strategy and the Sharpe ratio of its source portfolio both nuanced and complex. Levered strategies tend to have substantially higher transaction costs[3] than do traditional strategies.[4]

In the sections that follow, we develop an exact performance attribution for levered strategies that takes market frictions into account. We discuss the development of our performance attribution model and illustrate it in the context of a particular risk parity strategy that targeted a fixed volatility equal to the realized volatility of a 60% equity/40% bond (hereafter, 60/40) fixed-mix portfolio over the 84-year sample period, 1929–2012. We use the performance attribution model to compare a levered strategy with a variety of benchmarks. We then discuss how the levered strategies responded to changes in market conditions. We also look beyond risk parity strategies by considering a US government bond index levered to the volatility of US equities. Next, we revisit the covariance term from the viewpoint of volatility targeting. We conclude with a synopsis of our empirical findings and theoretical contributions.

For the reader's convenience, we provide a number of appendices that support our main narrative. Appendix A derives our approximation of geometric return from arithmetic return. As illustrated in our empirical examples, this approximation has a high degree of accuracy in practical situations. Appendix B provides a detailed overview of the literature on low-risk investing and leverage. Appendix C describes the data in enough detail to allow researchers to replicate our results. Appendix D describes our linear trading model.

The Impact of Leverage on Return to an Investment Strategy

Leverage magnifies return, but that effect is only one facet of the impact that leverage has on an investment strategy. Leverage requires financing and exacerbates turnover, thereby adding transaction costs. It amplifies the variance drag on cumulative return due to compounding. When leverage is dynamic, it can add substantial noise to strategy return. We provide an exact attribution of the cumulative return to a levered strategy that quantifies these effects.

A *levered strategy* is built from a fully invested *source portfolio* of risky assets, presumably chosen for its desirable risk-adjusted returns, and a *leverage rule*.[5]

An investor has a certain amount of capital, L. The investor chooses a leverage ratio, λ, borrows $(\lambda - 1)L$, and invests λL in the source portfolio.[6] In what follows, we assume $\lambda > 1$.

Attribution of Arithmetic and Geometric Return. The relationship between the single-period returns to a levered portfolio, r^L, and to its source portfolio, r^S, is given by

$$r^L = \lambda r^S - (\lambda - 1) r^b, \tag{1}$$

where the borrowing rate, r^b, is greater than or equal to the risk-free rate, r^f. Note that the excess return is given by

$$\begin{aligned} r^L - r^f &= \lambda r^S - (\lambda - 1) r^b - r^f \\ &= \lambda\left(r^S - r^f\right) - (\lambda - 1)\left(r^b - r^f\right). \end{aligned} \tag{2}$$

Excess return and volatility scale linearly in λ for $\lambda \geq 0$ if and only if $r^b = r^f$. In that case, the situation is essentially the same as in the single-period CAPM except that the source portfolio need not be the market portfolio.

When $r^b > r^f$, volatility still scales linearly in $\lambda \geq 0$, but Equation 2 indicates that excess return scales sublinearly. As a consequence, the Sharpe ratio is a *declining* function of λ. Note that the excess borrowing return of the levered strategy is

$$r^L - r^b = \lambda\left(r^S - r^b\right). \tag{3}$$

It is the excess *borrowing* return and volatility that scale linearly in leverage for $\lambda \geq 1$. The bar for leverage to have a positive impact on return has gotten higher: The excess borrowing return, $r^S - r^b$, must be positive.

The expected return to a levered strategy is estimated by rewriting Equation 3 as

$$r^L = r^S + (\lambda - 1)\left(r^S - r^b\right), \tag{4}$$

and taking the expectation over multiple periods, $E(r^L)$, we get

$$\begin{aligned} E\left(r^L\right) &= E\left(r^S\right) + E\left[(\lambda - 1)\left(r^S - r^b\right)\right] \\ &= E\left(r^S\right) + E(\lambda - 1) E\left(r^S - r^b\right) \\ &\quad + \operatorname{cov}\left(\lambda, r^S - r^b\right). \end{aligned} \tag{5}$$

We use the term *magnified source return* to denote the sum of the first two terms on the right side of Equation 5. That formula shows that the expected return to a levered strategy is equal to the magnified source return plus a covariance correction. We found empirically that, even when the correlation between leverage and excess borrowing return is quite small, the covariance correction can be substantial in relation to the magnified source return.

We can interpret the expectation and covariance in Equation 5 in two ways: prospectively and retrospectively. Prospectively, they represent the expectation and covariance under the true probability distribution. Retrospectively, they represent the realized mean and realized covariance of the returns.[7]

Also important over multiple periods is the cost of trading, which imposes a drag, r^{TC}, on any strategy. To take account of this effect, we extend Equation 5 as follows:

$$\begin{aligned} E\left(r^L\right) &= E\left(r^S\right) + E(\lambda - 1) E\left(r^S - r^b\right) \\ &\quad + \operatorname{cov}\left(\lambda, r^S - r^b\right) - E\left(r^{TC}\right) \\ &= E\left(r^S\right) + E(\lambda - 1) E\left(r^S - r^b\right) \\ &\quad + \operatorname{cov}\left(\lambda, r^S - r^b\right) \\ &\quad - \left[E\left(r^{TCS}\right) + E\left(r^{TCL}\right)\right], \end{aligned} \tag{6}$$

where r^{TC} is expressed as a sum of trading costs resulting from turnover in the source portfolio and trading costs resulting from leverage-induced turnover:

$$r^{TC} = r^{TCS} + r^{TCL}. \tag{7}$$

Estimates of r^{TC} and its components rely on assumptions about the relationship between turnover and trading cost. We assumed that cost depended linearly on the dollar value that turned over, and we used Equations D11 and D12 to estimate r^{TC} in our empirical studies. More information is in Appendix D.

Equation 6 is based on arithmetic expected return, which does not correctly account for compounding. The correction for compounding imposes a variance drag on cumulative return that affects strategies differentially. For any given source portfolio, the variance drag is quadratic in leverage. If a levered strategy has high volatility, the variance drag may be substantial.

If we have returns for months $t = 0, 1, \ldots, T - 1$, the realized geometric average is

$$G(r) = \left[\prod_{t=0}^{T-1}\left(1 + r_t\right)\right]^{1/T} - 1, \tag{8}$$

where r_t is the arithmetic return in month t. Given two strategies, the one with the higher realized geometric average will have higher realized cumulative return. In Appendix A, we show that the following holds to a high degree of approximation:[8]

$$G(r) \sim \left[1+E(r)\right]e^{-\mathrm{var}(r)/2} - 1. \quad (9)$$

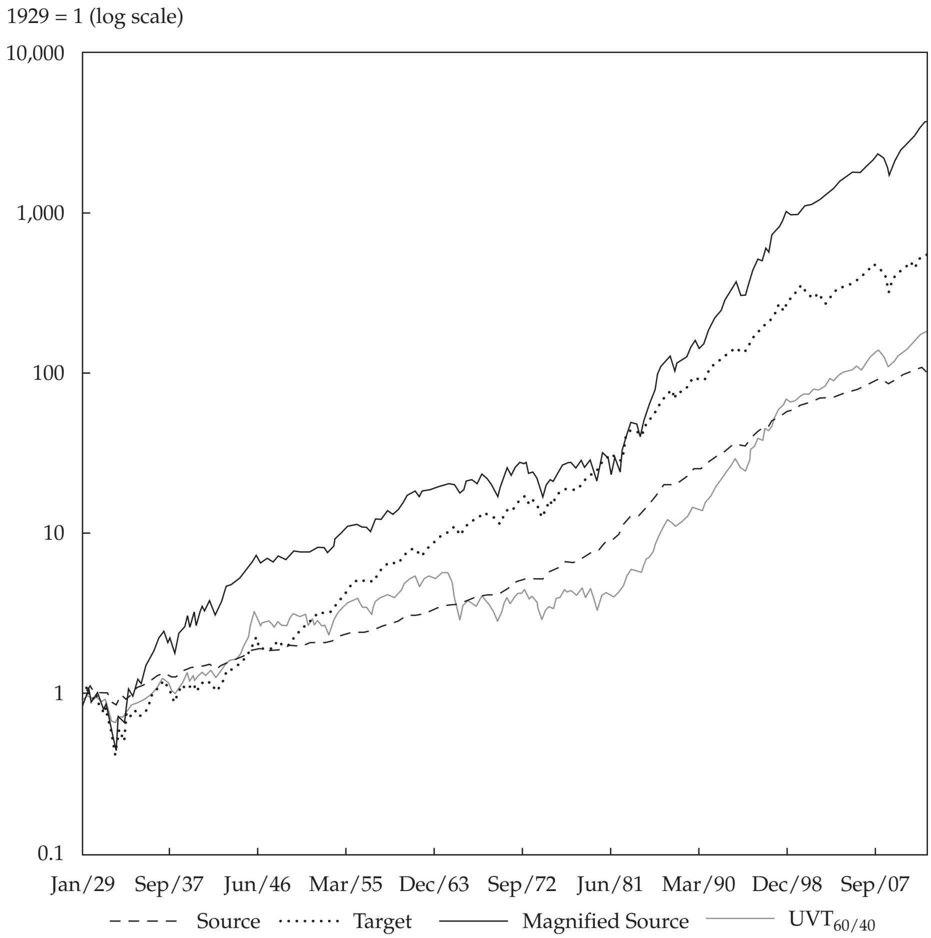
Note that the correction depends on only the realized variance of return.[9] Booth and Fama (1992) provided a correction for compounding based on continuously compounded return; our correction for the geometric average of monthly returns in Equation 9 is slightly simpler.

Thus, in comparing the realized returns of strategies, the magnified source return of the levered strategy must be adjusted for three factors that arise only in a multi-period setting—the covariance correction, the variance drag, and trading costs.[10]

Empirical Example: Performance Attribution of a Levered Risk Parity Strategy. We next demonstrate the utility of the performance attribution detailed previously in the context of a risk parity strategy that was rebalanced monthly and levered to an *unconditional volatility target* equal to the realized volatility, 11.59%, of the 60/40 fixed mix between January 1929 and December 2012; this strategy is named $\mathrm{UVT}_{60/40}$.[11] The source portfolio was unlevered risk parity based on two asset classes, US equity and T-bonds. Foresight was required in order to set this target: The volatility of the 60/40 strategy was not known until the end of the period.[12]

Figure 1 shows the magnified source return and the realized cumulative return to $\mathrm{UVT}_{60/40}$ and the realized cumulative return to its source portfolio (fully invested risk parity) and target (60/40 fixed mix). All the computations assumed that leverage was financed at the three-month

Figure 1. Cumulative Returns to the Source (Unlevered Risk Parity), Magnified Source, Levered Risk Parity (with Volatility Target 60/40), and 60/40, 1929–2012



Note: Magnified source return is an idealized return that cannot be achieved in practice; the curve depicts what would have been earned if we had achieved a geometric return equal to the arithmetic magnified source return.

Eurodollar deposit rate. The realized cumulative returns were based on the additional assumption that trading was penalized according to the linear model described in Appendix D and took into account the covariance correction and variance drag on cumulative return. The magnified source return of $UVT_{60/40}$ easily beat the cumulative return of both the source and the target; however, the realized cumulative return of $UVT_{60/40}$ was well below the realized cumulative return of the 60/40 target portfolio (with essentially equal volatility, 11.58%) and was only slightly better than the return of the *unlevered* risk parity source portfolio, which had much lower volatility (4.20%). The volatilities are reported in **Table 2**.

The return decomposition equations, Equations 6 and 9, provide a framework for analyzing the performance of $UVT_{60/40}$. **Table 3** provides the required information. Consider first the magnified source return. The source portfolio had an annualized arithmetic return of 5.75%, gross of trading costs.[13] Leverage added an extra 3.97% to annualized return from the magnification term, the average excess borrowing return to the source portfolio multiplied by average leverage minus 1. The annualized magnified source return was thus 9.72%. The covariance between

Table 2. Historical Performance, 1929–2012

Portfolio	Arithmetic Total Return	Geometric Total Return	Average Leverage	Volatility	Arithmetic Excess Return	Sharpe Ratio	Skewness	Excess Kurtosis
60/40	8.18%	7.77%	1.00	11.58%	4.69%	0.40	0.19	7.44
Risk parity	5.68	5.74	1.00	4.20	2.20	0.52	0.05	4.92
$UVT_{60/40}$	6.85	6.37	3.66	11.54	3.37	0.29	–0.43	2.23

Notes: The source portfolio was risk parity with volatility target 60/40; the borrowing rate, r^b, is the three-month Eurodollar rate, with trading costs. Arithmetic returns were estimated from monthly data and were annualized by multiplying by 12. Geometric returns were annualized by $[1 + G(r)]^{12} – 1$. Volatility was measured from monthly returns and was annualized by multiplying by $\sqrt{12}$. Annualized excess return and annualized volatility were used to calculate Sharpe ratios.

Table 3. Performance Attribution, 1929–2012

	$UVT_{60/40}$	
		Element Return (%)
Total source return (gross of trading costs)		**5.75**
Leverage	2.6600	
Excess borrowing return	0.0149	
Levered excess borrowing return		3.97
Magnified source return		**9.72**
Volatility of leverage	0.0772	
Volatility of excess borrowing return	0.0422	
Correlation (leverage, excess borrowing return)	–0.0566	
Covariance (leverage, excess borrowing return)		**–1.84**
Source trading costs	–0.0007	
Leverage-induced trading costs	–0.0096	
Total trading costs		**–1.03**
Total levered return (arithmetic)		**6.85**
Compounded arithmetic return (gross)	1.0707	
Variance correction	0.9934	
Variance drag		**–0.48**
Approximation error		0.00
Total levered return (geometric)		**6.37**

Notes: The source portfolio was risk parity with volatility target 60/40, with trading costs. The performance attribution was based on Equations 6 and 9. Borrowing was at the three-month Eurodollar deposit rate, and trading costs were based on the linear model in Appendix D. Arithmetic returns were estimated from monthly data and annualized by multiplying by 12. Geometric returns were annualized by $[1 + G(r)]^{12} – 1$. Formulas corresponding to the words in the performance attribution are presented in **Exhibit 1**.

Exhibit 1. Performance Attribution Terminology

Description	Formula
Total source return (gross of trading costs)	$E(r^S)$ (gross of trading costs)
Leverage	$E(\lambda - 1)$
Excess borrowing return	$E(r^S - r^b)$
Levered excess borrowing return	$E(\lambda - 1)[E(r^S - r^b)]$
Magnified source return	$E(r^S) + E(\lambda - 1)[E(r^S - r^b)]$
Volatility of leverage	$\sigma(\lambda)$
Volatility of excess borrowing return	$\sigma(r^S - r^b)$
Correlation (leverage, excess borrowing return)	$\rho(\lambda, r^S - r^b)$
Covariance (leverage, excess borrowing return)	$\text{cov}(\lambda, r^S - r^b)$
Source trading costs	$-E(r^{TCS})$
Leverage-induced trading costs	$-E(r^{TCL})$
Total levered return (arithmetic)	$E(r^L)$
Compounded arithmetic return (gross)	$\left[1 + \frac{E\left(r^L\right)}{1200}\right]^{12}$
Variance correction	$\exp\left(\frac{-\sigma^2_{r^L}}{2}\right)$
Variance drag	$\left\{\left[1 + \frac{E\left(r^L\right)}{1200}\right]^{12}\left[\exp\left(\frac{-\sigma^2_{r^L}}{2}\right) - 1\right]\right\}(100) - E\left(r^L\right)$
Approximation error	$G\left(r^L\right) - \left\{\left[1 + \frac{E\left(r^L\right)}{1200}\right]^{12}\left[\exp\left(\frac{-\sigma^2_{r^L}}{2}\right) - 1\right]\right\}(100)$
Total levered return (geometric)	$G(r^L)$

Note: The term "*E*" denotes expectation.

leverage and excess borrowing return, however, reduced the annualized return by 1.84%, trading costs reduced it by 96 bps, and the variance drag reduced it by a further 48 bps. Together, these three effects ate up 3.28 percentage points, or 82.6%, of the 3.97% contribution of leverage to the magnified source return.

Assumptions about Transaction Costs and Their Impact on Empirical Results

The return calculations in our empirical examples relied on assumptions about transaction costs over the study period, 1929–2012. Comparisons between levered and unlevered strategies were sensitive to these assumptions, but comparisons between strategies that were comparably levered were much less sensitive to them. For transparency, we have included the details of our assumptions about transaction costs in Appendices C and D. Here, we explain some of the reasoning that led to the choices we made, and we discuss the impact of our choices on the results.

One guideline is that trading became less expensive over time, so we assessed a greater cost to turnover at the beginning of the study period than at the end. Specifically, we assumed that the portfolio was rebalanced monthly.[14] We also assumed that trading cost 1% of the dollar amount of a trade between 1929 and 1955, 0.5% between 1956 and 1971, and 0.1% between 1972 and 2012. Because turnover tends to be higher in a levered strategy than in an unlevered strategy, higher trading costs tended to do more damage to a levered strategy than to an unlevered strategy.

As a borrowing rate, we used the three-month Eurodollar deposit rate, for which we had data back to the beginning of 1971. Prior to 1971, we used the three-month T-bill rate plus a spread of 60 bps, which was 40 bps less than the average spread between the Eurodollar deposit rate and the T-bill rate between 1971 and 2012. This choice improved the performance of our levered strategies relative to what it would have been had we used the average spread. Of course, a lower borrowing rate would have further improved the performance of the levered strategies.[15] Because the levered strategies involved borrowing and the unlevered strategies did not, more assumptions underlie the empirical results for levered strategies than for unlevered strategies. As a consequence, our uncertainty about results for levered strategies is greater than for unlevered strategies.

Of course, we could have included empirical results based on a wider range of assumptions about transaction costs. Doing so would have been misleading, however, because it would have conveyed the impression that we had done a thorough study of the issue. We had not. We chose a streamlined approach of providing examples that are based on a single set of assumptions consistent with published literature and that rely on readily available data. The purpose of these examples is to illustrate the efficacy of our performance attribution framework. We encourage practitioners and scholars to use their own estimates of trading and borrowing costs in applying the framework in order to evaluate strategies and to facilitate the decision to lever.

Benchmarks for a Levered Strategy

In this section, we consider benchmarks that are fully invested and benchmarks using fixed leverage and leverage determined by conditional volatility targeting.

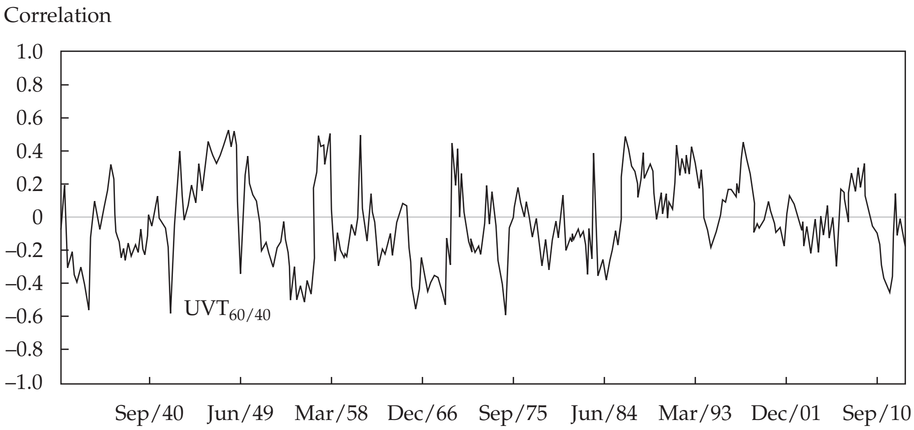
Fully Invested Benchmarks. Recall the annualized arithmetic and geometric returns, volatility, and Sharpe ratio of the $UVT_{60/40}$ strategy, its source, and its target that are presented in Table 2. Because $UVT_{60/40}$ was levered whereas the source and target were not, these comparisons were subject to uncertainty about historical financing and trading costs. $UVT_{60/40}$ had an annualized geometric return only 63 bps higher than that of the source portfolio, unlevered risk parity.[16] At the same time, the source portfolio had a much lower volatility (4.20%). As a result, $UVT_{60/40}$ had a Sharpe ratio of 0.29, compared with 0.52 for the unlevered risk parity portfolio. Note that the high Sharpe ratio of unlevered risk parity was obtained at the cost of low expected return.

The 60/40 portfolio and the $UVT_{60/40}$ portfolio had essentially equal volatilities. Under our assumptions on historical financing and trading costs, 60/40 delivered an annualized geometric return of 7.77% and a realized Sharpe ratio of 0.40; the analogous figures for $UVT_{60/40}$ were 6.37% and 0.29.

Investors who are considering a risk parity strategy or any levered strategy can populate Tables 2 and 3 with their forward-looking estimates of the components of the strategy's return. This analysis can inform the decision to invest in a levered strategy instead of the fully invested source or target portfolio.

Fixed-Leverage and Conditional-Leverage Benchmarks. In this section, we focus on comparisons of realized returns among levered strategies that were constructed in different ways. These comparisons were less sensitive to the assumptions about historical financing and trading costs than comparisons between levered and unlevered strategies. Like any volatility-targeting strategy, $UVT_{60/40}$ was dynamically levered. As discussed in the section on performance attribution of a levered risk parity strategy, however, the covariance between leverage and excess borrowing return diminished annualized arithmetic return by 1.84%. Deeper insight into this cost is provided in Table 3, which decomposes these covariances into products of correlation and standard deviations. Note that the magnitude of the correlation between leverage and excess borrowing return was small—only –0.056. **Figure 2**, which

Figure 2. Correlation between Leverage and Excess Borrowing Return for $UVT_{60/40}$, 1929–2012



Note: Correlation was computed from monthly data with the use of a trailing 36-month window.

shows rolling 36-month estimates of the correlation between leverage and excess borrowing return, indicates that the sign of the correlation flipped repeatedly at short horizons. At investment horizons of three to five years, the main effect of the covariance term appeared to be to add noise to the returns.

When leverage is fixed, the covariance between leverage and excess borrowing return must be zero. We considered two fixed-leverage strategies: $FLT_{60/40,\lambda}$ matched the average leverage of $UVT_{60/40}$ but had higher volatility, whereas $FLT_{60/40,\sigma}$ matched the volatility of $UVT_{60/40}$ but had lower leverage.

Another alternative to UVT is a conditional volatility-targeting strategy: $CVT_{60/40}$ levered fully invested risk parity such that the projected volatility (based on the previous 36 months' returns) equaled the volatility of the target 60/40 portfolio over the previous 36 months.[17]

Table 4 provides performance attributions for the $UVT_{60/40}$, $FLT_{60/40,\lambda}$, $FLT_{60/40,\sigma}$, and $CVT_{60/40}$ strategies. Note that the table is a version of Table 3 applied to the four levered strategies. All four levered strategies made use of the same source portfolio and hence had the same source arithmetic

Table 4. Performance Attribution of Realized Geometric Return for Four Levered Strategies, 1929–2012

	$UVT_{60/40}$		$FLT_{60/40,\lambda}$		$FLT_{60/40,\sigma}$		$CVT_{60/40}$	
		Element Return (%)		Element Return (%)		Element Return (%)		Element Return (%)
Total source return (gross of trading costs)		**5.75**		**5.75**		**5.75**		**5.75**
Leverage	2.6600		2.6900		1.7500		2.3100	
Excess borrowing return	0.0149		0.0149		0.0149		0.0149	
Levered excess borrowing return		3.97		4.02		2.61		3.45
Magnified source return		**9.72**		**9.77**		**8.37**		**9.20**
Volatility of leverage	0.0721		0.0000		0.0000		0.0508	
Volatility of excess borrowing return	0.0422		0.0422		0.0422		0.0422	
Correlation (leverage, excess borrowing return)	–0.0566		0.0000		0.0000		–0.0299	
Covariance (leverage, excess borrowing return)		**–1.84**		**0.00**		**0.00**		**–0.64**
Source trading costs	–0.0007		–0.0007		–0.0007		–0.0007	
Leverage-induced trading costs	–0.0096		–0.0051		–0.0027		–0.0093	
Total trading costs		**–1.03**		**–0.58**		**–0.34**		**–1.00**
Total levered return (arithmetic)		**6.85**		**9.19**		**8.03**		**7.56**
Compounded arithmetic return (gross)	1.0707		1.0959		1.0833		1.0783	
Variance correction	0.9934		0.9881		0.9934		0.9926	
Variance drag		**–0.48**		**–0.91**		**–0.41**		**–0.53**
Approximation error		0.00		0.01		0.01		0.00
Total levered return (geometric)		**6.37**		**8.29**		**7.62**		**7.03**

Notes: The source portfolio was risk parity with volatility target 60/40. $FLT_{60/40,\lambda}$ had constant leverage of 3.69, matching the average leverage of $UVT_{60/40}$, whereas $FLT_{60/40,\sigma}$ had constant leverage of 2.75, chosen to match the volatility of $UVT_{60/40}$. The performance attribution is based on Equations 6 and 9. Borrowing was at the Eurodollar deposit rate, and trading costs were based on the linear model in Appendix D. Arithmetic returns were estimated from monthly data and were annualized by multiplying by 12. Geometric returns were annualized by $[1 + G(r)]^{12} - 1$. Formulas corresponding to the words in the performance attribution are presented in Exhibit 1.

return. Leverage contributed substantially and at roughly the same level to the magnified source return, with similar contributions to the return of $UVT_{60/40}$, $FLT_{60/40,\lambda}$, and $CVT_{60/40}$ because those three strategies had similar average leverage. The contribution to the return of $FLT_{60/40,\sigma}$ was significantly lower because that strategy had lower average leverage. The covariance term reduced the annualized arithmetic return of $UVT_{60/40}$ by 1.84% but led to a much smaller reduction in the return of $CVT_{60/40}$ and, by design, had no effect on the return of the two FLT strategies. Trading costs reduced the return of $UVT_{60/40}$ and $CVT_{60/40}$ by about 95 bps but had a smaller effect on the two FLT strategies.[18] The variance drag reduced the geometric returns of $UVT_{60/40}$, $FLT_{60/40,\sigma}$, and $CVT_{60/40}$ by similar amounts because these strategies had similar variances. The effect on $FLT_{60/40,\lambda}$ was greater as a result of its higher volatility. Table 4 shows that when all the effects were taken into account, the geometric returns of $FLT_{60/40,\lambda}$, $FLT_{60/40,\sigma}$, and $CVT_{60/40}$ exceeded the geometric return of $UVT_{60/40}$ by, respectively, 192 bps, 125 bps, and 66 bps.

Attributes of Levered Strategies

The parameters of the UVT and FLT levered strategies were set with foresight. The dynamically levered strategy $UVT_{60/40}$ was based on the realized volatility of a 60/40 fixed mix between January 1929 and December 2012. Even though that volatility was known only at the end of the period, it was used to make leverage decisions throughout the period. The $FLT_{60/40,\lambda}$ leverage was set to match the average leverage of $UVT_{60/40}$, and the $FLT_{60/40,\sigma}$ leverage was set such that the volatility matched the volatility of $UVT_{60/40}$.

The strategy $CVT_{60/40}$ introduced in the previous section did not rely on future information to set leverage.[19] As a result, its realized volatility failed to match the realized volatility of the target. At each monthly rebalancing, $CVT_{60/40}$ was levered to match the volatility of the 60/40 fixed mix. Both volatilities were estimated from a 36-month rolling window.

All else equal, $UVT_{60/40}$, $FLT_{60/40,\lambda}$, $FLT_{60/40,\sigma}$, and $CVT_{60/40}$ called for additional investment in the source portfolio when its price rose. A decline in the value of the source portfolio reduced the net value of the levered portfolio while keeping the amount borrowed constant. Leverage had increased, and rebalancing required selling the source portfolio to return to leverage λ. Similarly, an increase in the value of the source portfolio resulted in taking on more debt and using the proceeds to buy more of the source portfolio. In this sense, the UVT, FLT, and CVT strategies with $\lambda > 1$ were momentum strategies. As **Exhibit 2** shows, UVT, FLT, and CVT strategies responded differently to changes in asset volatility.

Changing the Volatility Target

In this section, we explore the relationship between UVT and CVT strategies—in particular, their sensitivity to the volatility target. In addition to 60/40, we used the "market portfolio" (i.e., the value-weighted portfolio of stocks and bonds that had a higher volatility than 60/40) as the volatility target. UVT_{MKT} and CVT_{MKT} denote unconditionally levered and conditionally levered risk parity strategies with the market as the volatility target. Return comparisons of UVT_{MKT} with CVT_{MKT} and of $UVT_{60/40}$ with $CVT_{60/40}$ were not sensitive to our assumptions about historical financing and trading costs, and the comparisons of UVT_{MKT} with $UVT_{60/40}$ and of CVT_{MKT} with $CVT_{60/40}$ were only slightly sensitive to those assumptions.

Each term in the return attribution of the UVT risk parity strategies was sensitive to the choice of MKT or 60/40 as the volatility target. In contrast, the magnified source returns, covariance terms, and trading costs of CVT_{MKT} were quite similar to those of $CVT_{60/40}$. The only large difference between the two CVT strategies was the variance drag. This finding indicates that CVT strategies were more stable than UVT strategies.

As shown in **Table 5**, the geometric returns of UVT_{MKT} (6.53%) and CVT_{MKT} (6.52%) were virtually the same, whereas $CVT_{60/40}$ outperformed $UVT_{60/40}$ by 66 bps.

Changing the Source Portfolio

Thus far, we have illustrated our performance attribution model with a variety of risk parity strategies with a common source portfolio—unlevered risk parity. That approach allowed us to isolate the impact of various leverage rules on performance. In this section, we examine the impact of the source portfolio on performance: We consider strategies that levered

Exhibit 2. Strategy Responses to Changes in Market Conditions

Trigger	FLT Response	UVT Response	CVT Response
Increase in target volatility	No change	No change	Leverage ↑
Increase in source volatility	No change	Leverage ↓	Leverage ↓
Increase in price of source	Buy source	Buy source	Buy source

Table 5. Performance Attribution of Realized Geometric Return of Levered Strategies UVT_{MKT}, $UVT_{60/40}$, CVT_{MKT}, and $CVT_{60/40}$ in Terms of Their Common Source Portfolio (Risk Parity), 1929–2012

	UVT_{MKT}		$UVT_{60/40}$		CVT_{MKT}		$CVT_{60/40}$	
		Element Return (%)		Element Return (%)		Element Return (%)		Element Return (%)
Total source return (gross of trading costs)		**5.75**		**5.75**		**5.75**		**5.75**
Leverage	3.7100		2.6600		2.5800		2.3100	
Excess borrowing return	0.0149		0.0149		0.0149		0.0149	
Levered excess borrowing return		5.55		3.97		3.85		3.45
Magnified source return		**11.30**		**9.72**		**9.60**		**9.20**
Volatility of leverage	0.0995		0.0772		0.0532		0.0508	
Volatility of excess borrowing return	0.0422		0.0422		0.0422		0.0422	
Correlation (leverage, excess borrowing return)	–0.0566		–0.0566		–0.0321		–0.0299	
Covariance (leverage, excess borrowing return)		**–2.37**		**–1.84**		**–0.72**		**–0.64**
Source trading costs	–0.0007		–0.0007		–0.0007		–0.0007	
Leverage-induced trading costs	–0.0140		–0.0096		–0.0113		–0.0093	
Total trading costs		**–1.47**		**–1.03**		**–1.20**		**–1.00**
Total levered return (arithmetic)		**7.45**		**6.85**		**7.68**		**7.56**
Compounded arithmetic return (gross)	1.0771		1.0707		1.0796		1.0783	
Variance correction	0.9891		0.9934		0.9872		0.9926	
Variance drag		**–0.92**		**–0.48**		**–1.11**		**–0.53**
Approximation error		0.00		0.00		–0.05		0.00
Total levered return (geometric)		**6.53**		**6.37**		**6.52**		**7.03**

Notes: The source portfolio was risk parity with volatility targets of the value-weighted market portfolio and 60/40. The performance attribution was based on Equations 6 and 9. Borrowing was at the Eurodollar deposit rate, and trading costs were based on the linear model in Appendix D. Arithmetic returns were estimated from monthly data and were annualized by multiplying by 12. Geometric returns were annualized by $[1 + G(r)]^{12} - 1$. Formulas corresponding to the words in the performance attribution are presented in Exhibit 1.

an index of US government bonds to target the volatility of US equities. As in the previous examples, we consider both a dynamically levered volatility-targeting strategy, $UVTB_{STOCKS}$, and fixed-leverage benchmarks—namely, $FLTB_{STOCKS,\lambda}$ (with the same average leverage as $UVTB_{STOCKS}$) and $FLTB_{STOCKS,\sigma}$ (with the same volatility as $UVTB_{STOCKS}$). The details, presented in **Table 6**, are qualitatively similar to what we saw for the risk parity strategies in Tables 4 and 5: An attractive magnified source return was diminished substantially by transaction costs for all levered strategies and by the covariance term for the dynamically levered strategy, $UVTB_{STOCKS}$. However, because the source portfolio had lower volatility than the unlevered risk parity portfolio and the target volatility was higher than those of the 60/40 and value-weighted market portfolios, leverage was higher and the effects were more dramatic.

The covariance term for $UVTB_{STOCKS}$ of –4.23% per year imposed a larger drag on return than did the covariance terms for $UVT_{60/40}$ (–1.84%) and for UVT_{MKT} (–2.73%). Despite the fact that the volatility target in $UVTB_{STOCKS}$ was *fixed*, the leverage was highly variable because of changes in the inverse of the volatility of the source portfolio of T-bonds.[20] The correlation between leverage and excess borrowing

Table 6. Performance Attribution of the Realized Geometric Return of the Levered Strategies $UVTB_{STOCKS}$, $FLTB_{STOCKS,\lambda}$, and $FLTB_{STOCKS,\sigma}$ in Terms of Their Common Source Portfolio (Treasury Bonds), 1929–2012

	$UVTB_{STOCKS}$		$FLTB_{STOCKS,\lambda}$		$FLTB_{STOCKS,\sigma}$	
		Element Return (%)		Element Return (%)		Element Return (%)
Total source return (gross of trading costs)		**5.08**		**5.08**		**5.08**
Leverage	6.4300		6.4900		4.8000	
Excess borrowing return	0.0082		0.0082		0.0082	
Levered excess borrowing return		5.29		5.34		3.95
Magnified source return		**10.37**		**10.42**		**9.03**
Volatility of leverage	0.1742		0.0000		0.0000	
Volatility of excess borrowing return	0.0327		0.0327		0.0327	
Correlation (leverage, excess borrowing return)	–0.0742		0.0000		0.0000	
Covariance (leverage, excess borrowing return)		**–4.23**		**0.00**		**0.00**
Source trading costs	0.0000		0.0000		0.0000	
Leverage-induced trading costs	–0.0259		–0.0162		–0.0093	
Total trading costs		**–2.59**		**–1.62**		**–0.93**
Total levered return (arithmetic)		**3.55**		**8.80**		**8.10**
Compounded arithmetic return (gross)	1.0361		1.0916		1.0841	
Variance correction	0.9820		0.9707		0.9823	
Variance drag		**–1.80**		**–2.84**		**–1.61**
Approximation error		–0.05		–0.02		0.00
Total levered return (geometric)		**1.70**		**5.39**		**6.49**

Notes: The source portfolio was US Treasury bonds with volatility target US equities. $UVTB_{STOCKS}$ was levered to the volatility of stocks (18.93%) over the period January 1929–December 2012. $FLTB_{STOCKS,\lambda}$ had fixed leverage of 8.72, equal to the average leverage of $UVTB_{STOCKS}$; $FLTB_{STOCKS,\sigma}$ had fixed leverage and the same volatility (22.47%) as $UVTB_{STOCKS}$. The performance attribution was based on Equations 6 and 9. Borrowing was at the Eurodollar deposit rate, and trading costs were based on the linear model in Appendix D. Arithmetic returns were estimated from monthly data and were annualized by multiplying by 12. Geometric returns were annualized by $[1 + G(r)]^{12} - 1$. Formulas corresponding to the words in the performance attribution are presented in Exhibit 1.

return to the source portfolio was –0.07. As in the case of the dynamically levered risk parity strategies, a small correlation resulted in a large return drag.

Historical Performance of the Various Levered and Fully Invested Strategies

Table 7 summarizes the historical performance of our source portfolios (unlevered risk parity and T-bonds), volatility targets (fully invested 60/40, value-weighted market, and stocks), and the various levered strategies we considered. Unlevered risk parity has the highest Sharpe ratio (0.52), followed closely by T-bonds (0.49). Both exhibited low volatility and low excess return, however, making them unattractive as asset allocations for most investors.[21] Levered strategies are attractive as an asset allocation only if the Sharpe ratio survives leverage.

Two features of a levered strategy contribute to the difference between its Sharpe ratio and the Sharpe ratio of its source portfolio. The first is transaction costs. Both leverage-induced trading costs and financing costs diminished Sharpe ratios. The second is the covariance term. Because the covariance term was negative in the examples considered in this

Table 7. Historical Performance, 1929–2012

Portfolio	Arithmetic Total Return (%)	Geometric Total Return (%)	Average Leverage (%)	Volatility (%)	Arithmetic Excess Return (%)	Sharpe Ratio	Skewness	Excess Kurtosis
60/40	8.18	7.77	1.00	11.58	4.69	0.40	0.19	7.44
Value-weighted market	8.12	7.24	1.00	14.93	4.63	0.31	0.61	14.39
Stocks	10.43	9.00	1.00	18.93	6.95	0.37	0.18	7.46
Risk parity	5.68	5.74	1.00	4.20	2.20	0.52	0.05	4.92
Bonds	5.08	5.14	1.00	3.26	1.59	0.49	0.03	4.74
$UVT_{60/40}$	6.85	6.37	3.66	11.54	3.37	0.29	–0.43	2.23
$FLT_{60/40,\lambda}$	9.19	8.29	3.69	15.53	5.70	0.37	–0.01	4.78
$FLT_{60/40,\sigma}$	8.03	7.62	2.75	11.57	4.54	0.39	0.00	4.80
$CVT_{60/40}$	7.56	7.03	3.31	12.22	4.07	0.33	–0.41	7.13
UVT_{MKT}	7.45	6.53	4.71	14.88	3.97	0.27	–0.44	2.23
CVT_{MKT}	7.68	6.52	3.58	16.13	4.19	0.26	–0.75	15.62
$UVTB_{STOCKS}$	3.55	1.70	7.43	19.10	0.07	0.00	–0.55	4.75
$FLTB_{STOCKS,\lambda}$	8.80	5.93	7.49	24.47	5.31	0.22	–0.08	4.68
$FLTB_{STOCKS,\sigma}$	8.10	6.49	5.80	18.95	4.61	0.24	–0.07	4.66

Notes: Various source portfolios and targets are used. Borrowing was at the three-month Eurodollar deposit rate, and trading costs were based on the linear model in Appendix D. Arithmetic returns were estimated from monthly data and were annualized by multiplying by 12. Geometric returns were annualized by $[1 + G(r)]^{12} - 1$. Volatility was measured from monthly returns and was annualized by multiplying by $\sqrt{12}$. Sharpe ratios were calculated from annualized excess arithmetic return and annualized volatility.

article, it lowered the Sharpe ratios of the dynamically levered strategies relative to the Sharpe ratios of their source portfolios and comparably calibrated fixed levered strategies. As indicated in Figure 2, however, the correlation between leverage and the return to the source portfolio, which is the driver of the covariance term, can be highly unstable at horizons of three to five years. So unless a leverage-seeking investor has a specific reason to believe this correlation will be positive over a particular period for a particular dynamically levered strategy or unless he or she enjoys the coin-flip-like risk illustrated in Figure 2, the investor may prefer a fixed-leverage strategy.

The Covariance Term Revisited

The most novel part of our analysis is its focus on the covariance between leverage and excess borrowing return. In this section, we examine the covariance term from the standpoint of volatility targeting. We have already noted that leverage reduces the Sharpe ratio if the borrowing rate exceeds the risk-free rate or if trading incurs costs. In a multi-period setting, however, leverage has an impact on the Sharpe ratio, even in the absence of those market frictions, via the covariance term.

To focus on the covariance term, we made the highly unrealistic assumptions that borrowing is at the risk-free rate (i.e., $r^b = r^f$), which is fixed, and that trading costs are zero. We found in applying UVT that leverage does change the Sharpe ratio even under these assumptions.[22] Variable leverage, as used in UVT, is "an unintended market-timing strategy."[23]

Under these unrealistic assumptions, the excess return of the levered strategy is given by

$$r^L - r^f = \lambda\left(r^S - r^f\right). \tag{10}$$

Suppose we pick a fixed-volatility target, V; then we must set $\lambda = V/(\text{Volatility of source})$. Thus, we have

$$\begin{aligned}
&\text{Sharpe ratio of levered strategy}\\
&= \frac{E\left(r^L - r^f\right)}{V}\\
&= \frac{E\left[\lambda\left(r^S - r^f\right)\right]}{V}\\
&= \frac{E\left[\left(\dfrac{V}{\text{Volatility of source}}\right)\left(r^S - r^f\right)\right]}{V}\\
&= \frac{E\left[V\left(\dfrac{r^S - r^f}{\text{Volatility of source}}\right)\right]}{V}\\
&= E\left(\frac{r^S - r^f}{\text{Volatility of source}}\right)\\
&= E\left(r^S - r^f\right)E\left(\frac{1}{\text{Volatility of source}}\right)\\
&\quad + \text{cov}\left(r^S - r^f, \frac{1}{\text{Volatility of source}}\right).
\end{aligned} \tag{11}$$

Equation 11 makes it clear that a covariance term will affect the Sharpe ratio of any strategy that involves levering a source portfolio of variable volatility to a volatility target. Our empirical examples show that the covariance in Equation 11 has a material effect on realized return and realized Sharpe ratio. Recall that the leverage was especially volatile in $UVTB_{STOCKS}$, which levered a source portfolio of T-bonds to the volatility of stocks. Even though the target volatility was constant, the leverage was very volatile precisely because the inverse of the volatility of bonds was high.

Conclusion

In this article, we developed a platform that supports both backward-looking performance attribution and forward-looking investment decisions concerning levered strategies. Specifically, in Equation 6, we expressed the difference between arithmetic expected return to a levered strategy portfolio and its source portfolio as a sum of four terms:

$$E\left(r^{L}\right)=E\left(r^{S}\right)+E(\lambda-1)E\left(r^{S}-r^{b}\right)$$
$$+\operatorname{cov}(\lambda, r^{S}-r^{b})$$
$$-\left[E\left(r^{TCS}\right)+E\left(r^{TCL}\right)\right].$$

The first two terms, whose sum we call *magnified source return*, are the ones that most easily come to mind in the context of a levered strategy. As we showed empirically, however, other factors have a material effect on the cumulative return to a levered strategy. These terms are the covariance of leverage with the excess borrowing return, trading costs, and compounding effects.

Equation 6 accounted for both the covariance term and transaction costs, but it neglected the effect of compounding, which imposes a variance drag on cumulative return that is not captured in arithmetic expected return. If the levered strategy has high volatility, the variance drag may be substantial. Hence, a more accurate decision rule depends on the geometric expected return in the approximation given in Equation 9:

$$G(r)\sim\left[1+E(r)\right]e^{-\operatorname{var}(r)/2}-1.$$

We used Equations 6 and 9 to examine the realized performance of fixed-leverage (FLT) strategies and two dynamically levered strategies: unconditional volatility targeting (UVT) and conditional volatility targeting (CVT). Some scholars have expressed the view that CVT strategies are poor alternatives to UVT strategies (e.g., Asness et al. 2013). Their view is not supported by the results reported in Tables 4 and 5. In fact, it is the leverage that was implicitly determined by the volatility targets in $UVT_{60/40}$ and $CVT_{60/40}$—not the volatility itself—that interacted with the return to the source portfolio to determine strategy performance. In the 1929–2012 period, $CVT_{60/40}$ outperformed $UVT_{60/40}$. Future research is required to determine whether the sign of the covariance term is predictable at longer horizons.

In the examples we considered, the cumulative effects of borrowing and trading costs, the variance drag, and the covariance term offset much of the benefit of return magnification. Leverage, both fixed and dynamic, substantially lowered Sharpe ratios. In addition, dynamic leverage added noise to returns. Over the 84-year time horizon, fixed-leverage strategies outperformed volatility-targeting strategies and levered strategies had lower Sharpe ratios than their unlevered source portfolios.

Asness et al. (2012) argued that risk parity (levered to the volatility of the market) outperforms 60/40 over a long horizon.[24] Our analysis, however, does not support this argument.[25] Risk parity performed relatively well over the period 2008–2012, which featured Fed-supported interest rates that were extraordinarily low by historical standards. But that performance need not indicate how risk parity will perform in other regimes. Rising interest rates tend to raise the cost of funding a levered strategy and, at the same time, lower the prices of bonds in risk parity portfolios. Rising interest rates also have the potential to limit corporate profits and, thereby, exert downward pressure on equity prices. These considerations should be incorporated into any decision to lever low-risk portfolios when interest rates are unusually low.

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This article qualifies for 1 CE credit.

Appendix A. Geometric Return

To analyze the effects of compounding, Booth and Fama (1992) expressed continuously compounded return in terms of arithmetic return. We have chosen to analyze the effects of compounding by using the geometric average of monthly returns. Equation A3 for the geometric average of monthly returns is somewhat simpler than the Booth–Fama formula for continuously compounded return. Both derivations rely on the second-order Taylor expansion approximation of the logarithm.

Let L_t denote the equity in a strategy at month t, where $t = 0, 1, \ldots, T$. The correct ranking of realized strategy performance is given by $G(r)$, the geometric average of the monthly returns minus 1, which takes compounding into account:

$$\begin{aligned} G(r) &= \left(\frac{L_T}{L_0}\right)^{1/T} - 1 \\ &= \left(\prod_{t=0}^{T-1} \frac{L_{t+1}}{L_t}\right)^{1/T} - 1 \\ &= \left[\prod_{t=0}^{T-1} (1+r_t)\right]^{1/T} - 1. \end{aligned} \tag{A1}$$

Because the logarithm is strictly increasing, $\log[1 + G(r)]$ induces exactly the same ranking of realized strategy returns as $G(r)$. The ranking is different from the one induced by $E(r)$ and $\log[1 + E(r)]$. Thus, it requires a correction term involving $\mathrm{var}(r)$:

$$\begin{aligned} \log\left[1+G(r)\right] &= \frac{1}{T}\sum_{t=0}^{T-1} \log(1+r_t) \\ &\sim \frac{1}{T}\sum_{t=0}^{T-1}\left[r_t - \frac{(r_t)^2}{2}\right] \\ &= \frac{1}{T}\sum_{t=0}^{T-1} r_t - \frac{1}{T}\sum_{t=0}^{T-1}\frac{(r_t)^2}{2} \\ &= E(r) - \frac{\mathrm{var}(r) + \left[E(r)\right]^2}{2}. \end{aligned} \tag{A2}$$

$$\log\left[1+G(r)\right] \sim \log\left[1+E(r)\right] - \frac{\mathrm{var}(r)}{2}. \tag{A3}$$

$$G(r) \sim \left[1+E(r)\right]e^{-\mathrm{var}(r)/2} - 1. \tag{A4}$$

Equations A2 and A3 approximate the logarithm by its quadratic Taylor polynomial. When $r_t > 0$, the Taylor series for the logarithm is alternating and decreasing in absolute value for $|r_t| < 1$, so the error in the approximation of $\log(1 + r_t)$ in Equation A2 is negative and bounded above in magnitude by $|r_t|^3/3$ for each month t. When $r_t < 0$, the error is positive and may be somewhat larger than $|r_t|^3/3$.

Because the monthly returns are both positive and negative, the errors in months with negative returns will substantially offset the errors in months with positive returns, so the errors will tend not to accumulate over time. The approximation error in annual geometric return was at most 1 bp in our risk parity examples (see Table 4) and 5 bps in our levered bond examples (see Table 6).

Appendix B. Related Literature

We present our discussion of the related literature in seven areas: the CAPM, the measurement of risk and nonlinearities, motivations for leverage, levered low-risk strategies, the empirical evidence on levered low-risk investing, the effect of leverage on markets, and arithmetic versus geometric returns.

The CAPM

Finance continues to draw heavily on the capital asset pricing model developed in Treynor (1962), Treynor and Black (1976), Sharpe (1964), Lintner (1965a, 1965b), and Mossin (1966) and extended in Black and Litterman (1992).[26] In the CAPM, leverage is a means to adjust the level of risk in an efficient portfolio and nothing more. Markowitz (2005) illustrated another facet of leverage, however, in the context of a market composed of three coconut farms. In this disarmingly simple example, some investors were leverage-constrained and others were not. The market portfolio was mean–variance inefficient; as a result, no mean–variance investor would choose to hold it and expected returns of assets did not depend linearly on market betas.

Measurement of Risk and Nonlinearities

An impediment to a clear understanding of leverage may be the way we measure its risk. Standard risk measures—such as volatility, value at risk, expected shortfall, and beta—scale linearly with leverage. But as we know from the collapse of Long-Term Capital Management, the relationship between risk and leverage can be nonlinear; see, for example, Jorion (2000). Föllmer and Schied (2002) and Föllmer and Schied (2011, ch. 4) described risk measures that penalize leverage in a super-linear way. Recent experience suggests that these measures may be useful in assessing the risk of levered strategies.

A contribution of this article is an explanation of how the interaction between leverage and market frictions creates specific nonlinearities in the relationship between leverage and return. Understanding these specific nonlinearities provides a practical framework to guide the decision about whether and how to lever.

Motivations for Leverage

If investors are overconfident in their predictions of investment returns, they may find leverage attractive because it magnifies the returns when times are good and because they underestimate the risk of bad outcomes.[27]

In a CAPM world, investors with below-average risk aversion will choose to lever the market portfolio.[28]

Perfectly rational investors may also be attracted to leverage by the low-risk anomaly, the apparent tendency of certain low-risk portfolios to have higher risk-adjusted returns than high-risk portfolios have. An investor who believes in the low-risk anomaly will be tempted to lever low-risk portfolios in the hope of achieving high expected returns at acceptable levels of risk.

The low-risk anomaly provides a rational argument for investors with typical risk aversion to use leverage. Indeed, the low-risk anomaly is arguably the only rational reason for an investor to use leverage in an investment portfolio composed of publicly traded securities.[29]

Differences in risk aversion may explain why some investors choose higher expected return at the price of higher volatility, but a rational investor has little reason to choose *leverage* unless the source portfolio being levered offers superior *risk-adjusted* returns along with a volatility below the investor's risk tolerance.

Levered Low-Risk Strategies

Low-risk investing refers to a diverse collection of investment strategies that emphasize low beta, low idiosyncratic risk, low volatility, or downside protection. The collection of low-risk strategies includes broad asset allocations but also narrower strategies restricted to a single asset class. Markowitz (1952) made an early reference to low-risk investing, commenting that a minimum-variance portfolio is mean–variance optimal if all asset returns are uncorrelated and have equal expectations. But low-risk strategies typically require leverage in order to meet expected return targets. In an exploration of this idea, Frazzini and Pedersen (2014) echoed some of the conclusions in Markowitz (2005), and they complemented theory with an empirical study of an implicitly levered equity risk factor that was long low-beta stocks and short high-beta stocks. This approach descended from Black, Jensen, and Scholes (1972), who provided evidence that the CAPM may not properly reflect market behavior.

Empirical Evidence on Levered Low-Risk Investing

A growing volume of empirical literature indicates that market frictions may prevent investors from harvesting the returns promised by a frictionless analysis of levered low-risk strategies. Anderson et al. (2012) showed that financing and trading costs can negate the abnormal profits earned by a levered risk parity strategy in a friction-free market. Li, Sullivan, and Garcia-Feijóo (2014) and Fu (2009) showed that market frictions may impede the ability to scale up the return of low-risk strategies through leverage.[30]

Asset allocation based on capital weights has a long and distinguished history; see, for example, Graham (1949) and Bogle (2007). Rules-based strategies that allocate risk instead of, or in addition to, capital are of a more recent vintage. Risk-based investing is discussed in, for example, Lörtscher (1990), Kessler and Schwarz (1996), Qian (2005), Clarke, de Silva, and Thorley (2011, 2013), Shah (2011), Sefton, Jessop, De Rossi, Jones, and Zhang (2012), Cowan and Wilderman (2011), Anderson et al. (2012), Bailey and Lopez de Prado (2012), and Goldberg and Mahmoud (2013).

Strategies that target volatility are also gaining acceptance, although the literature is still sparse. Goldsticker (2012) compared volatility-targeting strategies with standard allocations, such as fixed-mix portfolios, and found that the relative performance of the strategies was period dependent.

The Effect of Leverage on Markets

Another important issue is the extent to which leverage may contribute to market instability. See, for example, Brunnermeier and Pedersen (2009), Adrian and Shin (2010), and Geanakoplos (2010). We do not address this issue in this article because we took the distribution of the underlying asset returns as given and restricted our analysis to the effect of leverage on the return of investment strategies.

Arithmetic vs. Geometric Return

Despite the extensive literature on the importance of compounding to investment outcomes, analyses of investment strategies are often based on arithmetic expected return. Background references on compounding and geometric return include Fernholz (2002) and MacLean, Thorp, and Ziemba (2011). Perold and Sharpe (1988) discussed how the interplay among volatility, rebalancing, and compound return causes a fully invested fixed-mix portfolio or portfolio insurance strategy to behave differently from a buy-and-hold strategy with the same initial mix. Booth and Fama (1992) worked out the relationship between the compound return to a fixed-mix portfolio and its constituents, and Willenbrock (2011) applied their results to portfolios that included commodities. Markowitz (2012) compared six different mean–variance approximations to geometric returns.

Appendix C. Data

The results we have presented were based on CRSP stock and bond data from January 1929 through December 2012. The aggregate stock return is the CRSP value-weighted market return (including dividends) from the table "Monthly Stock-Market Indices (NYSE/AMEX/NASDAQ)"—variable name *vwretd*. The aggregate bond return is the face value outstanding (cross-sectionally) weighted average of the unadjusted return for each bond in the "CRSP Monthly Treasury (Master)" table. In this table, the variable name for the unadjusted return is *retnua* and for the face value outstanding, *iout1r*. Each bond in the table was used unless one or both of the values of *retnua* and *iout1r* were missing.

The proxy for the risk-free rate was the "US Government 90-day T-bills Secondary Market" rate provided by Global Financial Data for the period January 1929 through December 2012. The proxy for the cost of financing leverage was the "U.S. 3-Month Euro-Dollar Deposit" rate downloaded from the US Federal Reserve (www.federalreserve.gov/releases/h15/data.htm). The three-month Eurodollar deposit data are available for January 1971 through December 2012. Prior to January 1971, we added a constant of 60 bps to the 90-day T-bill rate.[31]

Trading costs were calculated by using the procedure described in Appendix D. We assumed that the cost of trading was 100 bps from 1926 to 1955, 50 bps from 1956 to 1970, and 10 bps from 1971 onward.

The construction of the unlevered and levered risk parity strategies was carried out exactly as detailed in Anderson et al. (2012). The construction of the bonds levered to stock strategies was the analog for the case of a single asset class.

Anderson et al. (2012), following Asness et al. (2012), used the volatility of the market as the target for risk parity. In this article, we used the volatility of the 60/40 portfolio as the target because it provides a more appropriate comparison with traditional strategies used by institutional investors. The return of UVT strategies is particularly sensitive to the volatility target.

Appendix D. Trading Costs

We estimated the drag on return that stems from the turnover-induced trading required to maintain leverage targets in a strategy that levers a source portfolio, S.

At time t, the strategy calls for an investment with a leverage ratio of λ_t. We made the harmless assumption that the value of the levered strategy at t, denoted L_t, is \$1.00.[32] Then, the holdings in the source portfolio, or assets, are $A_t = \lambda_t$. The debt at time t is given by $D_t = \lambda_t - 1$.

The task is to find holdings A_{t+1} in the portfolio at time $t + 1$ that are consistent with leverage target λ_{t+1}. This task turns out to be a fixed-point problem because the trading costs must come out of the investor's equity. Between times t and $t + 1$, the value of the source portfolio changes from S_t to S_{t+1} and the strategy calls for rebalancing to achieve leverage λ_{t+1}. Just prior to rebalancing, the value of the investment is

$$A'_t = \lambda_t\left(1 + r_t^S\right), \tag{D1}$$

the liability has grown to

$$D'_t = \left(\lambda_t - 1\right)\left(1 + r_t^b\right), \tag{D2}$$

and the investor's equity is

$$L'_t = A'_t - D'_t = \lambda_t\left(1 + r_t^S\right) - \left(\lambda_t - 1\right)\left(1 + r_t^b\right). \tag{D3}$$

Note that in Equations D1 and D3, we used r_t^S, the source return gross of trading costs.

Let $\mathbf{w}_t = (w_{t1}, \ldots, w_{tn})^{\mathrm{T}}$ denote the vector of relative weights assigned to the n asset classes in the source portfolio at time t so that $\sum_{i=1}^{n} w_{ti} = 1$ for all t. Just prior to rebalancing, the weights have changed to

$$\mathbf{w}'_t = \left(w'_{t1}, \ldots, w'_{tn}\right)^{\mathrm{T}}, {}^{\mathrm{T}} \tag{D4}$$

where

$$w'_{ti} = \frac{w_{ti}\left(1 + r_i^t\right)}{1 + r_t^S}. \tag{D5}$$

At time $t + 1$, the strategy is rebalanced according to its rules, which produces holdings of $A_{t+1}\mathbf{w}_{t+1}$ in the n asset classes. We let $\mathbf{x}_t = (x_{t1}, \ldots, x_{tn})^{\mathrm{T}}$ denote the vector of dollar amounts of the changes in value as a result of rebalancing so that

$$\mathbf{x}_t = A_{t+1}\mathbf{w}_{t+1} - A'_t\mathbf{w}'_t. \tag{D6}$$

If we assume a linear model, the cost of trading, $\mathbf{x}_t$, is $\kappa\|\mathbf{x}_t\|_1 = \sum_{i=1}^{n}|x_{ti}|$ for some $\kappa \geq 0$. The cost reduces the investor's equity to

$$\begin{aligned} L_{t+1} &= L'_t - \kappa\|\mathbf{x}_t\|_1 \\ &= \lambda_t\left(1 + r_t^S\right) - \left(\lambda_t - 1\right)\left(1 + r_t^b\right) - \kappa\|\mathbf{x}_t\|_1. \end{aligned} \tag{D7}$$

Now, let

$$\begin{aligned} g(\alpha) &= \frac{\alpha}{L_{t+1}} - \lambda_{t+1} \\ &= \frac{\alpha}{L'_t - h(\alpha)} - \lambda_{t+1}, \end{aligned} \tag{D8}$$

where $g(\alpha)$ denotes the leverage implied by holding $\alpha\mathbf{w}_{t+1}$ in the n assets, taking into account the effect of trading costs on equity L_{t+1}, minus the desired leverage, and $h(\alpha) = L'_t - L_{t+1} \geq 0$. Assuming that g is defined on the whole interval $\left(0,\lambda_{t+1}L_t'\right)$, it is continuous, $g(0) = -\lambda_{t+1} < 0$ and $g\left(\lambda_{t+1}L_t'\right) \geq 0$. So, by the Intermediate Value Theorem, there exists α_{t+1} such that $g(\alpha_{t+1}) = 0$.[33] The value of α_{t+1} can readily be found by a bisection algorithm, which worked well in all the empirical situations studied for this article.[34]

We set

$$A_{t+1} = \alpha_{t+1} \tag{D9}$$

so that the holdings of the n assets are given by

$$A_{t+1}\mathbf{w}_{t+1} = \alpha_{t+1}\mathbf{w}_{t+1}. \tag{D10}$$

The reduction in return as a result of trading costs is given by

$$r^{TC} = \kappa\left\|\alpha_{t+1}\mathbf{w}_{t+1} - A_t'\mathbf{w}_t'\right\|_1. \tag{D11}$$

We computed the trading cost incurred by the source portfolio, $E(r^{TCS})$, in the same way and defined the trading cost resulting from leverage as

$$E\left(r^{TCL}\right) = E\left(r^{TC}\right) - E\left(r^{TCS}\right). \tag{D12}$$

Notes

1. Sullivan (2010) discussed the risks that a pension fund incurs by using a levered strategy.
2. This statement is true for this specific example but is not true in general.
3. Investment returns are often reported gross of fees and transaction costs. That practice may be reasonable in comparing strategies with roughly equal fees or transaction costs, but it is inappropriate when comparing strategies with materially different fees or transaction costs.
4. By traditional strategies, we mean the strategies that have typically been used over the last 50 years by pension funds and endowments. These strategies invest, without leverage, in a relatively fixed allocation among asset classes.
5. The source portfolio can be long–short in the risky assets. It must have a nonzero value, however, so that return can be calculated. Because we want to model leverage explicitly, we do not allow the source portfolio to contain a long or short position in a riskless asset, such as T-bills, a money market account, or commercial paper.
6. Leverage may be achieved through explicit borrowing. It may also be achieved through the use of derivative contracts, such as futures. In these derivative contracts, the borrowing cost is implicit rather than explicit, but it is real and is typically at a rate higher than the T-bill rate. For example, Naranjo (2009) found that the implicit borrowing cost of using futures is approximately the applicable LIBOR applied to the notional value of the futures contract.
7. Note that we took the realized covariance, obtained by dividing by the number of dates, rather than the realized sample covariance, which would be obtained by dividing by 1 less than the number of dates. We used the realized covariance because it makes Equation 5 true.
8. The magnitude of the error is estimated from Equation A4. Note that G and E denote realizations of, respectively, the geometric and average arithmetic returns. The term var(r) denotes the realized variance of r rather than the realized sample variance.
9. In an earlier version of this article, we indicated, incorrectly, that both the level and the variability of volatility determine the magnitude of the variance drag.
10. Note that the source and target portfolios may incur their own trading costs as well as benefit from volatility pumping. The performance attribution of Equation 6 uses the source return and magnified source return, *gross* of trading costs. When we report historical arithmetic returns to the source and target portfolios, we report them *net* of trading costs and inclusive of any benefit from volatility pumping. When we report cumulative returns to the source and target portfolios, we report them net of the variance drag.
11. The leverage was chosen such that the volatility, gross of trading costs, would be exactly 11.59%. When trading costs were taken into account, the realized volatility was slightly lower—11.54%. $UVT_{60/40}$ was constructed, effectively, in the same way as the levered risk parity strategy in Asness, Frazzini, and Pedersen (2012), with one main difference: They levered risk parity to match the volatility of the market, which had higher volatility than the 60/40 portfolio. In the section "Changing the Volatility Target," we consider risk parity levered to the volatility of the market.
12. The sensitivity of strategy performance to the volatility target is discussed in the section "Changing the Volatility Target."
13. Trading costs subtracted only 7 bps per year from the source return.
14. In practice, trading costs can be lowered by reducing the frequency or completeness of rebalancing, at the cost of introducing tracking error. Furthermore, trading costs may be higher for some asset classes than for others. In our empirical examples, however, financing costs were more important than trading costs.
15. We considered using one-month rates, but doing so would have engendered a more complex extrapolation because the one-month T-bill rate began only in 2001. The difference between the one-month and three-month Eurodollar deposit rates averaged 20 bps between 1971 and 2013. This difference was offset by the 40 bps we subtracted in our extrapolation.
16. The annualized geometric return of the source portfolio, 5.74%, slightly exceeded 5.68%, the annualized arithmetic return of the source portfolio net of trading costs. This result is an artifact of the annualization procedures for arithmetic and geometric returns. The source portfolio had a monthly arithmetic return of 47.3 bps net of transaction costs. This return was annualized by multiplying by 12, for a result of 5.68%. Annualized geometric return, however, takes into account compounding: $1.00473^{12} - 1 = 5.83\%$. The variance drag reduced this figure by 9 bps to 5.74%. The variance drag on the source return was much smaller than the variance drag on the levered portfolios because the source portfolio was so much less volatile and the variance drag is quadratic in volatility.
17. $CVT_{60/40}$ was introduced in Anderson et al. (2012).
18. As discussed in the next section, even maintaining a fixed leverage requires trading. In principle, the trading needed to adjust leverage to meet a volatility target could offset some of the trading required to maintain fixed leverage, but this possibility strikes us as unlikely in typical situations. Had we assumed lower trading costs, the gap in trading costs among the strategies would have narrowed but the ranking of those costs would not have changed.
19. The foresight used in the definitions of the UVT and FLT strategies allowed them to exactly match their volatility or leverage targets, gross of trading costs. Because $CVT_{60/40}$ did not rely on foresight, it could not exactly match the realized target volatility, gross of trading costs. Both UVT and $CVT_{60/40}$ volatility and FLT leverage were further affected by trading costs.

20. See "The Covariance Term Revisited" section for an analysis of the covariance term from the standpoint of volatility targeting. Had we made the unrealistic assumptions that financing was at the risk-free rate and that trading costs were zero, the two FLT strategies would still have easily outperformed the UVT strategy.
21. Of course, bonds are often used as one asset class in an asset allocation, such as 60/40 or the value-weighted market portfolio; 60/40 has been widely used as an asset allocation, and risk parity has been proposed as an alternative asset allocation; see, for example, Asness et al. (2012).
22. This issue has been misunderstood in the published literature. For example, Asness et al. (2013) wrote, "Scaling the returns to any stable risk target (or not scaling them at all) cannot mathematically affect the Sharpe ratio, or the *t*-statistic of the alpha of our levered portfolios, because we are multiplying the return stream by a fixed constant" (p. 14). Their analysis conflated single-period models with multi-period models and misstated the construction of the UVT_{MKT} strategy used in Asness et al. (2012).
23. Asness et al. (2013) asserted that variable *volatility*, rather than variable leverage, is "an unintended market-timing strategy" (p. 14).
24. Comparing the performance of the value-weighted market with risk parity levered to the volatility of the market and comparing the performance of 60/40 with risk parity levered to the volatility of 60/40 are reasonable approaches. Comparing the performance of 60/40 with that of risk parity levered to the volatility of the market does not, however, seem reasonable; we are grateful to Patrice Boucher for this insight.
25. See, however, the discussion in the section "Assumptions about Transaction Costs and Their Impact on Empirical Results."
26. French (2003) provided a history of the CAPM elucidating Treynor's role in its development.
27. A positive relationship between overconfident CEOs and firm leverage is documented in Malmendier, Tate, and Yan (2011). Shefrin and Statman (2011) identified excessive leverage taken by overconfident bankers as a contributor to the global financial crisis.
28. Note, however, that the market portfolio in the CAPM includes bonds and other risky asset classes, not only stocks. Levered strategies include the use of margin, futures, and other derivatives to assemble levered equity-only portfolios, which behave quite differently from levered portfolios in the CAPM.
29. There are, of course, other rational reasons for using leverage in other contexts. The leverage provided by a mortgage may be the only feasible way for a household to buy a house, which provides a stream of consumption benefits and tax advantages in addition to facilitating an investment in the real estate market. Companies leverage their shareholder equity with borrowing to finance operations, for a variety of reasons, including differences in risk aversion, informational asymmetries, and tax implications.
30. Ross (2004) provided an example of the limits to arbitraging mispricings of interest-only strips of mortgage-backed securities.
31. The average difference between the 90-day T-bill rate and the three-month Eurodollar deposit rate for 1971–2012 was 102 bps. So, our estimate of 60 bps was relatively conservative.
32. This assumption is harmless in a linear model of trading costs, which we develop here. It would be inappropriate for a realistic model of market impact.
33. Typically, α_{t+1} is uniquely determined; if not, choose the largest value that satisfies the equation.
34. If *no* α such that $g(\alpha) = 0$ exists, the reason is that the equity of the strategy is so low that the transaction costs in getting to the desired leverage wipe out the equity. We did not observe any such severe drawdowns in our empirical examples, but they would clearly be possible with extreme leverage or a very volatile source portfolio.

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