The Implied Futures Financing Rate

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1 Abstract

We explore the cost of implicit leverage associated with an S&P 500 Index futures contract and derive an implied financing rate. While this implicit financing rate has often been attractive relative to market rates on explicit financing, the relationship between the implicit and explicit financing rates has been volatile and varied considerably based on legal and economic regimes.

2 Introduction

What is the true financing cost of a levered equity investment strategy? Generally speaking, an investor who wishes more exposure than her free cash permits has two choices: she can borrow and add the debt proceeds to her cash investment, or, as discussed in this article, she can obtain implicit leverage through the derivatives market.1 In this paper, we explore the cost of implicit leverage associated with a prominent equity index futures contract. In summary, our research shows that the related implicit financing rate has often been attractive relative to market rates on explicit financing; however, the relationship between the implicit and explicit financing rates has been volatile and varied considerably based on legal and economic regimes.

1Two of the present authors have explored explicit leverage in a previous paper. [Anderson et al., 2014]
The purchase of an S&P 500 Index\(^2\) futures contract\(^3\) results in exposure to a notional amount of the underlying in exchange for a current payment much less than that notional amount.\(^4\) Intuitively, this suggests financing; a levered purchase of the underlying equities provides similar exposure and cash flows. In this paper, we consider a stock and two futures contracts on the stock, based on which we derive an associated implied forward financing rate. This rate represents the implicit cost of financing associated with such an investment in a futures contract. The spread between this implied rate and market interest rates evolved dynamically over a series of four regimes that we identified during the period January 1996 to August 2013. The Commodity Futures Modernization Act of 2000 (“CFMA”) reduced this spread, which was subsequently altered by the 2008 financial crisis and recovery.

### 3 Interest Rates Implied by Futures

#### 3.1 General Case without Dividends

The basic equations for futures and implied financing rates used in this paper come from the well-known relationship that in the absence of arbitrage relates the futures price \(f(t, T)\) at the current time \(t\) to the expected stock price \(S(T)\) at the maturity date \(T\) of the relevant futures contract, conditioned on the information at time \(t\):

\[
f(t, T) = E_t[S(T)] \equiv E_Q[S(T)|\mathcal{F}_t]
\]  

In Equation (1), \(E_Q[S(T)|\mathcal{F}_t]\) denotes the expectation of \(S(T)\) conditional on the information available at an earlier time \(t\) under the risk neutral measure \(Q\) associated with riskless investment in a money market account.

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\(^2\)The term ‘S&P 500 Index’ is a registered copyright of Standard & Poors, Inc.

\(^3\)Unless otherwise stated, the terms futures contract in this article refers to such an S&P 500 Index futures contract. The terms of such futures contracts are specified by the Chicago Mercantile Exchange (CME) for the big (or floor) and the small (or E-mini) futures contracts traded on that exchange. Current terms for such futures contracts are available on the CME’s website. The CME has other futures contracts related to the S&P 500 Index, including contracts that reflect dividends on the index components. These other contracts lack the volume and the history of the traditional contract we choose.

\(^4\)The payment required to execute a futures contract consists principally of initial and variational margin; leverage of 10:1 or higher is often possible. See footnote 3.

\(^5\)See, e.g., [Anderson and Kercheval, 2010, Chapter 4]. \(Q\) is the risk-neutral measure.
we denote by $M$. We will assume there is a short rate process $r(t)$ associated with this riskless investment and related (stochastic) discount factor (or just discount) $D$:

$$M(t_0, t_1) = e^{\int_{t_0}^{t_1} r(s) ds}, \quad D(t_0, t_1) = \frac{1}{M(t_0, T_1)}, t_0 < t_1$$

We assume there is such an $r_t$ that is attainable\textsuperscript{6} from the stock, the near futures contract and the next futures contract on the S&\P 500 Index.\textsuperscript{7} We call such a short rate implied by the futures and the stock the \textit{futures implied rate}, or \textit{FIR} for short, and define it formally below. Such a short rate process need not be equivalent to a market interest rate, where two rates are equivalent if they may be swapped into each other at no cost.\textsuperscript{8} Instead, it is the rate of riskless investment attainable through a self-financing portfolio in the stock and the futures, and as such represents the true rate of financing available through futures.

We assume there is no arbitrage and, at first, also that $S$ pays no dividends. Accordingly $S(t)$ equals its expected discounted future value under the standard Martingale Pricing Formula (MPF).\textsuperscript{9}

$$S(t) = E_t[S(T)D(t, T)], T > t$$

Using this and the well-known relationship between the covariance and the expectation of a product of two random variables, the \textit{inverse futures ratio}

\textsuperscript{6}A claim is \textit{attainable} from other claims if there is a self-financing trading strategy in those other claims that has the same payoffs as the first claim. \textit{See, e.g.} [Anderson and Kercheval, 2010, Chapter 2.3].

\textsuperscript{7}These contracts are traded on the Chicago Mercantile Exchange. In market terminology, the \textit{near} contract is the one with the earliest possible maturity after the observation time, and the \textit{next} contract is the one with next earliest maturity. The difference in the maturity dates of two consecutive S&\P 500 Index futures contracts, such as the near and the next, is typically approximately three months. In this research, we have used the \textit{big} or \textit{floor} contract because of its longer existence; in later years the volume of the \textit{E-mini} is typically larger but the closing prices are almost always the same.

\textsuperscript{8}Two equivalent rates are said in the market to have a \textit{zero basis} between them. ‘Equivalence’ in this sense requires a static hedge, so that each rate is attainable from the other in a very simple way using market instruments. \textit{Compare} Footnote 6 above and Section 4 below.

\textsuperscript{9}The MPF states that the current price of a security or other self-financing trading strategy equals the conditional expectation of its discounted future cash flows, including its terminal future value, discounting by the riskless investment or other numeraire. [Anderson and Kercheval, 2010, Chapter 3]. In the absence of dividends, the terminal future value is the only future cash flow, as in the equation in the text.
(IFR) is given by:

$$\text{IFR} \equiv \frac{S(t)}{f(t,T)} = \frac{E_t[S(T)D(t,T)]}{f(t,T)}$$

$$= \frac{E_t[S(T)]E_t[D(t,T)]}{f(t,T)} + \frac{\text{cov}_t(S(T), D(t,T))}{f(t,T)}$$

$$= E_t[D(t,T)] + \frac{\text{cov}_t(S(T), D(t,T))}{E_t[S(T)]}$$

$$= E_t[D(t,T)] + \text{cov}_t\left(\frac{S(T)}{E_t[S(T)]}, D(t,T)\right)$$

(2)

(3)

(4)

Equation (3) follows from Equation (2) by Equation (1) and Equation (4) follows from Equation (3) since $E_t[S(T)]$ is known at time $t$.

The term IFR is the inverse of the ratio of the futures price to the current stock price. Equation (4) says simply that the IFR equals the expected discount plus a covariance term. The first term on the right in Equation (4) is the expected discount factor over $[t,T]$, equal to the zero price at time $t$. The final term on the right in Equation (4) equals the covariance of the stock price at time $T$ and the (stochastic) discount factor, in each case viewed from time $t$.

It is instructive to compare Equation (4) with the analogous equation for a fair forward contract with forward price $F_0(t,T)$. Since the price of a fair forward contract should be zero, we have from the MPF:

$$S(t) = F_0(t,T)E_t[D(t,T)]$$

or equivalently,

$$\frac{S(t)}{F_0(t,T)} = E_t[D(t,T)]$$

(5)

Comparing Equations (4) and (5) and recalling the definition of the IFR, we see that the covariance term may be thought of as a convexity adjustment or bias.\(^{10}\)

Lacking a covariance term, the determination of an implied financing rate from a forward appears superficially more appealing than the more complex determination from a futures contract. More basically, an equity forward on a stock has two cash flows: a current fixed cash payment and a future equity-linked payment on the notional amount at maturity. Because there are only two cash flows, the forward-implied financing rate can be directly observed. By contrast, a futures contract that has daily, often substantial...
For empirical reasons,\textsuperscript{11} it is easier to work with the inverse \textit{forward} futures ratio than the inverse futures ratio itself. We will also need the forward discount

\[ D(t, T_1, T_2) \equiv \frac{D(t, T_2)}{D(t, T_1)} \]

We define the \textit{inverse forward futures ratio} or \textit{IFFR} by:

\[ IFFR \equiv f(t, T_2, T_1) \equiv \frac{f(t, T_1)}{f(t, T_2)}, \]

where \( T_2 > T_1 \) and the final quotient is the futures price for a contract with maturity \( T_1 \) divided by the futures price for a contract with maturity \( T_2 \), in each case observed at time \( t \).

In this article, \( T_1 \) and \( T_2 \) are the maturities of the near and next futures contracts, respectively, and, consistent with our data, we assume that

\[ \Delta T \equiv T_2 - T_1 = 0.25 \quad \text{(A1)} \]
\[ T_2 \leq 0.5 \quad \text{(A2)} \]

We measure time in years, so \( \Delta T = 0.25 \) of one year, or one-quarter.\textsuperscript{12} By Assumption (A1), Assumption (A2) is equivalent assuming \( T_1 \leq 0.25 \).

\textsuperscript{11}These reasons include high price volatility, asynchronicity between the close of the cash and the futures markets, market segmentation and the substantial volume of daily transactions in both the spot and the futures markets. A more detailed description of our methodology appears in the Computational Appendix Section 7.1.

\textsuperscript{12}See note 7 above. These assumptions come from the approximately quarterly calendar of futures contracts.
3.2 The Expected Discount and the IFFR

We can now begin the derivation of our basic equation. We start with the IFFR definition:

\[ \text{IFFR} = \frac{f(t, T_1)}{f(t, T_2)} = \frac{E_t[S(T_1)]}{f(t, T_2)} \]

\[ = \frac{E_t[E_{T_1}[S(T_2)D(t, T_1, T_2)]]}{f(t, T_2)} = \frac{E_t[S(T_2)D(t, T_1, T_2)]}{f(t, T_2)} \]

\[ = \frac{E_t[S(T_2)]E_t[D(t, T_1, T_2)]}{f(t, T_2)} + \frac{\text{cov}_t(S(T_2), D(t, T_1, T_2))}{f(t, T_2)} \]

\[ = E_t[D(t, T_1, T_2)] + \text{cov}_t \left( \frac{S(T_2)}{E_t[S(T_2)]}, D(t, T_1, T_2) \right) \]

where Equation (7) follows from the Tower Law and Equation (9) follows from Equation (8) since \( f(t, T_2) = E_t[S(T_2)] \) which is known at time \( t \).

Thus the inverse forward futures ratio \( f(t, T_2, T_1) \) is the expected forward discount factor \( E_t[D(t, T_1, T_2)] \) plus the covariance term

\[ \text{CT} \equiv \text{cov}_t \left( \frac{S(T_2)}{E_t[S(T_2)]}, D(t, T_1, T_2) \right) \]

By analogy to Equations 4 and 5, we see that \( \text{CT} \) in Equation (9) is substantially the same as a forward forward-futures bias.\(^{13}\)

Condensing, we have our basic equation:

\[ \text{IFFR} = E_t[D(t, T_1, T_2)] + \text{CT} \implies E_t[D(t, T_1, T_2)] = \text{IFFR} - \text{CT} \]

Equation (10) expresses the expected forward discount as the observable IFFR less \( \text{CT} \). The expected forward discount is closely related to the FIR, as we shall we below, but \( \text{CT} \) is not observable. In the next section, we will address this by developing bounds on the size of unobservable quantities in order to treat them as small errors.

3.3 General Case with Dividends

The previous equations assumed that the stock \( S(t) \) pays no dividends. However, many stocks in the S&P 500 Index pay dividends that have been

\(^{13}\) Compare Equations (4), (5) and Footnote 10 and accompanying text.
significant in many periods. We must therefore modify Equation (10) to reflect dividends.

We obtained implied dividend rates \( r_{\text{div}}(t) \) on the S&P 500 from the option markets, based on put-call parity.\(^{14}\) As derived in the Computational Appendix,\(^{15}\) the required modification to Equation (10) is:

\[
E_t[D(t, T_1, T_2)] = k_{\text{div}}(t) \times IFFR - CT
\]  

(11)

where\(^{16}\)

\[
k_{\text{div}}(t) = 1 - r_{\text{div}}(t) \Delta T
\]

\( k_{\text{div}}(t) \) lies between 0 and 1.\(^{17}\) Thus in the presence of dividends the left-hand side of Equation (10) decreases by \( k_{\text{div}}(t) \). We will frequently need to refer to \( \frac{1}{\Delta T} \), equal to the number or frequency of intervals of length \( \Delta T \) in a year, so we introduce the notation

\[
\omega_T \equiv \frac{1}{\Delta T}
\]

We have assumed \( \Delta T \) is 25\% of a year,\(^{18}\) so \( \omega_T \) correspondingly equals 4.

Ignoring \( CT \), we see that the dividend rate reduces the expected discount \( E_t[D(t, T_1, T_2)] \) and, as discussed in the Computational Appendix, correspondingly increases the forward FIR.\(^{19}\)

We may also rewrite Equation (11) to reflect dividends as:

\[
E_t[D(t, T_1, T_2)] = k_{\text{div}}(t) \times IFFR + \epsilon_{CT}
\]  

(12)

where we have treated \( CT \) as an error term, \( \epsilon_{CT} \equiv -CT \). The difference between Equation (11) and Equation (12) is merely a change of viewpoint on \( CT \).

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\(^{14}\)Wharton Research Data Services’ OptionMetrics contains a time series of estimated implied dividend rates \( r_{\text{div}}(t) \) obtained from a regression with the property that the present value at time \( t \) of the implied dividends over \( (t, T) \) equals \( r_{\text{div}}(t) \times S(t) \times (T - t) \), where \( S(t) \) is the price of the associated stock at time \( t \). See Ivy DB File and Data Reference Manual 30, Version 3.0, revised May 19, 2011.

\(^{15}\)Computational Appendix [7.2.].

\(^{16}\)Time is measured in full years, so a six-month interval measures 0.5.

\(^{17}\)Since (1) \( \Delta T \) is on the order of a quarter and (2) dividend rates on the S&P 500 have always been positive but less than 400\% (an extremely high rate).

\(^{18}\)Assumption (A1) above.

\(^{19}\)We define a forward FIR, which we will compare with forward market interest rates, for the reasons referred to in note 11 above.
3.4 FIR Definition and Error Terms

We now define the FIR formally through the following two equivalent equations, where \( G_{FIR} \) denotes growth under the FIR over the (approximately quarterly) period \( \Delta T \) and \( M(t, T_1, T_2) \equiv \frac{1}{D(t, T_1, T_2)} \):20

\[
(FIR(t) + 1)^{\Delta T} \equiv G_{FIR}(t) \equiv E_t[M(t, T_1, T_2)] \\
FIR(t) \equiv (G_{FIR}(t))^{\Delta T} - 1
\] (13)

FIR, \( G_{FIR} \) are both functions of \( t \), although for notational convenience we will suppress the dependence where the context makes it clear.

While Equation (13) defines the FIR as an annualized rate of expected growth, what theory gives us to estimate from our data is an expected discount, as indicated in Equations (10) and (11). As described in the Computational Appendix,21 the expected growth and expected discount are nearly inverse, with negligible error in the case at hand.

We therefore have the alternative equation, with small error \( \epsilon_{invert} \) from inverting around the expectation:

\[
G_{FIR} \equiv E_t[M(t, T_1, T_2)] = \frac{1}{E_t[D(t, T_1, T_2)]} + \epsilon_E^{invert} \] (14)

where from Equation (12)

\[
E_t[D(t, T_1, T_2)] = k_{div}(t) \times IFFR + \epsilon_{CT} \] (15)

We want to estimate \( G_{FIR} \) from the two observables we have:

\[
\hat{G}_{FIR}(t) \equiv (k_{div}(t) \times IFFR(t))^{-1} \] (16)

The estimate \( \hat{G}_{FIR}(t) \) is also a function of \( t \).

Equations (14) and (15) together introduce two sources of error, \( \epsilon_{CT} \) from the covariance term \( CT \) and \( \epsilon_E^{invert} \) from inverting the expectation of \( D \)

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20The discount is often expressed simply as the inverse of the money-market return, with no separate definition, since \( D \times M = 1 \). The FIR is, however, defined as an expected return, and thus we need to distinguish between expected growth and expected discount, since \( E[D] \times E[M] = 1 - \text{cov}(D, M) \neq 1 \).

21Section 7.4.1. See also Footnote 20 above.

22Id.
rather than taking the expectation of $M$. Combining them introduces a third source of error, from a Taylor series approximation to $1/x$:

$$G_{FIR}(t) - e_{E}^{invert}(t) = \frac{1}{k_{div}(t) \ast IFFR + e_{CT}(t)}$$

$$= \hat{G}_{FIR}(t) - e_{CT}(t) + e_{Taylor}(t)$$

where the error terms are also functions of $t$

We may rewrite this:

$$\hat{G}_{FIR}(t) = G_{FIR}(t) - e_{E}^{invert}(t) + e_{CT}(t) - e_{Taylor}(t)$$

$$\equiv G_{FIR}(t) + e_{consolidated}(t) \quad (17)$$

where the consolidated error $e_{consolidated} \equiv e_{CT} - e_{E}^{invert} - e_{Taylor}$ consolidates all sources of error.

Equation (17) requires only the absence of arbitrage and such general assumptions about the distributions of $S(t), r(t), f(t, T)$ as are necessary for Equation (1) and the MPF. However, there are three separate error terms that we must bound individually and then combine to determine a consolidated error bound in Equation (17).

Using a conditionally lognormal model discussed in the Computational Appendix,\(^{23}\) we may simplify the consolidated error term and its consolidated bound. Deferring the details to the Computational Appendix, here we state the bounds on the consolidated error term derived from this model that we will use:

$$b_{consolidated} = 0.8 \text{ bps} \quad (18)$$

$$b_{consolidated, annualized} = 3.5 \text{ bps} \quad (19)$$

As discussed in the Computational Appendix, these are both Hybrid-Average bounds. The latter is annualized based on quarterly compounding.

The consolidated bound $b_{consolidated}$ will give us the following empirical equation for the FIR which will be central to our FIR estimate:

$$G_{FIR} = \hat{G}_{FIR} - e_{consolidated}$$

$$= \frac{1}{k_{div}(t) \ast IFFR - e_{consolidated}} \quad (20)$$

$$|e_{consolidated}| \leq 0.8 \text{ bps}$$

\(^{23}\)Section 7.3.3.
Following Equation (20), we will estimate the FIR from 
\((k_{div}(t) \times IFFR)^{-1}\) and conclude that the true FIR lies within \(b_{\text{consolidated}} = 0.8 \text{ bps}\) of our estimate and the annualized FIR correspondingly lies within \(b_{\text{consolidated,annualized}} = 3.5 \text{ bps}\) of our annualized estimate.

4 FIR Spread Features and Arbitrage Limits

In this section, we discuss the reasons that the FIR and observable market interest rates\(^{24}\) may differ, yet have common statistical attributes. The difference accounts for the nonzero spread between the FIR and market interest rates, a spread that we shall see can be volatile. The commonality provides some plausibility for Assumptions (A4) and (A5) that we will use in the Computational Appendix to develop error bounds, including the bounds described in Section 3.4 above.\(^{25}\) Those assumptions are based on empirical observations of market interest rates, which, for the reasons discussed below, we then assume may apply to the unobservable FIR.

In elementary financial models,\(^{26}\) the FIR defined by Equation (13), were it attainable in the market using just stock and futures, would of necessity be equivalent to any market riskless interest rate. This is an elaboration of the Law of One Price.\(^{27}\)

The original, and familiar, argument for this equality is that two positions with identical payoffs must have the same price, or market actors will execute the lower-price position and short the higher-price position and earn without more (\(i.e.\) statically) a riskless profit, which is impossible in the absence of arbitrage.\(^{28}\) This argument extends to two positions that may be costlessly hedged into each other; they must also have the same price.\(^{29}\) The argument may be extended to dynamic hedges that must be constantly adjusted. The argument may be extended further to the case where the hedge has a cost or receipt, provided the inverse hedge is also available and has an opposite

\(^{24}\)A rate on traded debt instruments.
\(^{25}\)The referenced assumptions appear in Section 7.4.
\(^{26}\)Assuming, \textit{inter alia} the absence of arbitrage and all transaction costs, and also, if needed in addition, trader indifference to gamma and other residual hedging risks, as discussed in greater detail in the text below.
\(^{27}\)See [Anderson and Kercheval, 2010, Chapter 2].
\(^{28}\)According to a standard equilibrium argument the market actors drive the lower price higher and the higher price lower until they become equal, eliminating the arbitrage.
\(^{29}\)Compare note 8 above
receipt or cost. The price of the second position must equal that of the first plus the cost of carry.\textsuperscript{30}

However, even in the simple case of a static hedge (defined below), the extension is more complex in practice than this familiar argument suggests.\textsuperscript{31}

Consider a trader who wishes to provide her customer with a long forward on a stock with no dividends; under the Law of One Price the forward price should equal the stock’s spot price grown at the term interest rate until maturity.

The customary analysis proceeds by noting that the trader can execute the following hedge transaction with the same payoff as the stock forward: a spot stock purchase financed by a zero coupon term financing with maturity equal to the forward settlement date. Once put in place, the hedge need not be modified.\textsuperscript{32} Such a hedge is called static, by contrast with dynamic hedges that must be regularly or constantly adjusted.\textsuperscript{33}

This static hedge, however, rarely occurs. Among other considerations, most stock traders lack institutional authority to engage in a term borrowing; zero coupon financings are unconventional and thus costly; stock held in inventory has regulatory costs, including regulatory capital. As a result of these and other factors, the static hedge described in the previous paragraph would be prohibitively cumbersome and expensive.

A cheaper alternative is the following, which also starts with a spot purchase. The trader buys the stock and immediately lends it to another market participant, who sells it short, transferring cash or other collateral of

\textsuperscript{30}If the combined payoffs of the first position and the hedge, including the hedge’s cost or receipt, replicate the second position’s payoffs, and the inverse of the hedge is also a market transaction with opposite payoffs and cost or receipt, then the two positions must have the same price, adjusted for the hedge cost or receipt (the cost of carry). Plain vanilla interest rate swaps and short-dated exchange traded options are examples of such hedges that may be inverted at (nearly) opposite cost or receipt, given the tight bid-ask spreads associated with such highly liquid instruments.

\textsuperscript{31}See also, e.g., [Shleifer et al., 1997] (”Even the simplest realistic arbitrages are more complex than the textbook” arbitrage that ”requires no capital and entails no risk.”)

\textsuperscript{32}Absent events typically excluded in simple transaction models of this type, such as a borrower or lender bankruptcy.

\textsuperscript{33}The Black-Scholes portfolio in a stock and a bank account that replicates a stock option is one canonical example of a dynamic hedge. The efficacy of that replication assumes continuous adjustment and, at least in the simplest case, a known constant volatility. In the absence of these stylized and unrealistic assumptions, the hedge leaves residual open market risk.
comparable worth to the trader, which she immediately uses (if cash) or re-hypothecates (if collateral) to provide the funding for the original spot purchase. This transaction is more complex but can generally be concluded entirely by the trader without involving senior managers and consists of conventional components that can be executed simultaneously, efficiently and inexpensively.

As a result, this second transaction, with variations, is the preferred structure for hedging a customer stock forward. However, it leaves the trader with a variety of open (unhedged) risks. For example, prior to the forward settlement date, the stock borrower might return the stock and demand the collateral, in which case the trader must find a new stock borrower and associated collateral, usually on only a few days’ notice. In addition, the financing obtained through the collateral rehypothecation is typically at a floating rate, usually reset daily. The trader accordingly has interest rate risk. The forward price under the customer contract is set at the outset based on then-current market interest rates. If rates rise prior to the forward settlement date, the trader will have a corresponding loss, and if rates fall, a corresponding gain.34

The forward price is thus set based on market interest rates plus transaction costs, including a premium to compensate the trader for the open risk she takes and cannot efficiently hedge. The Law of One Price must include these transaction costs, a significant portion of which is the open risk the trader cannot eliminate, and the forward-implied rate may significantly exceed the market interest rate for the corresponding term when the associated risk premium is large. The excess may be significant in certain periods where, for example, interest rate hikes are anticipated or the stock is in high demand, the interest in shorting it low, and the corresponding return on the stock loan meager.

Alternatively, it is possible that the stock will be in high demand for shorting, in which case the trader can earn a premium on the stock loan, reducing transaction costs and permitting a forward-implied rate that can be significantly less than the corresponding market interest rate.

These considerations affect contractually-simpler forwards as well as fu-

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34Because the forward price is constant, a spike in interest rates could leave the trader with a loss. Hedging the floating rate into a compound fixed rate would reduce or eliminate this risk. Nevertheless, few traders hedge their interest rate risk, perhaps because an efficient market rate hedge would still leave meaningful basis risk against the bespoke collateral rate.
tures. However, while a forward contract has a single future cash flow at maturity, a futures contract has daily cash flows, equal to the unpredictable daily changes in futures prices. The residual open risk from hedging a futures position is accordingly considerably greater than from hedging a forward.

In addition, Equation (4) (for example) shows that a futures contract contains covariance risk not present in a forward contract. As a result, futures contracts present qualitatively greater risk than forwards, and in particular there can be no static hedge - not even a cumbersome and expensive one - of stock and futures contracts into a riskless investment.

For these and other reasons, the FIR need not correspond very closely to prevailing market interest rates. If it nevertheless differs for too long too dramatically in level or volatility from a market rate of interest, traders will be strongly motivated to take the open risk required to arbitrage away much the difference between the two rates.\footnote{Suppose, for example, the FIR rate were many points higher than LIBOR. Since both rates can provide instantaneously riskless investment and financing, a trader could borrow at LIBOR and invest in the FIR and earn the spread. A sufficiently high spread could compensate for the open risk required to replicate a FIR investment from market instruments (stock and futures). See Footnote 6 and accompanying text. A similar, if more complex, argument applies were FIR and LIBOR volatility to be sufficiently different.}

These considerations support, but of course do not confirm, the following assumption on the statistical similarity of the FIR to a market interest rate, which is fundamental to Equation (20) and the associated bound on $\epsilon_{consolidated}$, and thus to our estimate of the FIR. From this assumption, we expected to see mean reversion in the spreads between the FIR and market interest rates, which we did, as shown in Figure 4.

Assumption. In our research, we assumed that market actors’ motivation to profit from, and thereby eliminate, arbitrage was sufficient to conform (1) the variance of the FIR to the variance of market interest rates, which we took for these purposes to be Eurodollar rates, and (2) the covariance of the S&P 500 Index with the FIR to that index’s covariance with market interest rates. We further assumed a level at which the motivation was sufficient. More specifically, we assumed, as described in Assumption (A4) and Assumption (A5) in Section 7.3.3, that FIR volatility was less than 1% per annum and its correlation with the S&P 500 Index was less than 50% in absolute value.\footnote{These bounds were taken from observations of market rates.}

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5 Results

Based on the foregoing, we estimated the FIR as the annualized forward rate given by Equation (13), where we substitute our estimate for quarterly growth \( \hat{G}_{FIR} \):

\[
\hat{\text{FIR}}(t) \equiv \left( \hat{G}_{FIR}(t) \right)^{\omega_T} - 1
\]

Figure (1) shows our estimated FIR. For comparison purposes, we also included contemporaneous annualized market Eurodollar and Treasury forward rates.\(^{37}\) We then used an error bound around this estimate given by Equation (19).\(^{38}\)

The FIR is thus represented in Figure (2) as an annualized rate somewhere in the band (“FIR Band”) between the red and aqua lines labeled ‘Max FIR’ and ‘Min FIR’, respectively.

Visual inspection of this band indicates something happened a little before the beginning of 2001. Prior to that, the FIR Band was consistently above the higher Eurodollar market rate.\(^{39}\) Afterwards, even the top of the FIR Band frequently dipped below the lower Treasury market rate.

The FIR Band climbed consistently above the Eurodollar rate again by 2006, only to fall below it by 2008, where, with increased volatility, it has remained until recently.

5.1 Regimes

Motivated by this visual review, we defined four separate regimes, determined by reference to the passage of The Commodity Futures Modernization Act of 2000 (“CFMA”) and the 2008 financial crisis:

\(^{37}\) See Footnote 11 for a reference to a summary of our methodology.

\(^{38}\) Id.

\(^{39}\) It is unclear whether the Eurodollar rate represented a true market rate during the financial crisis, that is to say, a rate at which transactions actually occurred. See, e.g., Barclays Bank PLC Admits Misconduct Related to Submissions for the London Interbank Offered Rate and the Euro Interbank Offered Rate and Agrees to Pay $160 Million Penalty, Department of Justice. 27 June 2012. Retrieved November 15, 2016. One author, in private practice at the time, saw rates reported during this period that were significantly lower than the rates at which transactions were in fact available, if and to the extent transactions were available at all.
1. From 1996 until the passage of the CFMA, which we set at the end of 2000.

2. After the CFMA but before the financial crisis, which we treated as beginning in July of 2007, the month in which “Bear Stearns disclosed that ... two [of its] subprime hedge funds had lost nearly all of their value amid a rapid decline in the market for subprime mortgages.”

3. The financial crisis, which we treated as ending in March 2009, and


Figure (3) shows the same graph of the FIR Band with the four regimes superimposed.

5.2 Spread Analysis

Figure (4) shows a graph of the estimated FIR-Eurodollar and FIR-Treasury spreads with the four regimes superimposed. The spread series remain in a range between positive three and negative four percent, suggesting mean-reversion, which other features of the graphs also support.

We therefore treated the spread series in each regime as an Ornstein-Uhlenbeck process, which we estimated as a first order autoregressive (AR(1)) process, viewed as a discrete OU process.

Our objective was to answer basic questions about the spreads, such as whether the reversion mean (defined below) of the spread in a particular regime was greater, or less, than zero at a specified confidence interval.


41 The spread equals the FIR minus indicated market rate. Computational Appendix-Section 7.1 describes the methodology we used the determine the forward market rates and associated FIR spreads.

42 See also Section 4 above.

43 An Ornstein–Uhlenbeck (OU) process displays traditional mean-reversion.

44 Equivalently, was the difference between the FIR and the Eurodollar or Treasury rate statistically significant in that regime?
5.3 Ornstein-Uhlenbeck Estimation

An OU process satisfies:

\[ dx_t = \kappa (\mu - x_t) dt + \sigma dW_t \]  

where \( \kappa \) is the rate of mean reversion, \( \mu \) is the long-term or reversion mean\(^{45} \) and \( \sigma \) is the volatility of the process.

To discretize, we replace \( dt \) with \( \Delta t \) and \( dW_t \) with \( \sqrt{\Delta t} \epsilon_t, \epsilon_t \sim N(0,1) \), to derive the conventional form of an AR(1) process:

\[ x_{t+1} = \kappa \mu \Delta t + (1 - \kappa \Delta t)x_t + \sigma \sqrt{\Delta t} \epsilon_t \]

where:

\[ c = \kappa \mu \Delta t \]
\[ b = 1 - \kappa \Delta t \]
\[ a = \sigma \sqrt{\Delta t} \]

This represents a model for our two spread processes in which the reversion mean\(^{46} \) can be estimated via:

\[ \mu = \frac{c}{1 - b} \]  

5.4 Time Series Analysis

We begin our analysis with correlograms of the FIR spreads to Treasury and Eurodollar rates, which exhibit classic serial correlation expected from a mean-reverting process, as shown in Figure (5). As a result, simple confidence intervals constructed based on, for example, t-statistics are unreliable.

Instead, we fitted an AR(1) model to each of the two spread series in each of the four regimes, a total of eight subseries. We then estimated the reversion mean through Equation (23) above. To develop confidence intervals

\(^{45}\)Equivalently, the equilibrium level.
\(^{46}\)Computational Appendix Section 7.5 discusses the relationship between the reversion mean and the sample mean.
for this estimate, we used a procedure with 10,000 (block) bootstrapped (re)samples and a block length of 20 for each subseries.\footnote{See Footnote 11. The bootstrap procedure addressed the remaining serial correlation in the AR(1) residuals, but did not address the possibility that the volatility of the AR(1) residuals varied. Further research could include fitting AR(1) + GARCH(1,1) models to reflect this possibility.}

A graphical summary of our results appears in Figures (6) through (8). The first of these graphs shows the reversion means for both series in each of the four regimes. The next two show the bootstrapped confidence intervals in each regime for the FIR spreads to the market Eurodollar and Treasury rates, respectively.

5.5 Statistical Significance

A numerical summary of the reversion means and associated bootstrapped confidence intervals for the two series and four regimes, as well as for the total observation period from 1996 to August 2013, appears in Tables 1 and 2.

Table 1 shows the confidence intervals (Bootstrapped Confidence Intervals or BCI) resulting from our block bootstrap procedure around the FIR spreads determined using market Eurodollar and Treasury rates and annualized FIR estimates from Equation (21). Table 2 widens the confidence intervals to reflect the error bound from Equation (18) by adding the error bound to either end of the confidence interval.

The tables show that in all regimes and in the total period, the FIR exceeded the Treasury rate by a statistically significant spread, at the 95% confidence level.\footnote{With the possible exception of the financial crisis, where the lower limit of the 95% confidence interval is barely negative after reflecting the 3.5 bps error bound.} By contrast, the FIR was typically less than the Eurodollar rate by a (negative) spread that was also statistically significant at that confidence level. Prior to the passage of the CFMA, however, the FIR exceeded both Treasury and Eurodollar market rates.

The goals of the CFMA included ‘to streamline and eliminate unnecessary regulation for the commodity futures exchanges and other entities regulated under the Commodity Exchange Act’,\footnote{H.R.5660, Section 2, paragraph (2).} presumably at least in part to increase efficiency. The tables suggest the CFMA achieved this goal with respect to the futures contracts we studied: the spreads between the FIR
<table>
<thead>
<tr>
<th>Period</th>
<th>Eurodollar Rate</th>
<th>Treasury Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total (1996 to 2013/8)</td>
<td>(13)</td>
<td>39</td>
</tr>
<tr>
<td>BCI</td>
<td>(20)-(6)</td>
<td>33-45</td>
</tr>
<tr>
<td>1996 to 2000 (pre-CFMA)</td>
<td>16</td>
<td>67</td>
</tr>
<tr>
<td>BCI</td>
<td>10-25</td>
<td>61-74</td>
</tr>
<tr>
<td>2001 to 2007/6 (post-CFMA)</td>
<td>(8)</td>
<td>13</td>
</tr>
<tr>
<td>BCI</td>
<td>(19)-(2)</td>
<td>3-21</td>
</tr>
<tr>
<td>2007/7 to 2009/3 (financial crisis)</td>
<td>(109)</td>
<td>54</td>
</tr>
<tr>
<td>BCI</td>
<td>(126)-(58)</td>
<td>30-76</td>
</tr>
<tr>
<td>2009/4 to 2013/8 (recovery)</td>
<td>(29)</td>
<td>36</td>
</tr>
<tr>
<td>BCI</td>
<td>(38)-(22)</td>
<td>31-42</td>
</tr>
</tbody>
</table>

Table 1: Forward FIR Spreads to Two Market Rates in Four Regimes with Confidence Intervals Based on Block Bootstraps

<table>
<thead>
<tr>
<th>Period</th>
<th>Eurodollar Rate</th>
<th>Treasury Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total (1996 to 2013/8)</td>
<td>(13)</td>
<td>39</td>
</tr>
<tr>
<td>BCI</td>
<td>(23.5)-(2.5)</td>
<td>29.5-48.5</td>
</tr>
<tr>
<td>1996 to 2000 (pre-CFMA)</td>
<td>16</td>
<td>67</td>
</tr>
<tr>
<td>BCI</td>
<td>6.5-28.5</td>
<td>57.5-77.5</td>
</tr>
<tr>
<td>2001 to 2007/6 (post-CFMA)</td>
<td>(8)</td>
<td>13</td>
</tr>
<tr>
<td>BCI</td>
<td>(22.5)-1.5</td>
<td>(0.5)-24.5</td>
</tr>
<tr>
<td>2007/7 to 2009/3 (financial crisis)</td>
<td>(109)</td>
<td>54</td>
</tr>
<tr>
<td>BCI</td>
<td>(129.5)-(54.5)</td>
<td>26.5-79.5</td>
</tr>
<tr>
<td>2009/4 to 2013/8 (recovery)</td>
<td>(29)</td>
<td>36</td>
</tr>
<tr>
<td>BCI</td>
<td>(41.5)-(18.5)</td>
<td>27.5-45.5</td>
</tr>
</tbody>
</table>

Table 2: Forward FIR Spreads to Two Market Rates in Four Regimes with Confidence Intervals Based on Block Bootstraps including Error Bound

and both the Treasury and the Eurodollar forwards rates fell significantly beginning in 2001; the FIR-Eurodollar spread became negative.

As shown in Figures (9), (10) and (11), the Treasury and Eurodollar
spreads moved in opposite directions over the third and fourth regimes.\footnote{The dotted lines indicating 95\% confidence intervals do not include the 3.5 bps error bound.} In the financial crisis that started in 2007, the Treasury spread widened, though not to pre-CFMA levels, while the Eurodollar spread dropped to the lowest levels in our observation period. One reason for the drop in the Eurodollar spread could be the explicit and implicit collateral associated with the initial and variational margin required by the CME, especially the daily settlement procedures with respect to the latter.\footnote{See Footnotes (3), (4) and (10).} In particular, the daily mark-to-market arrangement implicit in variational margin posting eliminates most counterparty credit risk. This represented a very attractive credit profile during the financial crisis.\footnote{Though US Treasuries represented an even more attractive one. See Figures (4) and (9).}

6 Conclusions

We used financial theory to estimate the financing rate (the FIR) implied by US equity futures prices. Our method required the assumption that the FIR shared common statistical attributes with observable market interest rates, even while differing from them.\footnote{See Section 4.} By using forward rates, our method estimated the current FIR and associated FIR spread in real time.\footnote{Footnote 11 summarizes certain features that could make real-time estimation more difficult with spot rates.} The estimated FIR spreads were volatile and exhibited mean-reversion.\footnote{Figure (4).}

Our FIR estimate represented a financing rate that was almost always less attractive than Treasury rates, but typically more attractive than Eurodollar rates. Our estimates of the spreads between the FIR and these two market rates were both initially positive at the beginning of our observation period, but tightened significantly after the enactment of the CFMA in December 2000. During the financial crisis of 2007-2008, the spreads moved in opposite directions: the FIR-Eurodollar spread became considerably more negative, while the FIR-Treasury spread widened, but not to pre-CFMA levels. As the crisis passed, the two spreads reversed these moves: the FIR-Eurodollar spread rose, while remaining significantly negative, and the FIR-Treasury spread tightened, but not to pre-crisis post-CFMA levels.
7 Computational Appendix

7.1 Summary of our Methodology

We obtained daily futures price and interest rate data from Thomson Reuters Datastream (Datastream), and the implied dividend rate from Wharton Research Data Services’ OptionMetrics. We used the big (floor) contracts as having the longest history. The incorporation of the dividend rate is described in Section 7.2.

We downloaded as csv files daily observations on futures volume (VM) and closing price (PS) by futures contract maturity date and imported them into Python Pandas dataframes. We then combined these dataframes to create time series for pairs of consecutive futures, retaining only dates on which both contracts in a pair had volumes exceeding 50 contracts. Chaining the resulting pairs together to get a single series of (near, next) contract pairs, we computed a time series of FIR estimates from Equations (16) and (21).

We also obtained daily offered Eurodollar and Treasury interest rates from Datastream, generally at the benchmark maturities of one, three and six months and one year.

From each group of contemporaneous market rate observations we developed a zero curve using constant forward interpolation between the benchmark maturity dates. From this curve we computed the forward rate for that observation date from the maturity of the near to the maturity of the next futures contracts, and annualized that forward rate based on quarterly compounding for comparison with our contemporaneous FIR estimate from those contracts.

7.2 Effect of Dividends

We incorporated dividends into our model using daily implied dividend rates (≡ r_{div}(t)) from OptionMetrics.

---

56 See Footnote 14.
57 See Footnote 3. The closing prices were almost always the same as those of the E-Mini contract once the latter began trading.
58 See Footnote 7.
59 Their day count conventions were ACT/360 and bond basis, respectively.
OptionMetrics defined $r_{div}$ so the present value of dividends$^{60}$ equaled $r_{div}(t) \cdot S(t) \cdot (T - t)$.

With this definition of $r_{div}$, Equation (7) becomes

$$\text{IFFR} = \frac{E_t[S(T)]}{f(t, T_2)} = \frac{E_t[E_{T_1}[S(T_2)D(t, T_1, T_2) + r_{div}(t)S(T_1)\Delta T]]}{f(t, T_2)} \equiv \frac{\text{numerator}}{f(t, T_2)} \quad (24)$$

where $\Delta T \equiv T_2 - T_1$ and from Equation (24) and the Tower Law, $\text{numerator} \equiv E_t[S(T_1)] = E_t[S(T_2)D(t, T_1, T_2) + r_{div}(t)S(T_1)\Delta T]$.$^{61}$

Since $r_{div}(t), \Delta T$ are both known at time $t$:

$$\text{numerator} = E_t[S(T_2)D(t, T_1, T_2)] + r_{div}(t)E_t[S(T_1)]\Delta T$$

$$= E_t[S(T_2)D(t, T_1, T_2)] + r_{div}(t)(\text{numerator})\Delta T$$

from which we get

$$\text{numerator} = \frac{E_t[S(T_2)D(t, T_1, T_2)]}{1 - r_{div}(t)\Delta T}$$

or from Equation (24)

$$k_{div}(t) \times \text{IFFR} = \frac{E_t[S(T_2)D(t, T_1, T_2)]}{f(t, T_2)}$$

where $k_{div}(t) \equiv 1 - r_{div}(t)\Delta T$.

From this we get Equation (11) by replacing $\text{IFFR}$ in Equation (9) with $k_{div}(t) \times \text{IFFR}$ to reflect dividends.

$^{60}$Footnote 14. This present value is reflected in the difference between the prices of a call and a put, based on putcall parity. OptionMetrics uses this difference to determine $r_{div}$.

$^{61}$We assumed a flat 'dividend rate curve' so the forward dividend rate $r_{div}(T_1) = the$ spot dividend rate $r_{div}(t)$. Purely practically, there was no data on which to base a different assumption.
7.3 Conditionally Lognormal Model

7.3.1 Elementary Formulas

1. Lognormal Moments

Let \( X = e^{N(\mu, \sigma^2)} \).

Then the mean and volatility of \( X \) are given by:

\[
E[X] = e^{\mu + \sigma^2/2} \tag{25}
\]

\[
\sigma(X) = E[X]\sqrt{e^{\sigma^2} - 1} \tag{26}
\]

Proof: Well-known.

2. Covariance and Correlation of two Lognormal Random Variables.

Let:

\[
X_1 \sim e^{N_1(\mu_1, \sigma_1^2)}
\]

\[
X_2 \sim e^{N_2(\mu_2, \sigma_2^2)}
\]

with \( N_1, N_2 \) jointly normal having correlation \( \rho_{N_1, N_2} \). Then:

\[
cov_{X_1, X_2} = (e^{\rho_{N_1, N_2} \sigma_1 \sigma_2} - 1) \times E[X_1] \times E[X_2] \tag{27}
\]

\[
\rho_{X_1, X_2} = \frac{e^{\rho_{N_1, N_2} \sigma_1 \sigma_2} - 1}{\sqrt{e^{\sigma_1^2} - 1} \sqrt{e^{\sigma_2^2} - 1}} \tag{28}
\]

The result is elementary but we give a proof because it is fundamental to our error bounds. Write

\[
cov(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2] \tag{29}
\]

The moment-generating function of a multivariate normal \( N \) is

\[
E\left[e^{t'N}\right] = e^{t'\mu + \frac{1}{2}t'\Sigma t} \tag{30}
\]

The first cross-moment in Equation (29) above is just Equation (30) applied to the bivariate normal \( (N_1, N_2) \) with \( t = (1, 1) \), or

\[
e^{\mu_1 + \mu_2 + \frac{1}{2}(\sigma_1^2 + \sigma_2^2) + \rho \sigma_1 \sigma_2}
\]

Subtracting the final term in Equation (29) using Equation (25) gives \( \text{cov}(X_1, X_2) = (e^{\rho \sigma_1 \sigma_2} - 1)E[X_1]E[X_2] \), which is Equation (27). Equation (28) then follows by division using Equation (26).
7.3.2 Bivariate Brownian Motion

The model described in Section 7.3.3 below describes the evolution of stock prices and interest rates over time, which we model with a two-dimensional Brownian motion (as is common practice in the finance literature) and write \( \mathbf{B} = (B_1, B_2) \). Since we are interested in the correlation between a future stock price and a future discount factor, which depends on the pathwise integral of the interest rate, we introduce the following Theorem.\(^{62}\)

**Theorem: 7.1.** Let \( \mathbf{B} = (B_1, B_2) \) be a bivariate Brownian motion where \((B_1, B_2)\) are two standard Brownian motions with correlation \( \rho \). Then:

1. The endpoints \( (B_1(T + \frac{1}{4}), \int_T^{T + \frac{1}{4}} B_2(s)ds) \equiv (N_S, N_D) \) are jointly normal
2. The variance matrix of \((N_S, N_D)\) is

\[
\mathbf{V}(T + \frac{1}{4}) = \begin{bmatrix}
\sigma^2_{N_S} & \rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D} \\
\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D} & \sigma^2_{N_D}
\end{bmatrix}
\]

with

\[
\sigma^2_{N_S} = T + \frac{1}{4} \tag{31}
\]

\[
\sigma^2_{N_D} = \frac{1}{16} \left( T + \frac{1}{12} \right) \tag{32}
\]

\[
\rho_{N_S,N_D} = \rho \times \frac{T + \frac{1}{8}}{\sqrt{T + \frac{1}{4}} \times \sqrt{T + \frac{1}{12}}} \tag{33}
\]

The proof has two parts. First, we show the endpoints are jointly normal by expressing the integral as the limit of Riemann sums and using Levy’s Continuity Theorem. Then, we compute their variance matrix by exchanging expectation and integration.

We omit the details of the first part,\(^{63}\) assume \((N_S, N_D)\) are jointly normal and compute the variance matrix \( \mathbf{V}(T + \frac{1}{4}) \). We will need the well-known

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\(^{62}\)In the notation \((N_S, N_D)\), ”S” refers to stock while ”D” refers to discount. The rational will become clear further below.

\(^{63}\)For a proof see [Gunther, N., 2016]
formula for the covariance of Brownian motion with itself:

\[ E[B(t)B(s)] = \min(t, s) \quad (34) \]

\[ \forall s \leq T + \frac{1}{4}, E[B_1 \left( T + \frac{1}{4} \right) B_1(s)] = \min(s, T) = s \quad (35) \]

We also write

\[ B_2 = \rho \times B_1 + \sqrt{1 - \rho^2} B_{\perp} \quad (36) \]

with \( B_1, B_{\perp} \) independent.

We have:

1. \( V[1, 1] = T + \frac{1}{4} \), a standard result about Brownian motion.

2. We compute \( V[1, 2] = V[2, 1] \) by using Equation (36) and exchanging expectation an integration:

\[
E \left[ B_1 \left( T + \frac{1}{4} \right) \int_T^{T + \frac{1}{4}} B_2(s) ds \right] = \int_T^{T + \frac{1}{4}} E \left[ B_1 \left( T + \frac{1}{4} \right) B_2(s) \right] ds
\]

\[
= \int_T^{T + \frac{1}{4}} \rho s ds = \frac{\rho}{4} \left( T + \frac{1}{8} \right) \quad (37)
\]

3. We compute \( V[2, 2] = E \left[ \int_T^{T + \frac{1}{4}} B_2(s) ds \times \int_T^{T + \frac{1}{4}} B_2(s) ds \right] \) by starting with a more general computation:

\[
\sigma^2_{T_1, T_2} = E \left[ \int_{T_1}^{T_2} B_t dt \times \sigma \int_{T_1}^{T_2} B_s ds \right]
\]

\[
= \int_{T_1}^{T_2} \int_{T_1}^{T_2} E[B_u B_v] ds dt
\]

\[
= \int_{T_1}^{T_2} \int_{T_1}^{T_2} \min(s, t) ds dt \quad (38)
\]

\[
= \int_{T_1}^{T_2} \left( \int_{T_1}^{t} s ds + \int_{t}^{T_2} t ds \right) dt
\]

\[
= \frac{T_2^3}{3} + \frac{2T_1^3}{3} - T_2T_1^2 \quad (40)
\]

Remarks. Equation (38) exchanges the order of expectation and integration, while Equation (39) is Equation (34). Equation (40)
follows after some algebra and is the well-known formula $\frac{T^3}{3}$ when $T_1 = 0, T_2 = T$.

Specializing Equation (40) to the case $T_1 = T, T_2 = T + \frac{1}{4}$, most of the terms in the expression for $\sigma_{T_1, T_2}^2$ collapse and we have:

$$
\left(\frac{T_2^3}{3} + \frac{2T_1^3}{3} - T_2T_1^2\right) = \left(\frac{(T_1 + \frac{1}{4})^3}{3} + \frac{2T_1^3}{3} - (T_1 + \frac{1}{4})T_1^2\right)
$$

$$
\left(\frac{T_1^3}{3} + \frac{T_1^2}{4} + \frac{T_1}{16} + \frac{1}{192} + \frac{2T_1^3}{3} - T_1^3 - \frac{T_1^2}{4}\right)
$$

$$
= \frac{T_1}{16} + \frac{1}{192}
$$

$$
\Rightarrow \sigma_{T_1, T_1+\frac{1}{4}}^2 = \frac{1}{16} \times \left(T_1 + \frac{1}{12}\right) \quad (41)
$$

We complete the computation of $V(T + \frac{1}{4})$ as follows. We get Equations (31) and (32) for the diagonal entries of $V$ from the first and third (Equation (41)) items above. From these and the second item (Equation (37)), we get Equation (33) for the off-diagonal expression in Theorem 7.1:

$$
\rho_{N_S, N_D} = \frac{\text{cov}(N_S, N_D)}{\sqrt{T + \frac{1}{4}} \sqrt{\frac{1}{16} \times \left(T_1 + \frac{1}{12}\right)}} = \frac{\rho \left(T + \frac{1}{4}\right)}{\sqrt{T + \frac{1}{4}} \sqrt{\frac{1}{16} \times \left(T_1 + \frac{1}{12}\right)}}
$$

$$
= \rho \times \frac{T_1 + \frac{1}{4}}{\sqrt{T_1 + \frac{1}{4}} \times \sqrt{T_1 + \frac{1}{12}}}
$$

**Theorem: 7.2. Log Stock-Discount Distribution**

Let $(B_S^t, B_r^t)$ be a generalized bivariate Brownian motion where $B_S^t, B_r^t$ have volatilities $\sigma_S, \sigma_r$, respectively, and correlation $\rho$. Then the variance matrix $V(T + \frac{1}{4})$ of $(N_S, N_D)$ is given by

$$
\sigma_{N_S}^2 = \sigma_S^2 \left(T + \frac{1}{4}\right) \quad (42)
$$

$$
\sigma_{N_D}^2 = \sigma_r^2 \left(T + \frac{1}{12}\right) \quad (43)
$$

$$
\rho_{N_S, N_D} = \rho \times \frac{T + \frac{1}{8}}{\sqrt{T + \frac{1}{4}} \times \sqrt{T + \frac{1}{12}}} \quad (44)
$$
Proof: Theorem 7.2 follows from Theorem 7.1 because normality is closed under scaling, and the computation of $V(T + \frac{1}{4})$ is multiplicative under scaling.

**Lemma: 7.1. Stock-Rate Correlation Lemma**

Let $(B^S_t, B^r_t)$ be as in Theorem 7.2 and $0 \leq T \leq 0.25$. Then $\rho_{NS,ND}$ satisfies:

$$86.6\% |\rho| < |\rho_{NS,ND}| < 92\% |\rho|$$  \tag{45}$$

Proof: From Equation (44) we have

$$|\rho_{NS,ND}| = |\rho| \times \frac{T + \frac{1}{8}}{\sqrt{T + \frac{1}{4}} \times \sqrt{T + \frac{1}{12}}}$$

The fraction on the right-side is strictly increasing in $T$, as may be checked by differentiation or graphing. Evaluating it on $[0,0.25]$ gives the result in Equation (45).

### 7.3.3 The Model

To develop the consolidated bound $b_{consolidated}$, we assume, for each pair of futures contracts, a *conditionally joint lognormal* model for $S$ and $D$. In this model:

1. $S(t)$ evolves from $t$ to $T_2$ following geometric Brownian motion

$$\frac{dS}{S} = \mu_S dt + \sigma_S dB^S_t$$  \tag{46}$$

2. $D$ evolves according to an interest rate model described by the following SDE for the *instantaneous forward rate* $f(t, T)$ observed at time $t$ with maturity $T$, $T > t$.

$$d_t f(t, T) = \sigma_r dB^r_t + \sigma_r^2 (T - t) dt$$  \tag{47}$$

3. $(B^S_t, B^r_t)$ is a bivariate Brownian motion with unit volatilities and correlation $\rho$. 

26
From Equation (47) we may derive formulas for the instantaneous short rate $\equiv r_t$ and the stochastic discount $\equiv D(T_1, T_2)$:\textsuperscript{64}

$$r_t = \sigma_r B^r_t + f(0, t) + \frac{1}{2} \sigma_r^2 t^2$$ (48)

$$D(T_1, T_2) = e^{-\int_{T_1}^{T_2} r_t dt}$$

$$= e^{-(\sigma_r \int_{T_1}^{T_2} B^r_t dt + \int_{T_1}^{T_2} f(0, t) dt + \frac{1}{6} \sigma_r^2 (T_2^3 - T_1^3))}$$

$$= e^{\sigma_r \int_{T_1}^{T_2} B^r_t dt - \mu_r} = e^{\sigma_r \int_{T_1}^{T_2} B^r_t dt - \mu_r} \equiv e^{N_D^\mu}$$ (49)

with:\textsuperscript{65}

$$\mu_r \equiv \int_{T_1}^{T_2} f(0, t) dt + \frac{1}{6} \sigma_r^2 (T_2^3 - T_1^3) > 0$$

$$\implies D(T_1, T_2) = e^{N_D^\mu} < e^{\sigma_r \int_{T_1}^{T_2} B^r_t dt} \equiv e^{N_D}$$ (50)

Applying Theorem 7.2 to this model and fixing a pair of consecutive near and next futures with maturities $T_1 < T_2 = T_1 + \frac{1}{4}$, respectively, we have that $E_t[S(T_2)]$ and $E_t[D(t, T_1, T_2)]$ are jointly lognormal when conditioned on the information $\mathcal{F}_t$ at time $t$. Further, their normal exponents have volatilities and, from Lemma 7.1, correlation:\textsuperscript{66}

$$\sigma_{NS} = \sigma_S \times \sqrt{T_2} = \sigma_S \times \sqrt{T_1 + 0.25}$$ (51)

$$\sigma_{ND}^{\mu} \leq \sigma_{ND} = \frac{\sigma_r}{4} \times \sqrt{T_1 + \frac{1}{12}}$$ (52)

$$\rho_{NS, ND}^{\mu} = \rho_{NS, ND} < 92\% \times \rho$$ (53)

where $\hat{T} \equiv T - t$\textsuperscript{67} and under our model $\sigma_S, \sigma_r, \rho$ are the constant instantaneous volatilities of the stock and interest rate processes and their instantaneous correlation, respectively, conditional on this pair of futures.

From the lognormality of $S$ and $D$ we have:

\textsuperscript{64}In Equation (49) we have used that $B^r_t = -B^r_t$, by Brownian motion symmetry. See [Baxter and Rennie, 1996, Section 5.2] for a more detailed description of this simple model and a demonstration that Equation (48) follows from Equation (47).

\textsuperscript{65}We assume that forward interest rates are not significantly negative. The main consequence of the difference between $N_D^\mu$ and $N_D$ is therefore that $e^{N_D^\mu}$ is slightly smaller, and thus has slightly smaller volatility, than $e^{N_D}$. Scaling naturally has no effect on correlation.

\textsuperscript{66}See Footnote 65.

\textsuperscript{67}In light of Assumption (A1), $t$ and $\hat{T}$ each uniquely specify both each other and $\hat{T}_2$; we will use them interchangeably.

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Theorem: 7.3. Forward Discount

\[ E_t[D(t, T_1, T_2)] = k_{\text{div}}(t) \times f(t, T_2, T_1) \times e^{-\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D}^\mu} \]  

(54)

For notational convenience we have suppressed the dependence of \( \rho_{N_S,N_D}, \sigma_{N_S}, \sigma_{N_D}^\mu \) on \( \hat{T}_1 \).

Proof: From Equation (11) and the formula for the covariance of two jointly lognormal random variables in Equation (27):

\[
k_{\text{div}}(t) \times IFFR = E_t[D(t, T_1, T_2)] + CT \\
= E_t[D(t, T_1, T_2)] + \text{cov}(\frac{S(T_2)}{E_t[S(T_2)]}, D(t, T_1, T_2)) \\
= E_t[D(t, T_1, T_2)] + (e^{\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D}^\mu} - 1)E_t[D(t, T_1, T_2)] \\
= e^{\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D}^\mu} \times E_t[D(t, T_1, T_2)]  
\]  

(55)

where in Equation (55) we have used that \( E_t \left[ \frac{S(T_2)}{E_t[S(T_2)]} \right] = 1 \) to simplify Equation (27). From this Equation (54) follows directly.

While Equation (11) is general, Equation (54), which will be fundamental to our estimate of \( b_{\text{consolidated}} \), relies on the restrictive lognormal and other assumptions of our model.

7.4 Error Bounds

We make the following assumptions on these instantaneous volatilities and correlation:\footnote{Volatilities are annualized. While these assumptions are derived from observations on market prices and rates, \( r \) and \( D \) represent neither a market interest rate nor a market discount factor. Section 4 explores the basis for these assumptions.}

\[
\sigma_S \leq 20\% \quad \text{(A3)} \\
\sigma_r \leq 1\% \quad \text{(A4)} \\
|\rho| \leq 50\% \quad \text{(A5)} \\
\implies |\rho_{N_S,N_D}| < 46\% 
\]
7.4.1 Inversion Error

We will need the following additional assumption to complete our error bounds:

\[ 102\% \geq E_t[M(t, T_1, T_2)], \forall t, T_1, T_2 \] \hspace{1cm} (A6)

Using Assumptions (A4) and (A6) and the results of Section 7.3, we can now show the inversion bound \( \epsilon_E^{\text{invert}} \) satisfies:

\[ \epsilon_E^{\text{invert}} \leq 0.22 \text{ bps} \] \hspace{1cm} (56)

From Equation (14), we want to compare \( E_t[M(t, T_1, T_2)] \) with \( E_t\left[\frac{1}{D}\right] \). Let \( D = D(t, T_1, T_2) \), so the comparison we want is between \( E_t\left[\frac{1}{D}\right] \) and \( \frac{1}{E_t[D]} \).

We have

\[ E_t\left[\frac{1}{D}\right] - \frac{1}{E_t[D]} > 0 \]

by Jensen’s inequality applied to \( f(x) = \frac{1}{x} \). We may write the expression on the left as:

\[
\frac{E_t\left[\frac{1}{D}\right] E_t[D] - 1}{E_t[D]} = \frac{E_t\left[\frac{1}{D} \times D\right] - \text{cov}(\frac{1}{D}, D)}{E_t[D]} = -\text{cov}(\frac{1}{D}, D)
\]

\[ = \left(e^{-\rho \frac{1}{2} D \sigma_D} \sigma_D - 1\right) E_t\left[\frac{1}{D}\right] \leq \left(e^{0.25\%^2} \frac{1}{3} - 1\right) E_t\left[\frac{1}{D}\right] \]

\[ \leq 0.021 \text{ bps} \times 1.02 \leq 0.022 \text{ bps} \] \hspace{1cm} (58)

The first equality in Equation (57) follows from Equation (27). The second comes from the limit in Equation (63) below and the following two observations:

1. In our lognormal model we have from Equation (49) and Brownian motion symmetry\textsuperscript{70}

\[ \sigma_D \times \sigma_D = \sigma_{ND}^2 \]

and

\textsuperscript{69} This assumption is equivalent to assuming quarterly rates were not expected to exceed 2% and is consistent with our data. See Figure 1.\textsuperscript{,} See Footnote 64 and Equation (50).
2. Maximizing over $\rho, \sigma_{ND}$ we have

$$\max_{-1 \leq \rho \leq 1, 0 \leq \sigma_{ND} \leq \sigma_{\text{max}}} e^{\rho \sigma_{\text{max}}^2}$$

occurs at $\rho = 1, \sigma_{ND} = \sigma_{\text{max}}$, which follows because the exponential is strictly increasing in its exponent.

Equation (58) then follows from Assumption (A6).

**Conclusion.** The absolute value of the inversion error $|\epsilon_{\text{invert}}^E| \leq 0.022$ basis points.

### 7.4.2 Maximum Bound.

We seek a maximum bound $b_{\text{consolidated}}$ of the form in Equation (20).

Using Equations (14), (20) and (54), we may write:\footnote{From Equation (17) $\hat{G}_{FIR} \times (1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}) = \epsilon_{\text{Taylor}} - \epsilon_{\text{CT}}$. Equation (54), which follows from our lognormal model, allows us to combine two sources of error.}

$$\hat{G}_{\text{FIR}} = \frac{1}{E_t[D(t, T_1, T_2)]} + \epsilon_{\text{invert}}^E = \hat{G}_{\text{FIR}} \times e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu} + \epsilon_{\text{invert}}^E$$

$$= \hat{G}_{\text{FIR}} + \hat{G}_{\text{FIR}} \times (e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu} - 1) + \epsilon_{\text{invert}}^E$$

$$\Rightarrow \epsilon_{\text{consolidated}} = \hat{G}_{\text{FIR}} \times (1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}) - \epsilon_{\text{invert}}^E$$

$$\Rightarrow |\epsilon_{\text{consolidated}}| \leq \text{max}(\hat{G}_{\text{FIR}}) \times \text{max}|1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}|$$

using that $\hat{G}_{\text{FIR}} > 0$ and the triangle inequality.

We may thus take for our consolidated bound:

$$b_{\text{consolidated}} = \text{max}(\hat{G}_{\text{FIR}}) \times \text{max}|1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}| + \text{max}|\epsilon_{\text{invert}}^E| \quad (59)$$

$\hat{G}_{\text{FIR}}$ is observable, and on our data we have $\hat{G}_{\text{FIR}} \leq 1.02$.\footnote{Compare Footnote 69.} Section 7.4.1 showed that $\text{max}|\epsilon_{\text{invert}}^E| \leq 2.2 \times 10^{-6}$. Thus, all that remains to determine $b_{\text{consolidated}}$ is to determine max $|1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}|$.\footnote{Section 7.4.2 Maximum Bound. We seek a maximum bound $b_{\text{consolidated}}$ of the form in Equation (20). Using Equations (14), (20) and (54), we may write: $\hat{G}_{\text{FIR}} = \frac{1}{E_t[D(t, T_1, T_2)]} + \epsilon_{\text{invert}}^E = \hat{G}_{\text{FIR}} \times e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu} + \epsilon_{\text{invert}}^E$

$$= \hat{G}_{\text{FIR}} + \hat{G}_{\text{FIR}} \times (e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu} - 1) + \epsilon_{\text{invert}}^E$$

$$\Rightarrow \epsilon_{\text{consolidated}} = \hat{G}_{\text{FIR}} \times (1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}) - \epsilon_{\text{invert}}^E$$

$$\Rightarrow |\epsilon_{\text{consolidated}}| \leq \text{max}(\hat{G}_{\text{FIR}}) \times \text{max}|1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}|$$

using that $\hat{G}_{\text{FIR}} > 0$ and the triangle inequality.

We may thus take for our consolidated bound:

$$b_{\text{consolidated}} = \text{max}(\hat{G}_{\text{FIR}}) \times \text{max}|1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}| + \text{max}|\epsilon_{\text{invert}}^E| \quad (59)$$

$\hat{G}_{\text{FIR}}$ is observable, and on our data we have $\hat{G}_{\text{FIR}} \leq 1.02$. Section 7.4.1 showed that $\text{max}|\epsilon_{\text{invert}}^E| \leq 2.2 \times 10^{-6}$. Thus, all that remains to determine $b_{\text{consolidated}}$ is to determine max $|1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}|$.\footnote{Footnote 71 From Equation (17) $\hat{G}_{\text{FIR}} \times (1 - e^{\rho_{NS,ND} \sigma_{NS} \sigma_{ND}^\mu}) = \epsilon_{\text{Taylor}} - \epsilon_{\text{CT}}$. Equation (54), which follows from our lognormal model, allows us to combine two sources of error.}
Since volatility is non-negative, the sign of the exponent $\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D}$ in the second maximum is determined by the sign of $\rho$. Noting that Assumptions (A3) - (A5) define a strip that is symmetric around the $\rho$ axis, we have:

$$\left| 1 - e^{\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D}} \right| \leq (1 - e^{-\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D}}) \lor (e^{\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D}} - 1) \quad (60)$$

where on the left-hand side the range for $\rho$ is given by Assumption (A5) but on the right-hand side that range is also subject to the additional restriction that $\rho \geq 0$. With this additional restriction both terms on the left in Equation (60) are non-negative.

We have the following:

**Lemma: 7.2. Cosh Inequality**

$$e^x + e^{-x} \geq 2$$

which follows from the well-known result that $cosh(x) \geq 1$.

From this lemma, the second of the two terms on the right-hand side of Equation (60) is the larger. We therefore have:

$$\max \left| 1 - e^{\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D}} \right| \leq \max \left( e^{\rho_{N_S,N_D}\sigma_{N_S}\sigma_{N_D}} - 1 \right) \quad (61)$$

where we may replace Assumption (A5) with the more restrictive non-negative assumption:

$$0 \leq \rho_{N_S,N_D} \leq 46\% \quad (A5')$$

The maximum in the right-hand side of Equation (61) is over a term that, since two of the exponents are non-negative, is strictly increasing in all its three arguments $\rho$, $\sigma_{N_S}$, $\sigma_{N_D}$, and so the maximum occurs when each of these arguments is as large as possible. From Equations (50) to (53) and our four Assumptions (including Assumption (A5’)), this occurs when:

$$\hat{T}_{1,max} = 0.25 \implies \hat{T}_{2,max} = 0.5 \implies$$

$$\sigma_{S_{max}}^N = 20\% \ast \sqrt{\hat{T}_{2,max}} = 20\% \ast \sqrt{0.5} \approx 14.14\% \quad (62)$$

$$\sigma_{D_{max}}^{N'} \leq \sigma_{D,max}$$

$$\leq \frac{1\%}{4} \sqrt{\hat{T}_{1,max} + \frac{1}{12}} = 0.25\% \sqrt{\frac{1}{3}} \approx 0.1443\% \quad (63)$$

$$\rho_{N_S,N_D,\max} = 46\%$$

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from which we get the following consolidated error bound:

\[
\max \left| 1 - e^{\rho NS,ND\sigma NS\sigma ND} \right| \leq \left( e^{46}\sigma_{S,\max}^N\sigma_{D,\max} - 1 \right) \leq 0.94 \text{ bps}
\]

\[\Rightarrow |\epsilon_{\text{consolidated}}| \leq 1.02 \times 0.94 \text{ bps} + 2.2 \times 10^{-6} \leq 0.98 \text{ bps} \approx 1 \text{ bp} \quad (64)\]

We may therefore take as a maximum bound \(b_{\text{consolidated}} = 1 \text{ bp}\). This is a quarterly bound on the maximum error in estimating \(G_{FIR}\) from \(\hat{G}_{FIR}\).

### 7.4.3 Hybrid-Average Bound

The bound given by Equation (64) provides global maxima for \(|\epsilon_{\text{consolidated}}(t)| = |\epsilon_{\text{consolidated}}(T_1)|\). For each pair of futures contracts, these bounds are attained, if at all, on a single day when \(T_1, T_2 = 0.25, 0.5, \text{resp.}\), that is to say, when \(T_1, T_2,\) are at their maxima at the beginning of the period over which we observe the specific futures pair.

We have seen in Section 5.2 that the spread of the estimated to FIR to a market rate followed a mean-reverting process empirically. As discussed in Section 5.1, we are interested in the mean and other statistics of this process over four regimes, each regime comprising many months or years. Thus, an average error bound is more informative than a maximum. For example, if we look instead at the end of the observation period where \(T_1, T_2,\) are at their minima, \(T_1, T_2 = 0, 0.25, \text{resp.}\), we have:\n
\[
\sigma_{S,T_1=0,\max}^N = 20\% \ast \sqrt{0.25} = 10\%
\]

\[
\sigma_{D,T_1=0,\max}^D = 0.125\% \ast \sqrt{\frac{1}{3}} \approx 0.0722\%
\]

\[
\rho_{NS,N_D,\max} = 46\%
\]

\[
\max \left| 1 - e^{\rho NS,T_1=0,\max\sigma NS,T_2=0,\max} \right| \leq \left( e^{46}\sigma_{S,T_1=0,\max}^N\sigma_{D,T_2=0,\max}^D - 1 \right) \leq 0.32 \text{ bps}
\]

\[\Rightarrow |\epsilon_{\text{consolidated}}| \leq 1.02 \times 0.32 \text{ bps} + 2.2 \times 10^{-6} \leq 0.361 \text{ bps} \quad (65)\]

We may therefore take \(b_{\text{consolidated},T_1=0} = 0.361 \text{ bps}\). Here we have continued to use the maximum value for \(|\epsilon_{E_{\text{invert}}}|\), in effect relying on Equation (59) but

\[\text{Cf. Equations (63) and (63)}\]
replacing the bound on $|1 - e^{\sigma_{NS} N_D \sigma_{NS} \sigma_{ND}}|$ with the bound conditioned on $\tilde{T}_1 = 0$ discussed above.

It is tempting to simply average the two estimates given by Equation (64) and Equation (65) and get an "average" quarterly consolidated error bound of 0.671 bps.

Instead of such a simple arithmetic average, we compute a hybrid maximum average, to be both more precise and more conservative. We leave $\sigma_{NS}$ and the bound for $|\epsilon^{-\text{invert}}|$ at their maxima $\sigma_{NS,\max}$ and $2.2 \times 10^{-6}$, respectively, and integrate, over the interval $\tilde{T}_1 \in [0, 0.25]$, the term $e^{\rho \sigma_{NS,\max} \sigma_{ND}(\tilde{T}_1)} - 1$ from Equation (61), viewed as a function of $\tilde{T}_1$ through $\sigma_{ND}(\tilde{T}_1) = \frac{a_r}{4} \times \sqrt{\tilde{T}_1 + \frac{1}{12}}$.

The hybrid average error bound resulting from modifying Equation (59) as just described is:

$$b_{\text{consolidated},HAvg} = 0.8 \text{ bps} \quad (66)$$

This quarterly bound reflects the hybrid average value of $e^{\rho \sigma_{NS,\max} \sigma_{ND}(\tilde{T}_1)} - 1$, the maximum values of $\hat{G}_{FIR,\rho}$ and the $\epsilon^{-\text{invert}}$ bound of $2.2 \times 10^{-6}$. It is the bound we will use hereafter, annualized as described in the next section.

### 7.4.4 Annualization

We have established bounds around $\hat{G}_{FIR}$, which is a quarterly rate estimated from quarterly data. We want a bound around the FIR, which is an annualized rate.

Given

1. a quarterly rate $r_{\text{quarterly}}$,
2. a noisy estimate of that rate $\hat{r}_{\text{quarterly}}$ with quarterly error $\epsilon_{\text{quarterly}}$,
3. a quarterly bound $b_{\text{quarterly}}$ on that quarterly error, and
4. a bound $r_{\text{quarterly,\max}} \geq r_{\text{quarterly}}, \hat{r}_{\text{quarterly}}$

\[74\] Rounded up to the nearest tenth of a basis point from 0.768 bps. To compute the indicated integral, make a change of variable $u = 46\% \sigma_{NS,\max} \sigma_{r} \times \sqrt{t + \frac{1}{12}}$, which yields an integral of the form $\int e^u \times u \, du$ that can be integrated by parts.
under reasonable assumptions\textsuperscript{75} we may compute a corresponding annualized bound \( b_{\text{annualized}} \) on the error in estimating the corresponding annualized rate \( \hat{r}_{\text{annualized}} \) as follows.

\[
b_{\text{annualized}} = (1 + r_{\text{quarterly,max}} + b_{\text{quarterly}})^4 - (1 + r_{\text{quarterly,max}})^4
\]  

(67)

We make our final assumption:\textsuperscript{76}

\[
1.02 \geq \hat{G}_{\text{FIR}}, G_{\text{FIR}} \geq 0
\]

(A7)

From Equation (67) we may therefore take as our annualized bound:

\[
b_{\text{annualized}} = (1.02 + b_{\text{quarterly}})^4 - 1.02^4
\]

(68)

Using the maximum and hybrid-average bounds for \( b_{\text{quarterly}} \), we therefore have, in annualized basis points:

\[
b_{\text{consolidated,max,annualized}} \leq 4.17
\]

(69)

\[
b_{\text{consolidated,HAvg,annualized}} \leq 3.27
\]

(70)

In light of the conservative features of our hybrid averaging process, based as it was on, \textit{inter alia}, the maximum values of \( \hat{G}_{\text{FIR}}, \rho \) and \( \epsilon_E^{\text{invert}} \), and the empirical distribution of \( \hat{r}_{\text{quarterly}} \), we rounded up Equation (70) to the nearest one-half basis point and took for our consolidated annualized bound as:

\[
b \equiv b_{\text{consolidated,HAvg,annualized}} \equiv 3.5 \text{ bps}
\]

(71)

### 7.5 Relation Between Mean and Reversion Mean

Suppose \( x_t, t \in [0..T] \) is a univariate time series. We write \( x-, x+ \) for the two subseries \( x_{0..T-1} \) and \( x_{1..T} \), respectively. To compute an AR(1), we may regress \( x+ \) on \( x- \) to get:

\[
x+ = (a \times x-) + c
\]

where \( c = \bar{x+} - a \times \bar{x-} \)

\textsuperscript{75}See [Gunther, N., 2016].

\textsuperscript{76}The assumption that estimated and realized absolute growth were never negative, is mild and was consistent with our data. See Footnote 69.

\textsuperscript{77}See Footnote 76.
Suppose the averages of $x$, $x-$ and $x+$ are all the same and equal $\bar{x}$:

$$\bar{x} = \bar{x-} = \bar{x+} \quad \text{(72)}$$

Then we have the following expression for the reversion mean:

$$\mu = \frac{\bar{x} - a \times \bar{x}}{1 - a} = \bar{x}$$

One may therefore say that the difference between the reversion mean and the sample mean measures the extent to which Equation (72) fails to hold.
Figure 1: Forward FIR Estimate
Figure 2: Forward FIR Estimate with 3.5bps Covariance Band
Figure 3: Forward FIR Estimate, 3.5bps Covariance Band and Four Regimes
Figure 4: FIR Spread Mean Reversion in Four Regimes
Figure 5: Correlograms of FIR Spreads to Treasury and Eurodollar Rates
Figure 6: FIR Spread Means For Both Series in Four Regimes
Figure 7: Bootstrapped Confidence Intervals for Eurodollar Spread Series in Four Regions
Figure 8: Bootstrapped Confidence Intervals for Treasury Spread Series in Four Regimes
Figure 9: Boxplots of FIR-Treasury Spreads
Figure 10: Boxplots of FIR-Eurodollar Spreads
Figure 11: Treasury and Eurodollar Spread Boxplots.

Boxplots of Bootstrapped FR-BT Treasury Spread Means

Boxplots of Bootstrapped FR-BU Spread Means
References


