Static Models of Central Counterparty Risk

Samim Ghamami*

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Abstract

Following the 2009 G-20 clearing mandate, international standard setting bodies (SSBs) have outlined a set of principles for central counterparty (CCP) risk management. They have also devised formulaic CCP risk capital requirements on clearing members for their central counterparty exposures. There is still no consensus among CCP regulators and bank regulators on how central counterparty risk should be measured coherently in practice. A conceptually sound and logically consistent definition of the CCP risk capital in the absence of a unifying CCP risk measurement framework is challenging. Incoherent CCP risk capital requirements may create an obscure environment disincentivizing the central clearing of over the counter (OTC) derivatives transactions. Based on novel applications of well-known mathematical models in finance, this paper introduces a risk measurement framework that coherently specifies all layers of the default waterfall resources of typical derivatives CCPs. The proposed framework gives the first risk sensitive definition of the CCP risk capital based on which less risk sensitive non-model-based methods can be evaluated.

Keywords: Risk Management, Central Clearing Mandate, Central Counterparty Risk, Risk Capital, Stochastic Models, Copulas, Monte Carlo Simulation

*Federal Reserve Board, Washington, D.C. 20551, USA, email: samim.ghamami@frb.gov, and Center for Risk Management Research, University of California, Berkeley, CA 94720-3880, USA, email: samim_ghamami@berkeley.edu. I am grateful to Robert Anderson, Rama Cont, Jin-Chuan Duan, Kay Giesecke, Lisa Goldberg, Michael Gordy, Jose Lopez, and Oldrich Vasicek for their comments. I also thank the participants at the 2014 Seminar Series at the Center for Risk Management Research at University of California, Berkeley, the Center of Financial and Risk Analytics at Stanford University, the Federal Reserve Bank of New York, the Federal Reserve Bank of San Francisco, the participants at the 2014 Credit Risk Symposium at the Risk Management Institute at National University of Singapore, the participants at the 2014 Quant Congress in New York and the 2014 Global Derivatives USA in Chicago, and the participants at the 2014 Workshop on Systemic Risk Monitoring at the Isaac Newton Institute, Cambridge, England, and the 2015 Workshop on Systemic Risk and Financial Networks at the Institute for Pure and Applied Mathematics at University of California, Los Angeles. The opinions expressed in this paper are my own and do not necessarily reflect the views of the Board of Governors or its staff.

1 Introduction

The state of research on the *optimal* CCP design and the impact of derivatives CCPs on systemic risk is still not conclusive, and there is ongoing debate on whether central clearing through CCPs would make the financial system more stable (see, e.g., Pirrong [2013], Koeppl and Monnet [2013], Cont and Kokholm [2012], Duffie and Zhu [2011], and the references therein). Despite the uncertainties surrounding the CCPs' optimal structure and their impact on the financial system, the 2009 G-20 OTC derivatives reform program and the subsequent clearing mandate necessitate proper and practical central counterparty risk management.[1]

CCPs, which are intended to mitigate counterparty credit risk, have been widely employed in exchange-traded futures and options for decades. Consider a derivatives CCP that stands among a set of bilateral counterparties to OTC derivatives transactions – it becomes *the seller to every buyer and the buyer to every seller*. Suppose that the CCP has n direct clearing members, $CM_1, ..., CM_n$, and it holds a derivatives portfolio with each of them.[2] The financial resources of a CCP that are to protect it from the default of clearing members are often used in a pre-specified order and are commonly referred to as the *default waterfall* resources of the CCP as summarized below. Upon the default of a clearing member with which the CCP has positive exposure, the defaulter resources are used first. These are the defaulter-pay variation margin, VM_i, initial margin, IM_i, and the prefunded default fund contribution of CM_i, which is denoted by DF_i. Variation and initial margin, which are defined in Section 2, depend on the value of the derivatives portfolio that the CCP holds with the clearing member as it evolves over time and so are updated by the CCP on a frequent basis. Unlike variation and initial margins, the prefunded default funds are defined either on an ad hoc basis or qualitatively in practice and assumed as a priori-given quantities in the CCP research literature. The potential loss exceeding the defaulter-pay layer of the default waterfall is to be absorbed by the CCP itself and the surviving members' resources. These are the CCP's equity contributions, E, and the survivor-pay prefunded default fund contributions. The potential loss exceeding these layers of the default waterfall is protected by the surviving members' additional contributions, which are referred to as unfunded default funds denoted by $\tilde{DF}$, (see Pirrong [2011], Duffie et al. [2010], and the references therein for more on default waterfall resources of typical derivatives CCPs).

Derivatives CCPs collectively take a relatively well-defined and mathematical-model-based approach to specify the variation and initial margins of the clearing members. However, the remaining layers of CCP default waterfall resources, i.e., the prefunded and unfunded default funds, are often specified qualitatively or on an ad hoc basis.[3] This is mainly because, post financial crisis, the international standard setting bodies (SSBs) responsible for regulation of derivatives CCPs have outlined broad and non-mathematical-model-based principles for CCP

[1] In 2009, the G-20 leaders agreed that all standardised OTC derivative contracts should be cleared through CCPs. The clearing mandate has been embodied in the Dodd-Frank Wall Street Reform and Consumer Protection Act in the US and the European Market Infrastructure Regulation (EMIR) in Europe.

[2] Indirect (client) clearing is not considered in this paper.

[3] This observation has been made based on the author's involvement in the regulatory risk management oversight of derivatives CCPs. The CCP risk measurement framework introduced in this paper, to the best of the author's knowledge, has neither appeared in the literature nor been used by any CCP.

risk management; particularly, for the default waterfall resources beyond initial margin. In contrast to the CCP regulatory risk management principles, SSBs have taken a formulaic approach to define the CCP risk capital of clearing members after the clearing mandate.[4] As will be illustrated in this paper, the clearing members' CCP risk capital depends on all layers of default waterfall resources of a CCP in a complicated way, and a logically consistent and risk sensitive definition of the CCP risk capital in the absence of a mathematical-model-based CCP risk measurement framework is impossible. In the absence of a unifying framework for default waterfall resources, the CCPs' fragmented risk management framework could create risk capital inconsistencies among CCPs and their clearing members.

This paper introduces a CCP risk measurement framework that specifies the default waterfall resources of derivatives CCPs coherently by using well-known mathematical modeling approaches in finance. Specifically, the main contributions of this paper are two-fold as summarized below:

The proposed framework gives the first risk sensitive definition of the CCP risk capital based on which non-model-based methods can be evaluated.

As will be shown in Section 2.1, the regulatory specification of the clearing members' prefunded default funds (guarantee fund) is broad and risk insensitive in the sense that the credit quality of the clearing members and their correlation have not been taken in to account. Moreover, once the total guarantee fund is specified, the regulatory framework does not specify how the total prefunded defaults funds should be allocated to clearing members.

This paper's unifying default waterfall model gives the first risk sensitive definition of the total prefunded default funds and proposes a well-known risk allocation technique to specify the prefunded default fund contribution of each clearing member to the CCP.

Viewing the CCP as a financial institution that holds a portfolio of clearing members' portfolios, Section 2 considers the risk management problem that a CCP faces in the presence of clearing members' variation and initial margin as a portfolio counterparty credit risk problem. Then, Section 2 uses the static copula threshold portfolio credit risk approach to approximate the CCP's portfolio counterparty credit risk. Next, based on the one-period model, a two-step procedure is introduced. This procedure first specifies the total prefunded default funds of the CCP based on a particular risk measure and then uses the well known Euler principle to allocate the prefunded default fund to clearing members. After the prefunded default funds are specified, Section 2 uses the one-period model to define and formulate the unfunded default funds. The CCPs' strict margin requirements are often criticized due to the destabilizing procyclicality they may cause during times of financial stress. Section 3 illustrates how margins' procyclicality can

[4]The committee on payments and market infrastructures (CPMI) and the technical committee of the international organization of securities commissions (IOSCO) have developed regulatory principles for financial market infrastructures (FMIs), and the Basel committee on banking supervision (BCBS) has developed CCP risk capital charges in consultation with CPMI and IOSCO, (see CPMI and IOSCO [2012] and BCBS [2014a]).

be reduced in the one-period model. Section 4 derives the total potential losses incurred by clearing members due to their CCP exposures and then illustrates how the CCP risk capital can be rigorously defined based on the these total losses. Section 5 discusses the regulatory CCP risk capital and illustrates some of the challenges arising due to the absence of a model-based CCP risk measurement framework. Section 6 outlines how all the layers of the CCP default waterfall in the proposed one-period model can be estimated with Monte Carlo simulation in practice. The main purpose of Section 6 is to illustrate that some of the Monte Carlo procedures that have been successfully developed in the past decade can be used in our framework.

2 One-Period Models of Derivatives CCPs

Suppose that at a fixed time, $T > 0$, clearing member i, CM_i, defaults with probability p_i; $i = 1, ..., n$. The Bernoulli random variables $Y_1, Y_2, ..., Y_n$ are the default indicators in the one-period model; $E[Y_i] = P(Y_i = 1) = p_i$, $i = 1, ..., n$. Clearing member default indicators are made dependent by being defined as functions of n dependent underlying random variables, $X_1, ..., X_n$, using copula threshold models, (see, e.g., Chapter 8 of McNeil et al. [2005]). Based on the foundational work of Merton [1974], CM_i defaults at time T if X_i crosses a given threshold value denoted by x_i. In a t-copula threshold model specified below, $X_1, ..., X_n$, are dependent Student t random variables,

$$Y_i = 1\{X_i > x_i\}, \text{ and } X_i = \frac{\sum_{j=1}^{d} a_{ij} Z_j + a_{i0}\xi_i}{\lambda}; \quad i = 1, ..., n, \tag{1}$$

where x_i's are chosen to match the marginal default probabilities, p_i. The correlation among X_i's are specified through a *factor model*, where the *common risk factors*, $Z_1, ..., Z_d$, and the idiosyncratic risk factor, ξ_i, are independent standard normal random variables, and $a_{i0}, a_{i1}, ..., a_{id}$ are chosen such that the numerator becomes a standard normal random variable; $\lambda \equiv \sqrt{\frac{K}{v}}$, and K has distribution χ^2_v and is independent of $Z_1, ..., Z_d$, and ξ_i's. The $t-$copula model above becomes a normal copula by setting λ equal to 1 in which case $X_1, ..., X_n$ become correlated normal random variables.[5] Typical credit loss distributions are skewed with a relatively heavy upper tail; it is well-known that these empirical properties can not be captured by normal copula models as they – compared to other copula models, e.g., t-copulas – assign very small probabilities to simultaneous defaults, (see Sections 8.3.5 and 8.4.6 of McNeil et al. [2005] and the references there). As will be illustrated in the sequel, the CCP's default waterfall layers beyond variation and initial margin are defined based on clearing members default indicators. Simultaneous defaults of clearing members and the tail of the CCP's loss distribution should be modeled properly. Otherwise, CCP default waterfall resources could be underestimated; t-copula threshold models are preferable to normal copula threshold models in the proposed CCP risk measurement framework.[6]

[5]The one-factor normal copula threshold model was first introduced by Vasicek [1987] for loan portfolio modeling.

[6]t-copula threshold models can be calibrated similar to the widely used normal copula models as a multivariate t random vector can be viewed as a multivariate normal vector with a randomly scaled covariance matrix. The

Let C_i denote the CCP's collateralized credit exposure to CM_i at its default in the presence of the CM_i's variation and initial margins that have been posted to the CCP. The credit risk capital has been historically defined based on one-period models, where the exposure at default of an obligor in a loan portfolio is usually approximated by a constant. Basel II-III and risk capital researchers have also adapted one-period models for the counterparty credit risk capital by using the *loan equivalent approach* and average-type dynamic counterparty risk measures such as expected positive exposure (EPE) or effective EPE as loan-equivalent exposures at default, (see Chapter 14 of Crouhy et al. [2001] on the loan equivalent approach, Pykhtin and Zhu [2006] for application of this approach in Basel II, and Gregory [2010] for various risk measures used in counterparty credit risk management).[7] We use the similar approach and define C_i's based on the loan-equivalent exposures at default for the proposed one-period model.

Specifically, let $V_i^+(t) \equiv \max\{V_i(t), 0\}$ denote the positive part of the value of the derivatives portfolio that the CCP holds with CM_i at time $t > 0$, $i = 1, ..., n$. The CCP's collaterized exposure at time t in the presence of variation margin and initial margin can be defined as follows,

$$e_i(t) \equiv \max\{V_i^+(t + \Delta) - VM_i(t) - IM_i(t), 0\}, \tag{2}$$

where the VM available to the CCP at time t conditional on CM_i's time-t default, depends on the frequency of variation margin calls denoted by $\hat{\Delta}$. More specifically, $VM_i(t) \equiv V_i^+(t - \hat{\Delta})$. Often, $\hat{\Delta}$ is set equal to 1 day. The time interval, $\hat{\Delta} + \Delta$, is usually pre-specified and is referred to as the *margin period of risk* (MPOR). MPOR is to represent the amount of time from the last VM call that a CCP requires to replace a defaulting clearing member's derivatives portfolio, and it is usually set equal to 5 days for derivatives CCPs. In fact, given the concept of MPOR, it will be more conservative to define the uncollateralized replacement cost of the CCP due to a clearing member default at time t as $\max_{t \le u \le t+\Delta}\{V_i^+(u)\}$. However, as V could represent the value of hundreds or possibly thousands of derivatives contracts, accurate estimation of this maximum is impossible in practice. So, $V_i^+(t + \Delta)$ is used for the uncollateralized time-t replacement cost.

Initial margin is a widely used risk mitigant for CCPs that has also been recently imposed on OTC derivatives market participants for non-centrally cleared transactions (see BCBS and IOSCO [2012]). IM is often defined based on a particular risk measure, e.g. value at risk (VaR) or expected shortfall (ES), associated with the random variable $R_i \equiv V_i^+(t + \Delta) - V_i^+(t - \hat{\Delta})$. Given some confidence level $\tilde{\alpha} \in (0, 1)$, the VaR-based and ES-based $IM_i \equiv IM_i(t)$ are, [8]

$$IM_i \equiv VaR_{\tilde{\alpha}}(R_i) = \inf\{r \in \mathcal{R} : P(R_i > r) \le 1 - \tilde{\alpha}\}, \quad IM_i \equiv ES_{\tilde{\alpha}}(R_i) = E[R_i | R_i \ge VaR_{\tilde{\alpha}}(R_i)].$$

Bernoulli mixture representation of the one-factor version of the above-mentioned t-copula threshold model allows the modeler to conduct a formal statistical estimation of model parameters with an approximate Maximum Likelihood type procedure, (see Section 8.4 of McNeil et al. [2005] for the Bernoulli mixture characterization of copula-threshold models and their Section 8.6 for statistical parameter estimation of mixture models).

[7]Some authors use the term *credit-equivalent* approach (exposures) for loan-equivalent approach (exposures).

[8]For a discontinuous loss distribution function, the widely used definition of ES does not hold for all $\tilde{\alpha}$. The following term should be added to the ES in the continuous case, $(1 - \tilde{\alpha})^{-1} VaR_{\tilde{\alpha}}(1 - \tilde{\alpha} - P(R_i \ge VaR_{\tilde{\alpha}}))$.

Then, the CCP's EPE-based time-T loan-equivalent collateralized exposure due to the CM_i's default becomes,

$$C_i \equiv (1-\delta_i)EPE_i = (1-\delta_i)\int_0^T E[e_i(t)]dt, \tag{3}$$

where δ_i is the recovery rate of CM_i. For simplicity, we assume that δ_i's are constant. Similarly but more conservatively, C_i, can be defined based on effective EPE ,

$$eEPE_i = \int_0^T \max_{0\le u\le t} E[e_i(u)]dt.$$

Monte Carlo simulation is widely used to estimate various counterparty credit risk measures, (see, e.g., Ghamami and Zhang [2014]). Section 6 summarizes similar ideas for Monte Carlo EPE or eEPE-based estimation of C_i's. Loan-equivalent collateralized exposures need not be defined based on average-type counterparty credit risk measure. For instance,

$$\int_0^T VaR_{\hat{\alpha}}(e_i(t))dt,$$

can be considered as a quantile-based dynamic measure of counterparty credit risk at a given confidence level, $\hat{\alpha}$, and can be used to define C_i's. [9]

Note that, for simplicity, it has been assumed that the credit quality of the clearing members does not depend on the value of the cleared derivatives portfolios. That is, the so called *dependent risk – wrong way or right way risk* – is not captured in the above formulation, (see Ghamami and Goldberg [2014] for subtleties involved in modeling dependent risk in practice). This can be seen by referring to (1) and (3), and noting that the CCP's derivatives portfolio values with CM_i's, i.e., V_i's, are defined independent from CM_i's credit qualities represented by p_i's.

Section 2.1 illustrates how prefunded default fund contributions, $DF_1, ..., DF_n$ are specified based on the CCP's loan-equivalent collateralized exposure to clearing members and default indicators, $(C_1, Y_1), ..., (C_n, Y_n)$, under the proposed one-period model. For the remainder of this section suppose that prefunded default fund contributions are given. Let U_i denote the CCP's credit exposure to CM_i at its default that has been further mitigated by the CM_i's prefunded default fund contribution. That is,[10]

$$U_i = (C_i - DF_i)^+.$$

[9] CCPs can auction the defaulting member's portfolio to other clearing members. This feature does not exist for non-centrally cleared portfolios. Assuming a well-managed auction mechanism, one can argue that the replacement cost that a CCP faces due to a member's default is lower compared to the replacement cost incurred by a bilateral counterparty.

[10] As will be shown in the next section, DF_i is defined based on $(C_1, Y_1), ..., (C_n, Y_n)$ such that $C_i \ge DF_i$ for all $i = 1, ..., n$. So, the one period model gives $U_i = (C_i - DF_i)^+ = C_i - DF_i$. In this section, however, one can think of U_i's as model independent quantities.

Recall the CCP's default waterfall, the potential loss to the CCP after the defaulter-pay resources, VM, IM and DF, are used, is

$$L^{(1)} = \sum_{i=1}^{n} U_i Y_i.$$

This loss is then to be protected by the CCP's equity contribution, E, and the survivor-pay prefunded default fund contributions. The CCP's *second-level* potential loss, after the survivor-pay resources and the CCP's equity contribution are used, becomes,

$$L^{(2)} = \left(\sum_{i=1}^{n} U_i Y_i - E - DF_s \right)^{+}, \tag{4}$$

where,

$$DF_s \equiv DF - \sum_{i=1}^{n} DF_i Y_i,$$

is the prefunded default fund contribution of the surviving members and $DF \equiv DF_1 + ... + DF_n$. Note that $L^{(2)}$ represents the loss to the CCP that would exceed the defaulter-pay resources, the CCP's equity contribution, and the prefunded default funds of the surviving members. $L^{(2)}$ or part of it is allocated to the surviving members in the form of the CCP's unfunded default fund capital calls.

Next section introduces a 2-step procedure that specifies the prefunded default funds in the one-period model. Using the model-based characterization of $DF_1, ..., DF_n$, Section 2.2 shows how the unfunded default funds are specified.

2.1 A Two–Step Procedure to Define the Prefunded Default Funds

The regulatory CCP risk management framework is developed based on a set of broad principles referred to as the *Principles for Financial Market Infrastructures* (PFMI), (see CPMI and IOSCO [2012]). Derivatives CCPs often specify their total prefunded default funds (guarantee funds), DF, based on the so called *Cover 1/Cover 2* principle of PFMI, which is summarized below:[11]

A systemically important CCP or a CCP that is involved in activities with complex risk profiles should maintain financial resources to cover the default of two participants that would potentially cause the largest aggregate credit exposure for the CCP in extreme but plausible market conditions. Other CCPs should maintain financial resources to cover the default of the participant that would potentially cause the largest aggregate credit exposure for the CCP in extreme but plausible market conditions.

[11] CCP's are considered to be a particular type of FMIs by the PFMI.

As will be shown in this paper, at least three ambiguous and risk insensitive aspects of the *Cover 1/Cover 2* principle can be dealt with by introducing a mathematical model for the default waterfall resources of derivatives CCPs. First, when n, the number of clearing members, becomes relatively large and the portfolios that the clearing members hold with the CCP are relatively homogeneous, it is not clear whether defining the DF based on the *Cover 1/Cover 2* principle would protect the CCP sufficiently well compared to the case where the CCP has a smaller number of clearing members with more heterogenous portfolios. Second, the credit quality of the clearing members and the correlation among them do not play any role in the *Cover 1/Cover 2* principle. Third, the *allocation of DF* to clearing members has remained a subjective matter among derivatives CCPs. It is in the above-mentioned sense that we refer to the PFMI's treatment of the prefunded default funds as broad, risk insensitive, and qualitative.

Even if before the clearing mandate, CCPs could have some discretion in specifying their default waterfall resources beyond variation and initial margin, the regulatory CCP risk capital requirements on clearing members may create an ambiguous central clearing environment if prefunded and unfunded default funds continue to be specified qualitatively, (see and compare BCBS [2012]-BCBS [2014a] and CPMI and IOSCO [2012]).[12] As will be shown in Sections 4 and 5, clearing members' CCP risk capital depends on prefunded default funds; inconsistencies in how these funds are specified and allocated to clearing members will directly affect the CCP risk capital requirement of clearing members.

Introducing ideas from the financial risk management literature, this paper proposes a two-step procedure for specifying DF and $DF_1, ..., DF_n$ under the one-period model.

First, define the total prefunded default fund, DF, based on a particular risk measure – value at risk (VaR) or expected shortfall (ES) – associated with the total loss in the presence of only variation and initial margins. The CCP's total credit loss is denoted by L and specified as follows,

$$L = \sum_{i=1}^{n} C_i Y_i, \tag{5}$$

where C_i, defined in (3), is the CCP's loan-equivalent exposure to CM_i at it's default under the one-period model. Given some confidence level $\alpha \in (0, 1)$, the VaR-based DF becomes,

$$DF \equiv VaR_\alpha(L) \equiv VaR_\alpha = \inf\{l \in \mathcal{R} : P(L > l) \leq 1 - \alpha\}, \tag{6}$$

and the ES-based DF for the confidence level α becomes,

$$DF \equiv ES_\alpha(L) \equiv ES_\alpha = E[L | L \geq VaR_\alpha(L)]. \tag{7}$$

[12] As will be illustrated throughout the paper, the CCP risk capital requirements necessitate a unifying model-based framework for the default waterfall resources of derivatives CCPs. One might subjectively argue that in the absence of the CCP risk capital requirements and the desire to promote central clearing, a principle-based CCP risk measurement framework whose only mathematical-model-based component is clearing members' strict margin requirements may maintain the financial resiliency of derivatives CCPs.

It is well known from the axiomatic characterization of risk measures that ES is preferable to VaR as the former is a coherent risk measure while the latter is not, (see Chapter 6 and proposition 6.9 of McNeil et al. [2005]).

In the one-period model of central counterparty risk, the total prefunded default funds, DF, of a CCP is defined similar to the way the portfolio credit risk capital is defined for a loan portfolio. The CCP can be in fact viewed as a financial institution exposed to its portfolio counterparty credit risk, where the portfolio constituents are the CCP's clearing members' derivatives portfolios, and the CCP's exposure to each clearing member is collateralized by both variation and initial margin. The static model is then a one-period approximation of the portfolio counterparty credit risk as the potential default of each clearing member is allowed to occur at a single given time point denoted by T, (both Basel II and III use this approximation in defining the counterparty credit capital at a single counterparty level).

As it can be seen from the right side of (6) or (7), the prefunded default fund, DF, which is defined based on a risk measure associated with portfolio counterparty credit risk, depends on the collateralized exposures of the CCP to each clearing member and the credit quality of clearing members as represented by default indicators.

The Euler capital allocation principle, which has been successfully used in finance (see Chapter 6 of McNeil et al. [2005] et al. and the reference there), can be employed in the second step to allocate DF to clearing members and to specify their prefunded default fund contributions $DF_1, ..., DF_n$. More specifically, for the VaR-based DF, the prefunded default fund contribution of CM_i based on the Euler allocation principle becomes,

$$DF_i = C_i E[Y_i | L = VaR_\alpha], \tag{8}$$

and when DF is defined based on ES, the Euler allocation principle specifies DF_i as follows,

$$DF_i = C_i E[Y_i | L \geq VaR_\alpha]. \tag{9}$$

Euler allocation rules are defined based on the partial derivatives (sensitivities) of the risk measure under consideration and their derivation depends on technical conditions specified in Tasche [1999], (see, also, Section 6.3 of McNeil et al. [2005]). Note that both allocation rules satisfy the so called *full allocation property.* That is, in the case of VaR-based DF, we can write,

$$\sum_{i=1}^{n} DF_i = \sum_{i=1}^{n} C_i E[Y_i | L = VaR_\alpha] = E[L | L = VaR_\alpha] = VaR_\alpha = DF,$$

and for ES-based DF, we have,

$$\sum_{i=1}^{n} DF_i = \sum_{i=1}^{n} C_i E[Y_i | L \geq VaR_\alpha] = E[L | L \geq VaR_\alpha] = ES_\alpha = DF.$$

The full allocation property, a desirable property of any allocation principle, simply means that the total prefunded default fund is fully allocated to clearing members.

This paper proposes to form the total prefunded default fund based on the expected shortfall as in (7) and then allocate it using the Euler principle based on (9). Section 6 discusses how Monte Carlo simulation can be used to estimate $DF, DF_1, ..., DF_n$.

When the total prefunded default fund is specified and allocated under our proposed two-step procedure, the CCP's second-level loss accepts the following useful expression,

$$L^{(2)} = \left(\sum_{i=1}^{n} C_i Y_i - E - DF \right)^+ . \tag{10}$$

The derivation of the right side above uses the model independent definition of $L^{(2)}$ given in (4) and that based on the proposed prefunded default fund allocation rules in (8) and (9), we have $DF_i = C_i E[Y_i|O] \leq C_i$, with O denoting the event $\{L \geq VaR_\alpha\}$ or $\{L = VaR_\alpha\}$. Using the confidence level α associated with the VaR or ES-based $DF \equiv DF_\alpha$ and the CCP's equity contribution E, the probability of the event $\{L > E + DF\}$ can be made very small. That is, since $P(L > E + DF) \leq 1 - \alpha$, depending on E and DF_α, the CCP's potential loss that finds it way to the last layer of the default waterfall resources, i.e., the unfunded default funds, can be made very small.

Remark 1 Capital allocation is a well-studied topic in finance; we will not present an exhaustive literature survey on it in this paper. We refer only to a few influential papers on economic justification of the Euler allocation principle and on the comparison between VaR and ES in portfolio credit risk; all are applicable in our setting as we have taken a portfolio credit risk approach to define and allocate DF. Tasche [1999] gives the first economic justification of the Euler principle based on a return on risk-adjusted capital (RORAC) criterion, which is also applicable to portfolio credit risk. Denault [2001] uses cooperative game theory for an economic justification of the Euler principle. The axiomatic characterization of risk measures has been well received in the finance community; Kalkbrener [2005] introduces an axiomatic approach that formalizes a set of properties for good allocation rules; then, under some technical conditions, the Euler principle proves to be the only allocation rule having the formalized desirable properties. Kalkbrener et al. [2004] and Kurth and Tasche [2003] argue that ES is preferable to VaR particularly for portfolio credit risk; VaR does not encourage diversification for portfolio credit risk. Also, a set of numerical examples in Kalkbrener et al. [2004] and Kurth and Tasche [2003] indicate that ES contributions (see the right side of (9)) are more sensitive to concentration risk compared to VaR contributions (see the right side of (8)).

Remark 2 Suppose that the CCP stands between two counterparties to a derivatives transaction, e.g., an interest rate swap. If both clearing members default at the same time, the CCP, by its definition, would not incur any direct losses. In the proposed one-period model, however, when the above-mentioned *matched* pair of clearing members default, the loss to the CCP in the presence of only VM and IM would be the sum of $C_1 \geq 0$ and $C_2 \geq 0$, where C_1 and C_2 denote the CCP's loan-equivalent exposure to these clearing members at their default. That is, the one-period model does not capture this characteristic of a CCP's theoretical arrangement. In this sense the model produces conservative estimates for the default waterfall resources beyond variation and initial margin.

2.2 The Unfunded Default Funds

The following two cases are commonly considered in derivatives CCP risk management practice. When $L^{(2)}$, the CCP's potential loss exceeding variation and initial margins, CCP's equity contribution, and the prefunded default funds,

$$L^{(2)} = \left(\sum_{i=1}^{n} C_i Y_i - E - DF \right)^{+},$$

is fully allocated to the surviving members, the CCP's unfunded default fund capital calls on clearing members are referred to as *uncapped*. Otherwise, when the clearing members' unfunded default funds are capped by a multiple of their prefunded default funds, the CCP's unfunded default fund capital calls are *capped*.[13] In the former case, we assume that $L^{(2)}$ is allocated to the surviving members proportional to their prefunded default fund contributions.[14] That is, the CCP's uncapped unfunded default fund capital call on clearing member i becomes,

$$\tilde{DF}_i^{uc} \equiv L_i^{uc} = \frac{DF_i(1 - Y_i)}{DF_s} L^{(2)} \geq 0, \tag{11}$$

where $DF_s = \sum_{j=1}^{n} DF_j(1 - Y_j)$. Note that distributing $L^{(2)}$ proportional to prefunded default fund contributions gives,

$$\tilde{DF}^{uc} \equiv \tilde{DF}_1^{uc} + ... + \tilde{DF}_n^{uc} = L^{(2)}.$$

$L^{(2)}$ can also be allocated to the surviving members proportional to the loan-equivalent exposures, i.e., C_i's. However, allocation of $L^{(2)}$ proportional to the C_i's will not capture the credit quality, correlation, and tail dependence of the clearing members in the sense that is discussed in Appendix A. While the unfunded default funds of surviving clearing members are the CCPs' last layer of the default waterfall resources, they are unanticipated potential losses to surviving members. The double notation $\tilde{DF}_i^{uc} \equiv L_i^{uc}$ is to emphasize this point and simplify the communication of our results when moving from the CCP's perspective to that of a clearing member. When the CM_i's unfunded default fund is capped by a multiple of its prefunded default fund contribution, it becomes,

[13] We emphasize that the analysis and results of this paper do not depend on the above-mentioned "cap" being a multiple of the prefunded default funds. It could be defined as a given constant. It is out of the scope of our study to discuss the overall economic advantages of the capped versus uncapped case and to specify how and upper bound for the unfunded default funds should be derived. We also emphasize (again) that the paper's starting point is that the clearing mandate is in place and derivatives CCPs are operating under their typical default waterfall resources.

[14] This is a common assumption in practice. We discuss the risk management justification of this allocation rule in Appendix A.

$$\tilde{DF}_i \equiv L_i = \min\{L_i^{uc}, \beta DF_i(1 - Y_i)\}, \tag{12}$$

where $\beta > 0$.[15]

Clearly, the uncapped and capped unfunded default funds of CM_i are zero conditional on its default at time T. As will be shown in the next section, the CCP risk capital of clearing members depends on the unfunded default funds; the CM_i's CCP risk capital is an increasing function of L_i^{uc}. From the bank regulators' and the CM_i's perspective, it will be more conservative to specify the CM_i's CCP risk capital by defining the unfunded default funds assuming that CM_i survives at time T. That is, the CM_i's uncapped unfunded default fund assuming its time-T survival is defined as follows,

$$\tilde{DF}_i^{uc,s} \equiv L_i^{uc,s} = \frac{DF_i}{DF_{s,i}} \left(\sum_{j \neq i} U_j Y_j - E - DF_{s,i} \right)^+ = \frac{DF_i}{DF_{s,i}} \left(\sum_{j \neq i} C_j Y_j - E - DF \right)^+, \tag{13}$$

where $DF_{s,i}$ denotes the surviving members' prefunded default funds under the assumption that CM_i survives at time T,

$$DF_{s,i} \equiv DF - \sum_{j \neq i} DF_j Y_j, \tag{14}$$

and to derive the right side of (13), we use simple algebraic manipulations used in the derivation of (10). Similarly, assuming the CM_i's time-T survival, the CCP's unfunded default fund capital call on CM_i becomes,

$$\tilde{DF}_i^s \equiv L_i^s = \min\{L_i^{uc,s}, \beta DF_i\}, \tag{15}$$

where $\beta > 0$.

Remark 3 After the prefunded default funds are specified by the CCP, in defining $L_i^{uc,s}$ under the assumption that CM_i survives at time T, one should think of CM_i as the financial institution that, independent of its own credit quality, considers the credit quality of CM_j's, $j \neq i$, and represents them by dependent default indicators Y_j, $j \neq i$. Note that the two random variables $L_i^{uc,s}$, and L_i^{uc} conditional on $Y_i = 0$ are not equal in distribution as Y_i's are dependent random variables. That is, for any $x > 0$,

$$P(L_i^{uc} \leq x | Y_i = 0) \neq P(L_i^{uc,s} \leq x).$$

[15]To simplify the notation we have written $\tilde{DF}_i$ and L_i for $\tilde{DF}_i^c$ and L_i^c, respectively – superscript c denoting the capped case.

Remark 4 Given the proposed two-step procedure to specify DF and decompose it to DF_i's based on VaR or ES, we have,

$$\tilde{DF}_i^s \equiv L_i^s = \min\left\{\frac{DF_i}{DF_{s,i}}\left(\sum_{j\neq i} C_j Y_j - E - DF\right)^+, \beta DF_i\right\},$$

Note that

$$P\left(\sum_{j\neq i} C_j Y_j > E + DF\right) \leq P(L > DF) \leq 1-\alpha,$$

where α denotes the confidence level associated with DF. So, depending on DF_α and E, the probability that a surviving member would incur a loss at time T can be made very small.

3 The One-Period Model and Margins' Procyclicality

The CCPs' variation and initial margin requirements are often criticized as they might lead to destabilizing feedback mechanisms, (see, e.g., Pirrong [2013]). During times of financial stress, the market volatility increases. This in turn increases the initial margin requirements. High margin requirements during financial stress affect the funding and market liquidity, (see Brunnermeier and Pedersen [2009]); liquidity dry-ups further increase the market volatility. The resulting feedback mechanism can exacerbate and prolong the down times. There has been several regulatory proposals to mitigate the procyclicality associated with margin requirements. For instance, financial institutions subject to regulatory risk capital are being asked to use longer historical data containing stressed periods in the calibration of their margin models, (see, e.g., CGFS [2010]).[16] Also, Basel III has devised capital conservation and counter-cyclical buffers in its capital requirements, (see, e.g., Chapter 13 of Hull [2012]).

The proposed default waterfall model can be used to reduce margins' procyclicality in a more risk sensitive way. To show this, we first summarize the default waterfall layers in the one-period model. Recall a CCP's time-T collateralized exposure due to the CM_i's default,[17]

$$C_i = (1-\delta_i)\int_0^T E[e_i(t)]dt,$$

as defined in (3) with $e_i(t) \equiv \max\{V_i^+(t+\Delta) - VM_i(t) - IM_i(t), 0\}$; $i = 1, ..., n$; the IM is based on a given confidence level, $\tilde{\alpha}$. The the total prefunded default fund is specified and allocated to clearing members as follows,

$$DF_\alpha \equiv E[L|L > q], \text{ and } DF_i \equiv C_i E[Y_i|L > q],$$

[16] CGFS stands for the Committee on the Global Financial System.

[17] As stated in Section 2, C_i's can also be defined based on other dynamic counterparty credit risk measures.

with $L = \sum_{i=1}^{n} C_i Y_i$ and $q \equiv VaR_\alpha(\sum_{i=1}^{n} C_i Y_i)$. The CCP's potential loss in the presence of variation margin, initial margin, CCP's equity contribution, and prefunded default funds is,

$$L^{(2)} = (\sum_{i=1}^{n} C_i Y_i - E - DF_\alpha)^+.$$

Recall the CCP's unfunded default fund capital call on CM_i in the capped case,

$$\tilde{DF}_i = \min\left\{ \frac{DF_i(1-Y_i)}{DF_s} L^{(2)}, \beta DF_i(1-Y_i) \right\},$$

as defined in (12) and (11) with $DF_s = \sum_{j=1}^{n} DF_j(1-Y_j)$. Let $\tilde{DF} \equiv \sum_{i=1}^{n} \tilde{DF}_i$. Consider the CCP's expected potential loss in the presence of all layers of the default waterfall resources,

$$E[L^{(3)}] = E[(\sum_{i=1}^{n} C_i Y_i - E - DF_\alpha - \tilde{DF})^+].$$

Given the one-period model's characterization of the default waterfall resources, the CCPs and their regulators can reduce margins' procyclicality by lowering margin requirements while preserving the same expected loss, $E[L^{(3)}]$. Margin requirements can be reduced by, for instance, lowering the frequency of variation margin calls and the confidence level associated with members' initial margin. That is, the mix of margin requirements, CCPs equity contribution, and prefunded and unfunded funds can be chosen such that margins' procyclicality be reduced while the CCPs maintain the same level of financial resiliency. As will be shown in the next section, the CCPs' unfunded default fund capital calls are unanticipated losses on surviving members, against which bank regulators require risk capital on clearing members. Clearly, if the default waterfall is shifted heavily toward the unfunded default funds, these unanticipated losses during times of financial stress can become new sources of procyclicality. Also, high CCP risk capital charges on clearing members can disincentivize the central clearing of OTC derivatives transactions. To mitigate the margins' destabilizing feedback mechanism, CCPs can lower their margin requirements by increasing the prefunded default funds and the CCP's equity contribution to maintain the same level of expected second level loss, $E[L^{(2)}]$, and so the same level of expected unanticipated losses to clearing members.

4 The CCP Risk Capital of Clearing Members

Credit risk is the classical risk borne by a bank that is in the business of *deposit taking and loan making*. The Basel Committee on Banking Supervision (BCBS) has defined and required regulatory credit risk capital for more than two decades. Regulatory counterparty credit risk capital has also been devised for protection against potential counterparty credit losses.[18] From

[18] A counterparty in the above mentioned context refers to an OTC derivatives market participant that holds a derivatives portfolio with the financial institution under regulatory requirements.

the bank regulators' perspective, direct clearing members of CCPs should hold CCP risk capital against their central counterparty credit exposures.

To define the CCP risk capital of a clearing member, one should first consider different types of losses that the member could incur when clearing a derivatives portfolio with the CCP. Then, the bank regulators decide which types of losses the clearing member should be capitalized against. This decision is usually made based on the economic role of the cleared derivatives portfolio in the clearing member's business and based on various accounting considerations. Different types of losses a clear member may face when clearing a derivatives portfolio with the CCP can be described as follows. First, consider the losses to the clearing member due to the default of the CCP. If the CCP defaults, the clearing member could incur a replacement cost depending on the value of the derivatives portfolio it holds with the CCP. Also, if the initial margin posted to the CCP is not held in a bankruptcy remote manner, its market value could be considered a loss to the clearing member when the CCP defaults. Additionally, in the event of the CCP's default, it could be argued that the clearing member, whose prefunded default fund has been used by the CCP, incurs a loss equal to the market value of its prefunded default fund contribution. Second, consider the losses due to the default of other clearing members. As will be formulated below using the one-period model, depending on the size of the losses due to the default of a set of clearing members, the member under consideration, assuming its survival, could lose part or all of its prefunded default fund contribution to the CCP. Also, the surviving member could incur further unanticipated losses due to the unfunded default fund capital calls of the CCP. These are unfunded default fund contributions of the surviving member when the CCP's loss has exhausted the defaulters' resources, the CCP's equity contribution, and the prefunded default fund of the survivors.

In what follows the above-mentioned losses are formulated using the one-period model introduced in this paper. To model the losses due to the CCP's default, Appendix B formulates the CCP's default probability in the one-period model. The one-period model assumes that the CCP defaults when an uncovered loss remains at time T after all the CCP's default waterfall resources are exhausted. Using the results of Appendix B, Section 4.1 derives a clearing member's total potential loss due to its CCP exposures. Section 4.2 defines the CCP risk capital using expected and unexpected losses under the static model.

4.1 Total Losses to the Clearing Members

Hereafter, in the main body of the paper, we assume that a clearing member's total potential loss is defined under the assumption that it survives at time T. Appendix C derives the clearing member total losses in the absence of this assumption.

Recall that the CCP's partial or complete use of the DF_i can be viewed as losses to the CM_i. First, assuming CM_i's time-T survival, we use the one-period model's default waterfall model to formulate the losses to DF_i from the CM_i's perspective. Consider the following three cases. If the CCP's potential loss in the presence of only the defaulter-pay resources, i.e.,

$$\sum_{j\neq i}(C_j - DF_j)Y_j,$$

is less than the CCP's equity contribution, E, the prefunded default fund of CM_i, i.e. DF_i, will not be used. In the second case where the CCP's loss in the presence of the defaulter-pay resources exceeds the CCP's equity contribution but will be less than the survivor-pay resources plus the CCP's equity, i.e.,

$$E \leq \sum_{j \neq i} (C_j - DF_j) Y_j < E + DF_i + \sum_{j \neq i} DF_j (1 - Y_j),$$

part of the DF_i is used by the CCP to cover the defaulters' losses. Similar to our discussion on the risk management justifications of the allocation of unfunded default funds proportional to the prefunded default fund (see Appendix A and Section 2.2), we assume that the CCP's loss in the presence of the defaulter-pay resources and it's own equity in this case, $\sum_{j \neq i}(C_j - DF_j)Y_j - E$, is allocated to CM_i proportional to the prefunded default fund of the surviving members, i.e., proportional to $DF_i/DF_{s,i}$, where $DF_{s,i} \equiv DF - \sum_{j \neq i} DF_j Y_j$ as defined in (13)–(14). That is, under this second case, CM_i loses,

$$\frac{DF_i}{DF_{s,i}} \left(\sum_{j \neq i} (C_j - DF_j) Y_j - E \right),$$

of it's prefunded default fund. In the third case where the CCP's potential loss in the presence of the defaulter-pay resources exceeds the sum of the CCP's equity contribution and the prefuned default fund resources of the survivors, i.e., when,

$$\sum_{j \neq i} (C_j - DF_j) Y_j \geq E + DF_i + \sum_{j \neq i} DF_j (1 - Y_j),$$

the CCP's losses reach the last layer of the default waterfall which is the unfunded default fund contributions of the surviving members. That is, from the CM_i's perspective, its prefunded default fund contribution, DF_i is fully used (lost) under the third case.

The above-mentioned losses to the prefunded default fund of CM_i assuming it's time-T survival can be formulated as,

$$L_i^{df,s} \equiv \min \left\{ \frac{DF_i}{DF_{s,i}} \left(\sum_{j \neq i} (C_j - DF_j) Y_j - E \right)^+, DF_i \right\}. \tag{16}$$

Now, consider the uncapped case in which the CCP may default only when all clearing members default at time T. Assuming CM_i's survival at time T, the total potential loss in the uncapped case is equal to the losses to the CM_i's prefunded default fund derived in (16) plus the CM_i's unfunded default fund,

$$L_i^{t,uc,s} = L_i^{df,s} + L_i^{uc,s}, \tag{17}$$

where

$$L_i^{uc,s} = \frac{DF_i}{DF_{s,i}} \left(\sum_{j \neq i} C_j Y_j - E - DF \right)^+ ,$$

as derived in (13). When the CM_i's unfunded default fund is capped, the total potential loss is due to the possible default of other clearing members and the default of the CCP. Let $\tilde{Y}_i$ denote the CCP's default indicator from CM_i's perspective assuming it's survival at time T. In the capped case, the CM_i's total potential loss becomes,

$$L_i^{t,s} = L_i^{df,s} + L_i^s + \tilde{U}_i \tilde{Y}_i, \tag{18}$$

where $L_i^s = \min\{L_i^{uc,s}, \beta DF_i\}$, which is defined in (15), denotes the CM_i's potential loss due to the CCP's unfunded default fund capital calls. The CCP's default probability assuming CM_i's time-T survival, $P_{ccp,i} = E[\tilde{Y}_i]$, is derived in (30) of Appendix B. Also, $\tilde{U}_i$ denotes the CM_i's loan-equivalent exposure to the CCP at its default. Recall (3); an EPE-based exposure at default is defined as,

$$\tilde{U}_i = (1 - \delta) \int_0^{\tilde{T}} E[\tilde{e}_i(t)] dt, \tag{19}$$

where δ denotes the CCP's recovery rate, $\tilde{e}_i(t)$ is CM_i's collateralized exposure to the CCP at time t in the presence of variation margin, and $\tilde{T}$ specifies the time interval based on which clearing members' loan-equivalent exposure is estimated. CCPs do not usually post initial margin to the clearing members.[19] More specifically,

$$\tilde{e}_i(t) \equiv \max\{\tilde{V}_i^+(t) - V\tilde{M}_i(t), 0\},$$

where $\tilde{V}_i^+(t)$ is nonnegative part of the value of the derivatives portfolio that CM_i holds with the CCP at time t, and the VM available to the CM_i at time t depends on the frequency based on which the CCP posts variation margin to the clearing members. Let $\tilde{\Delta}$ denote the length of the time interval associated with the VM calls, then, $VM_i(t) \equiv V_i^+(t - \tilde{\Delta})$.[20] When the CCP's unfunded default fund capital calls are capped, the loss decomposition formula in (18) illustrates that the total loss of a clearing member is the sum of the loss caused by other members' default and the loss due to the CCP's default. In deriving (18), we have assumed that the CM_i's initial

[19] The time integral of expected exposures are also used to define the counterparty risk capital in Basel II and III. Since the credit quality of CCPs are usually viewed higher than other derivatives market participants, clearing members' loan-equivalent exposure to CCPs are based on shorter time intervals when compared to the non-central clearing loan-equivalent exposures.

[20] In cases where upon the CCP's default, the surviving members' initial margins are not returnable, (4.1) can be modified as follows, $\tilde{e}_i(t) \equiv \max\{\tilde{V}_i^+(t) - V\tilde{M}_i(t), IM_i(t), 0\}$, where $IM_i(t)$ is defined in Section 2.

margin posted to the CCP would be held in a bankruptcy remote manner. Otherwise, the CM_i's loss decomposition formula becomes,

$$L_i^{t,s} = L_i^{df,s} + L_i^s + (\tilde{U}_i + IM_i)\tilde{Y}_i, \tag{20}$$

where IM_i denotes the CM_i's initial margin posted to the CCP for the time period $[0, T]$.

4.2 The CCP Risk Capital

The definition of the portfolio credit risk capital is often based on unexpected losses.[21] When the one-period model is used for modeling the total loss, $\tilde{L}$, associated with a loan portfolio, the VaR-based unexpected portfolio credit risk capital at a given confidence level, α, becomes $VaR_\alpha(\tilde{L}) - E[\tilde{L}]$, (see, e.g., Chapter 12 of Hull [2012]). Differentiating expected and unexpected losses in the context of counterparty credit risk (CCR) is not straightforward. Basel II's counterparty risk capital is based on EPE or eEPE times the counterparty's default probability defined using a one-period model, (see, e.g., Pykhtin and Zhu [2006]). Basel III's CCR capital requirements consist of credit value adjustment (CVA) and Basel II's CCR risk capital. CVA, in contrast to Basel II's CCR risk capital, is not estimated based on one-period models; the definition of CVA allows the counterparty to default at any point in time in a given time interval. So, it is difficult to define a mathematically consistent framework for Basel III's counterparty credit risk capital differentiating expected and unexpected losses (see, e.g., Pykhtin [2011]).

The one-period model of the derivatives CCPs' default waterfall enables the banks, CCPs, and their regulators to coherently formulate different potential losses each clearing member could face when clearing a derivatives portfolio with the CCP. After deciding which classes of losses are to be capitalized against, the CCP risk capital of direct clearing members can be defined coherently. Given the financial purpose of a member's cleared derivatives portfolio in it's business, suppose that the bank regulators decide that all potential losses formulated in the previous section should be capitalized against. Assuming CM_i's time-T survival, recall the loss decomposition formula; for instance, in the capped case, $L_i^{t,s} = L_i^{df,s} + L_i^s + \tilde{U}_i\tilde{Y}_i$, as derived in (18). Then, the CCP risk capital of CM_i can be coherently defined based on total expected capped losses,

$$E[L_i^{t,s}] = E[L_i^{df,s}] + E[L_i^s] + \tilde{U}_i P_{ccp,i}, \tag{21}$$

where the CCP's default probability from the CM_i's standpoint, $P_{ccp,i}$, is derived in (30) of Appendix B Or, the CM_i's CCP risk capital can be based on unexpected losses,[22]

$$VaR_\alpha(L_i^{t,s}) - E[L_i^{t,s}],$$

[21] It is usually assumed that the interest income that banks earn on their credit portfolios covers the "expected" portion of credit losses, and so these are the "unexpected losses" that are to be capitalized against.

[22] The CCP risk capital can be also defined based on *unconditional* total losses to clearing members derived in Appendix C. The term "unconditional" refers to the case where we do not assume the clearing member's survival at time T.

given a confidence level $\alpha \in (0, 1)$.[23] Section 6 outlines Monte Carlo estimation of CCP risk capital.

5 The Regulatory CCP Risk Capital

Recognizing that banks differ in their level of sophistication to use mathematical models for risk capital calculations, the BCBS has often developed both non-model-based and more risk sensitive model-based methods for risk capital requirements. The credit risk framework of Basel II is the classical example where the credit risk capital can be calculated based on the non-model-based Standardized approach, or based on the model-based risk sensitive Foundation Internal Ratings Based (IRB) and the Advanced IRB approaches, (see, e.g., Chapter 12 of Hull [2012]).

The regulatory risk capital interim framework for banks exposures to CCPs was published by the BCBS [2012]. A consultative document, BCBS [2013], modified the interim framework and was supposed to be the basis for the finalized CCP risk capital. The finalized regulatory CCP risk capital requirements, BCBS [2014a], were published in April 2014.[24] None of the proposed CCP risk capital methods are based on a mathematical model of the default waterfall resources. That is, a model-based risk sensitive approach to CCP risk capital does not exist in the regulatory framework. As it has been shown throughout this paper, clearing members' CCP risk capital depends on all the layers of CCPs' default waterfall resources in a non-straightforward way. So, even a non-model-based risk insensitive but logically consistent definition of the CCP risk capital in the absence of a coherent default waterfall model is challenging. This section does not discuss the three-year historical evolution of the regulatory CCP risk capital in detail; it only highlights some of the major difficulties in defining the CCP risk capital in the absence of a unifying CCP risk measurement framework.[25]

Before discussing the regulatory framework, we emphasize that any risk sensitive CCP risk capital *rule* should differentiate the prefunded and unfunded default funds, and it should be based on a risk sensitive definition of the guarantee fund (total prefunded default funds) and a risk sensitive way of allocating it to the clearing members. It is in the latter sense that, as illustrated in Section 2.1, the PFMI's *Cover 1/Cover 2* principle can not be used for a risk sensitive model-based specification of the total prefunded default fund and its allocation to clearing members. The ambiguity surrounding DF and the absence of a risk allocation method to specify DF_i's will carry over to CCP risk capital rules as they do depend on the prefunded

[23] Clearing members usually earn interest income on their initial margin and prefunded default fund contributions to the CCP. An argument for using unexpected losses as opposed to expected losses for CCP risk capital is that the members' expected losses would be covered by the interest income on IM and DF and so the CCP risk capital is to only cover the "unexpected" portion of the losses.

[24] Three international SSBs: BCBS, CPMI, and IOSCO have been responsible for developing the finalized CCP risk capital rules.

[25] This section does not intend to promote model-based risk sensitive approaches to the CCP risk capital over non-model-based methods. Risk sensitive and risk insensitive methods should both exist as financial institutions subject to regulatory risk capital differ in their level of sophistication and available resources to use mathematical risk models. We are to show that a model-driven risk sensitive framework, which does not exist, can be used to better understand, formulate, and evaluate the less risk sensitive non-model-based methods.

default funds, (see the loss decomposition formula (18)).

The BCBS-CPMI-IOSCO's CCP risk capital of direct clearing members consists of two components: *default fund capital charges* due to the possible defaults of other clearing members or the CCP and *trade exposure capital charges* due to CCP's default risk. The CCP risk capital of a given clearing member is equal to the sum of these two components. The *trade exposure* component refers to the replacement cost associated with the cleared derivatives portfolio that a clearing member could incur when the CCP defaults. Consider the last term of the loss decomposition formula derived in (18),

$$L_i^{t,s} = L_i^{df,s} + L_i^s + \tilde{U}_i \tilde{Y}_i,$$

the *trade exposure* component of the CCP risk capital is to capture $\tilde{U}_i E[\tilde{Y}_i]$, where the CCP's default probability from CM_i's perspective, i.e., $E[\tilde{Y}_i] = P_{ccp,i}$ is derived in (30) of Appendix B. The CCP's default probability can not be modeled and formulated in the absence of a default waterfall model; so, the regulatory framework uses a risk weight of 2%, representing the credit quality of derivatives CCPs in a risk insensitive way. According to the BCBS [2014a], the collateralized exposure of a clearing member to CCP can be estimated using the Internal Model Method (IMM) of Basel III or the recently developed Standardized Approach for counterparty credit risk BCBS [2014b].

In what follow we now discuss the *default fund capital charge* component of the regulatory CCP risk capital. As it is evident from the very title "*default fund capital charges*", the regulatory framework does not differentiate prefunded defaults funds from unfunded default funds. In fact, it is impossible to differentiate the two in the absence of a coherent default waterfall model. The BCBS [2013] proposed two methods to calculate *default fund capital charges*, where one method was supposed to be more risk sensitive than the other. First, consider the more risk sensitive *Tranches approach* of BCBS [2013] for the default fund capital charge of clearing member i denoted by K_{cm_i},

$$K_{cm_i} = \frac{DF_i}{DF} \begin{cases} K_{ccp} - E & \text{if } DF + E < K_{ccp} \\ (K_{ccp} - E) + c_1(DF + E - K_{ccp}) & \text{if } E < K_{ccp} < E + DF \\ c_1 DF & \text{if } K_{ccp} < E, \end{cases} \tag{22}$$

where $c_1 = .16(\frac{K_{ccp}}{DF+E})$, and K_{ccp},

$$K_{ccp} = 8\% \times RW \times (\sum_{i=1}^{n} EAD_i),$$

is to represent the *CCP's hypothetical capital requirements* due to its members' counterparty credit risk exposures; EAD_i is the CCP's exposure to CM_i at its default in the presence of variation and initial margin. The risk weight RW is set equal to 20%,[26] and 8% is the minimum capital ratio or the so called Cooke ratio.[27] The BCBS's minimum capital ratio has been historically set to 8%.[28] Also, the BCBS has historically used *risk weights*, representing the credit quality of the counterparty (obligor) to the financial institution under consideration, for its less risk sensitive non-model based risk capital methods, (see, e.g., Chapter 2 of Crouhy et al. [2001]). Note that the non-model-based *CCP's hypothetical capital*, K_{ccp}, is a component of the Tranches approach, display (22), which is supposed to be the risk sensitive method for the default fund risk capital calculations.

To simplify (22) and facilitate a comparison between the Tranches approach and CCP risk capital in the proposed framework, suppose that the CCP's equity contribution, E, is zero. Given the proposed one period model, set $EAD_i \equiv C_i$, $i = 1, ..., n$. The Tranches approach then gives,

$$K_{cm_i} = \frac{DF_i}{DF} \begin{cases} K_{ccp} & \text{if } \; DF < K_{ccp} \\ \\ K_{ccp} + \frac{.16 K_{ccp}(DF - K_{ccp})}{DF} & \text{if } \; K_{ccp} < DF, \end{cases}$$

where $K_{ccp} = .016 \times (\sum_{i=1}^{n} C_i)$. Note that the CM_i's CCP risk capital increases when $DF > K_{ccp}$. This can not be rationalized. In the less risk sensitive *Ratio* approach of BCBS [2013], the default fund capital charge of clearing member i, denoted by K_{cm_i}, is defined as follows,

$$K_{cm_i} = \frac{DF_i}{DF} K_{ccp} = 8\% \times RW \times \frac{DF_i}{DF} \left(\sum_{i=1}^{n} EAD_i \right), \tag{23}$$

where $EAD \equiv C$ and the risk weight in the Ratio approach is larger that the risk weight in the Tranches approach.[29]

The Tranches and Ratio approaches are risk insensitive and logically inconsistent in that the formulation of K_{ccp} and its role in the CCP risk capital rules are unfounded. Not being able to capture the fact that prefunded default funds decrease the CCP's exposures to its clearing members, both approaches mainly allocate the CCP's total collateralized exposures, $C_1 + ... + C_n$, to clearing members proportional to their prefunded default funds, $DF_1, ..., DF_n$.

[26]According to BCBS [2012] and BCBS [2014a], the 20% risk weight is a minimum requirement, and the national supervisor of a bank can increase the risk weight.

[27]The Tranches approach of BCBS [2013] uses the "reference level of default fund resources" defined as $RLDF = \max\{DF^{cover^*}, K_{ccp}\}$ for K_{ccp}, where DF^{cover^*} denotes a CCP's calculation of prefunded default fund contributions based on the *Cover 1/Cover 2* principle, (see Pages 5 and 9 of BCBS [2013]). For simplicity, DF^{cover^*} is not considered in Section 5; it merely adds to the conceptual ambiguities that Section 5 attempts to highlight. The sole purpose of defining the above-mentioned quantity, $RLDF$, was to include the PFMI's principle-based specification of DF in the formulaic CCP risk capital rules.

[28]See Chapter 12, page 191, of Dermine [2009] for an interesting historical account of the minimum capital ratio: "8 percent is the result of negotiation by the members of the Bank for International Settlements in 1988".

[29]The risk weight in the Ratio approach is set equal to 12.5. Also, (23) is a simplified version of the Ratio approach. Similar to our comment in the previous footnote, we have not considered $RLDF = \max\{DF^{cover^*}, K_{ccp}\}$ as it does not seem to have a sound conceptual foundation.

The interim framework, BCBS [2012], also uses a similar K_{ccp}-centric approach introducing two methods; one was to be more risk sensitive than the other. However, in the interim framework, DF_i's are also taken into account in the definition of the *CCP's hypothetical capital requirements*, i.e.,

$$K_{ccp} = 8\% \times RW \times \left(\sum_{i=1}^{n} (EAD_i - DF_i)^+ \right). \tag{24}$$

In fact, the finalized BCBS-CPMI-IOSCO CCP risk capital rules also use the above definition of the K_{ccp} and set the *default fund capital charge* component of the CCP risk capital equal to

$$K_{cm_i} = \max \left\{ \frac{DF_i}{DF} K_{ccp} \ , \ 8\% \times 2\% \times DF_i \right\}. \tag{25}$$

Again, the finalized CCP risk capital rules can not be rationalized. It is unclear what the so-called K_{ccp}, defined in (24), is supposed to capture. If the first term inside the maximum on the left is an attempt to capture the CM_i's unfunded default funds, the second term is a risk insensitive way to capitalize against the exposure to the prefunded default funds of CM_i. It is logically inconsistent to take the maximum of a misrepresentation of the CM_i's unfunded default funds and a risk insensitive representation of its prefunded default funds. Note that the second term is risk insensitive because it assumes that the CCP, taking a risk weight of 2%, has defaulted and so CM_i loses all the its DF_i. The *default fund capital charge* component of the finalized CCP risk capital rules is unfounded and logically inconsistent; it is unable to give an adequate characterization of unfunded default funds and to differentiate them from prefunded defaults funds.

The CCP risk capital is a classical topic in risk management where mathematical modeling proves to be inevitable and invaluable: the proposed default waterfall model gives the loss decomposition formula (18), [30] which enables the regulators and SSBs to define the CCP risk capital coherently and in a conceptually sound way; whether it would be risk sensitive or risk insensitive. Suppose, as it is evident from this section's analysis of the regulatory framework, that the regulators and SSBs intend to capitalize a direct clearing member, say CM_i, against all the possible average-based losses it may incur due to its exposure to the CCP.[31] Then, the

[30] Or the modified version of it, i.e., the loss decomposition formula (20), when the CCP under consideration does not hold the clearing members' initial margin in a bankruptcy remote manner.

[31] In the first circulated version of this paper, we argued against asking clearing members to hold risk capital against their prefunded default contributions to the CCP. This version does avoid prescribing which loss components should be included in the CCP risk capital requirements. Since a main element of the G-20 clearing mandate is to incentivize central clearing against bilateral clearing, the topic that which loss components should be included in the CCP risk capital rules goes beyond merely economic and finance (academic) considerations; policy considerations and arguments are also involved, which are not the concern of this paper. We have introduced a framework based on which CCP risk capital can be defined in a logically consistent and risk sensitive way; regulators and policy makers would decide which loss components the clearing members should be capitalized against.

one-period model's expected loss formula given in (21),

$$E[L_i^{t,s}] = E[L_i^{df,s}] + E[L_i^s] + \tilde{U}_i P_{ccp,i},$$

can be used as a conceptually sound and risk sensitive definition of the CCP risk capital. It can also be used to develop and evaluate logically consistent non-model-based less risk sensitive methods. Recall (16)- (19) and (30), and note that the first and second terms on the right side of the proposed average-based CCP risk capital formula represent the CM_i's expected losses due to the CCP's use of all or parts of the CM_i's prefunded default funds, DF_i, and the CCP's unfunded default fund capital calls on CM_i, respectively. The last term is the CM_i's expected loss associated with the cleared derivatives portfolio's replacement cost due to the CCP's default risk.

6 Monte Carlo CCP Risk Measurement

Monte Carlo simulation is widely used by different types of financial institutions in market, credit, and counterparty credit risk measurement. Some of the well-established Monte Carlo schemes in risk management can be directly employed in the proposed CCP risk measurement framework to estimate all layers of the default waterfall resources. We, first, provide a short summary on how a typical derivatives CCP estimates the initial margin of its clearing members. Next, we briefly discuss and outline Monte Carlo estimation of prefunded default fund contributions and the CCP risk capital of clearing members.

It is well-known from the risk management common practice and academic literature that Monte Carlo derivatives portfolio risk estimation involves two steps; first the portfolio's underlying risk factors, e.g., equity prices, commodity prices, interest and exchange rates, are generated up to a given point on a discrete time grid. Then, conditional on the realization of the risk factors, portfolio constituents are valuated. As closed form derivatives pricing formulas are rarely available, the second step involves some approximations or additional layers of Monte Carlo simulation (see, e.g., Chapter 9 of Glasserman [2004], Broadie et al. [2011], Gordy and Juneja [2010], and the references there).[32]

Initial Margin is the only model-based component of CCPs' default waterfall resources. Derivatives CCPs' (initial) *margin models* define a clearing member's IM based on a particular risk measure, i.e., VaR or ES, associated with the clearing member's derivatives portfolio. For instance, consider the equidistant time grid $0 \equiv t_0 < t_1 < ... < t_n \equiv T$ with $h = t_j - t_{j-1}$; let $V \equiv V_i^+$ denote the positive part of the CCP's derivatives portfolio value with CM_i. The VaR-based $IM \equiv IM_i$ at time t_j estimated at time t_{j-1} for a given confidence level $\tilde{\alpha}$ and based on a fixed MPOR, h, is defined as,

$$IM_j \equiv VaR_{\tilde{\alpha},t_{j-1}}(\Delta V_j) = \inf\{r \in \mathcal{R} : P(\Delta V_j > r | F_{j-1}) \leq 1 - \tilde{\alpha}\},$$

where,

$$\Delta V_j = V(t_j) - V(t_{j-1}),$$

[32] The first step is usually done using the physical measure, and the valuation step uses the risk neutral measure.

and F_{j-1} denotes the information set (filtration) generated by the portfolio's underlying risk factors by time t_{j-1}. Suppose that CM_i joins the CCP at time zero; its IM is set equal to $IM_1 \equiv IM(t_1)$ and is estimated at time 0 by Monte Carlo. The CM_i's time dependent IM is then *updated* sequentially based on the above mentioned time grid. For instance, after IM_1 is estimated and when the market information is revealed at time t_1, i.e., conditional on F_1, the CCP estimates IM_2.

6.1 Monte Carlo Estimation of the Prefunded Default Funds

As stated in Section 2.1, we suggest using expected shortfall at a confidence level α to define the total prefunded default fund, $DF = E[L|L \geq VaR_\alpha(L)]$, where $L = \sum_{i=1}^n C_i Y_i$. Recall (3). First, we discuss Monte Carlo estimation of EPE-based C_i's,

$$EPE_i = \int_0^T E[e_i(t)]dt,$$

where $e_i(t)$ denotes the CCP's collateralized time-t exposure to CM_i at its default in the presence of variation an initial margin as defined in (2). Ghamami and Zhang [2014] introduce an efficient Monte Carlo framework for EPE, effective EPE and CVA estimation in the absence of initial margin that can be employed by CCPs. Monte Carlo estimation of EPE in the presence of initial margin will be computationally more demanding. To see this, recall (2), and set $R_i \equiv V_i^+(t+\Delta) - V_i^+(t-\hat{\Delta})$. Suppose that IM is defined based on $VaR_{\tilde{\alpha}}(R_i) \equiv VaR$. Note that expected collateralized exposure can then be written as,

$$E[e_i(t)] = E[(R_i - VaR)^+] = E[R_i \mathbf{1}\{R_i > VaR\}] - \tilde{\alpha} VaR.$$

Given the confidence level, $\tilde{\alpha}$, $P(R_i > VaR)$, which is less than or equal to $1-\tilde{\alpha}$ is usually small, in which case, crude Monte Carlo will lead to expected exposure estimates with large variance. Successful application of standard variance reduction techniques, e.g., importance sampling for rare event simulation, is usually problem specific and not straightforward at derivatives portfolio level. It, then, becomes quite crucial that CCPs increase their computational budget when Monte Carlo is used for EPE or eEPE-based estimation of C_i's.

After $C_1, ..., C_n$ are estimated, a derivatives CCP uses the two-step procedure outlined in Section 2 to specify DF based on expected shortfall and then decomposes DF to $DF_1, ..., DF_n$. That is, Monte Carlo is to be used to estimate,

$$q \equiv VaR_\alpha(L) \ , \ DF \equiv E[L|L > q], \text{ and } DF_i \equiv E[C_i Y_i | L > q], \ i = 1, ..., n,$$

given a confidence level $\alpha \in (0,1)$. Estimation of DF and DF_i's can be thought of as a two-phase procedure, where one first estimate VaR and then estimates the two conditional expectations above using the estimated VaR from the first phase in place of the true VaR, (see Section 9.2 of Glasserman [2004] and Brereton et al. [2013] for efficient Monte Carlo estimation of value at risk).

Since the loss probability, $P(L > q)$, which is less than or equal to $1-\alpha$, is typically small, crude Monte Carlo would require a large number of runs to achieve a satisfactory variance for the estimators of VaR and ES. Consider the ratio representation of expected shortfall,

$$E[L|L > q] = \frac{E[L\mathbf{1}\{L > q\}]}{P(L > q)}.$$

A Monte Carlo scheme that works well for estimation of the loss probabilities, $P(L > q)$, will often work well for estimating the numerator on the right side above because $E[L\mathbf{1}\{L > q\}]$ takes positive values only when the event $\{L > q\}$ occurs.

Consider the single-factor equivalent of the t-copula threshold model given in display (1),

$$Y_i = 1\{X_i > x_i\} \text{ and } X_i = \frac{a_i Z + \sqrt{1-a_i^2}\xi_i}{\lambda}, \ i = 1, ..., n.$$

Note that when λ takes values close to zero, all the X_i's are likely to be large, leading to many simultaneous defaults. Both empirically and using asymptotic approximations, Bassamboo et al. [2008] have illustrated when $P(L > q)$ is small – of order 10^{-3} or less based on their numerical examples – the event $\{L > q\}$ happens primarily when λ takes small values while Z and ξ_i's have little influence on the occurrence of large losses, (see Theorem 1 and Tables 1-5 of their paper). Bassamboo et al. have introduced an efficient exponential-twisting based importance sampling (IS) algorithm for estimating loss probabilities and expected shortfall. In words, their IS algorithm first generates Z based on its original distribution function and samples from λ based on its exponential-twisted distribution. Next, conditional on the Monte Carlo realizations in the previous step, the conditionally independent Bernoulli random variables, $Y_1, ..., Y_n$, are sampled from according to their original mean, $p_1, ..., p_n$, or they are being generated based on an exponentially-twisted IS probability mass function (see Section 4.1.1 and 4.2 of their paper).[33] When loss probabilities are of order 10^{-3} or less the above mentioned two-step IS algorithm outperforms crude Monte Carlo significantly and can be used by CCPs for estimating prefunded default fund contributions.

6.2 Monte Carlo Estimation of the CCP Risk Capital

This section assumes that CM_i, assuming its time-T survival, is to be capitalized against all possible losses in the capped case based on the loss decomposition formula derived in (18),

$$L_i^{t,s} = L_i^{df,s} + L_i^s + \tilde{U}_i \tilde{Y}_i,$$

due to its exposure to the CCP and that bank regulators use the average-based CCP risk capital, i.e.,

[33] Bassamboo et al. have also introduced a second importance sampling algorithm based on hazard rate twisting; their numerical examples indicate that the exponential-twisting algorithm outperforms the other for loss-probability estimation.

$$E[L_i^{t,s}] = E[L_i^{df,s}] + E[L_i^s] + \tilde{U}_i P_{ccp,i},$$

as specified in (21). The Monte Carlo estimated prefunded default funds, $DF, DF_1, ..., DF_n$, of the previous section are used in place of the true, $DF, DF_1, ..., DF_n$, in Monte Carlo estimation of the CCP risk capital in this section. Note that $\tilde{U}_i$, the CM_i's loan-equivalent exposure to the CCP at its default, which is defined in (19), can be estimated with Monte Carlo efficiently using the framework introduced by Ghamami and Zhang [2014]. First, consider the CM_i's expected loss due to the CCP's use of all or part of the CM_i's prefunded default fund, DF_i,

$$E[L_i^{df,s}] = E\left[\min\left\{\frac{DF_i}{DF_{s,i}}\left(\sum_{j\neq i}(C_j - DF_j)Y_j - E\right)^+, DF_i\right\}\right],$$

where the expression for $L_i^{df,s}$ has been derived in (16). Note that $L_i^{df,s}$ takes positive values only if,

$$\sum_{j\neq i}(C_j - DF_j)Y_j > E.$$

When the above mentioned event is a rare event, the importance sampling algorithm of Bassamboo et al. [2008], which places further probability mass on this event, can be used for efficient estimation of $E[L_i^{df,s}]$.

Next, consider the expected capped loss of CM_i,

$$E[L_i^s] = E\left[\min\left\{\frac{DF_i}{DF_{s,i}}\left(\sum_{j\neq i}C_jY_j - E - DF\right)^+, \beta DF_i\right\}\right].$$

Note that L_i^s takes positive values only if,

$$\sum_{j\neq i}C_jY_j > E + DF.$$

Similar to the above-mentioned probabilistic argument, when the above mentioned event is a rare event, the importance sampling algorithm of Bassamboo et al. [2008] can be used for efficient estimation of $E[L_i^s]$.

Now, consider $P_{ccp,i}$, the CCP's default probability from the CM_i's perspective, i.e., assuming its time-T survival,

$$P_{ccp,i} = P\left(\sum_{j\neq i}C_jY_j > E + DF + \tilde{DF}_i^s + \sum_{j\neq i}\tilde{DF}_j\right),$$

as derived in (30) of Appendix B. Note that $P_{ccp,i} \leq P(\sum_{j\neq i} C_j Y_j > E + DF)$, and so, again, the IS algorithm of Bassamboo et al. [2008] should work well for Monte Carlo estimation of $P_{ccp,i}$ as it puts more probability mass on the event $\{\sum_{j\neq i} C_j Y_j > E + DF\}$. Display (30) of Appendix B gives the conservative upper bound, $1-\alpha$, on $P_{ccp,i}$. So, a conservative approximation of $P_{ccp,i}$ can be also used based the confidence level α associated with DF.

7 Conclusion

The opacity of the OTC derivatives market has been widely considered as one of the major causes of the 2007-2009 financial crises, (see, e.g., Part 4 of Archarya and Richardson [2009]). Derivatives CCPs are to make the OTC markets safer by reducing this lack of transparency following the 2009 G20 clearing mandate. International regulatory risk management standards have historically influenced and to some extent shaped the dynamics of financial markets. Before the clearing mandate, a detailed model-based focus merely on CCPs' margin requirements may have served the financial system well. Post clearing mandate and with the possible proliferation of CCPs, however, it is crucial that the CCP risk would be measured based on a coherent framework across all the default waterfall resources. This is, in part, because CCP risk management rules and practices will impact the capital structure of clearing members through the regulatory CCP risk capital requirements; they will also affect the relative costs of central versus bilateral clearing. In the absence of a well defined CCP risk measurement framework, a non-unifiable central clearing environment may replace the opaque OTC markets.

The proposed framework assumes that a CCP's clearing members may default at the end of a fixed time interval and approximates the collateralized exposures at default in the presence of only variation and initial margin. The remaining default waterfall resources, i.e., the prefunded and unfunded default funds, are then defined based on the credit loss distribution of the CCP's portfolio of clearing members' portfolios. The PFMI's principle-based specification of the prefunded default funds, CPMI and IOSCO [2012], is risk insensitive in that the clearing members' credit quality and their correlation are ignored. Moreover, the allocation of the total prefunded default funds to clearing members is not considered by the PFMI. The one-period model gives the first risk sensitive definition of the prefunded default funds and uses the Euler principle to allocate them to clearing members.

It is impossible to develop risk sensitive and logically consistent CCP risk capital requirements for clearing members in the absence of a unifying default waterfall model of typical derivatives CCPs. The BCBS-CPMI-IOSCO's CCP risk capital rules, BCBS [2014a], are risk insensitive despite the SSBs' effort to develop risk sensitive methods along with less risk sensitive methods over the past three years. The proposed framework models possible losses a clearing member could face due to its exposure to the CCP and gives the first risk sensitive definition of the CCP risk capital. This model-based risk sensitive approach to the CCP risk capital can be used to develop and evaluate non-model-based approaches that are less risk sensitive.

Appendix

A The Allocation Rule for Unfunded Default Funds

Consider the total prefunded default funds based on expected shortfall at a given confidence level, i.e., $DF \equiv E[L|L > q]$, where $L = \sum_{i=1}^{n} C_i Y_i$ and $q \equiv VaR_\alpha(L)$, and the Euler-based decomposition of DF,

$$DF_i = C_i E[Y_i|L > q].$$

Consider the allocation of $L^{(2)} = (\sum_{i=1}^{n} C_i Y_i - E - DF)^+$ to CM_i proportional to DF_i's and C_i's,

$$\frac{DF_i \bar{Y}_i}{\sum_{j=1}^{n} DF_j \bar{Y}_j} L^{(2)}, \quad \text{and} \quad \frac{C_i \bar{Y}_i}{\sum_{j=1}^{n} C_j \bar{Y}_j} L^{(2)}, \tag{26}$$

where $\bar{Y}_j = 1 - Y_j$, $j = 1, ..., n$. In addition to the market risk represented by C_i, the CM_i's prefunded default fund also depends on Y_i, i.e., the CM_i's credit quality, and the correlation among clearing members through the event $\{L > q\}$, which makes DF_i's more suitable than C_i's for allocation of $L^{(2)}$. This risk management justification of the DF-based allocation of unfunded default funds can be also seen from the following stylized example.

Example 1 Consider a CCP with 3 clearing members where $C_i \equiv 1$, $i = 1, 2, 3$. Let $P_{k_1k_2k_3} \equiv P(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3)$; $k_j = 0, 1$, $j = 1, 2, 3$, and suppose that the joint probability mass function of the default indictors is as follows,

$$P_{000} = .64,\ P_{100} = .06,\ P_{010} = .06,\ P_{001} = .08,\ P_{110} = .01,\ P_{101} = .08,\ P_{011} = .03,\ P_{111} = .04,$$

which gives the marginal default probabilities, $p_1 = .19$, $p_2 = .14$, $p_3 = .23$. Let

$$w_i = E[Y_i|Y_1 + Y_2 + Y_3 \geq 2], \quad i = 1, 2, 3.$$

So, $w_1 = 13/16$, $w_2 = .5$, and $w_3 = 15/16$. First, consider the C-based allocation rule using the right side of (26). Conditional on CM_3's default and CM_1 and CM_2's survival at time T, i.e., conditional on the event $\{Y_1 = 0, Y_2 = 0, Y_3 = 1\}$, the CCP's potential loss will be distributed equally among CM_1 and CM_2 regardless of their credit quality and correlation. However, under the DF-based allocation rule given by the left side of (26), and conditional on $\{Y_1 = 0, Y_2 = 0, Y_3 = 1\}$, the CCP allocates $w_1/(w_1 + w_2) = 13/21$ of the potential loss to CM_1 and 8/21 of it to CM_2. This shows that the DF-based allocation rule is more desirable since, conditional on $\{Y_1 = 0, Y_2 = 0, Y_3 = 1\}$, CM_1, whose credit quality is lower than CM_2, is allocated more of the CCP's potential loss. Also, note that in addition to the marginal credit qualities, the CM_1's credit quality conditional on the event $\{Y_1 + Y_2 + Y + 3 \geq 2\}$ is lower compared to that of CM_2, i.e., $w_1 > w_2$.

B The CCP's Default Probability in the One-Period Model

Consider the CCP's default probability in the one-period model. Let A denote the event that the CCP defaults and N_d denote the number of defaults at time T. Conditioning on the number of defaults at time T, we have,

$$P(A) = P(A|N_d = n)P(N_d = n) + P(A|N_d < n)P(N_d < n). \tag{27}$$

Note that when the unfunded default funds are uncapped, $P(A|N_d < n)$ is zero, and so the CCP's default probability in the uncapped case becomes,

$$P(\sum_{i=1}^{n} C_i > E + DF|N_d = n)P(N_d = n). \tag{28}$$

When the unfunded default fund contributions are capped by a multiple of the clearing members' prefunded default funds, $P(A|N_d < n)$ could be positive and is expressed as follows,

$$P(A|N_d < n) = P(L^{(2)} > \tilde{DF}|N_d < n) = P(\sum_{i=1}^{n} C_i Y_i > E + DF + \tilde{DF}|N_d < n), \tag{29}$$

where $L^{(2)}$ is defined in (10), the capped unfunded default contribution of CM_i, denoted by $\tilde{DF}_i \equiv L_i$, is defined in (12), and $\tilde{DF} \equiv \sum_{i=1}^{n} \tilde{DF}_i$.

As stated in Section 2.2, it will be more conservative in CCP risk capital calculations to define a clearing member's potential losses due to the possible default of other members assuming its survival at time T. Under this assumption we have, $P(N_d = n) = 0$ and $P(N_d < n) = 1$ in (27). That is, from CM_i's perspective, $P_i(A) \equiv P(A|N_d < n)$. Consequently, in the uncapped case, a surviving member assigns zero to the central counterparty's default probability. The CCP's default probability in the capped case from the CM_i's perspective assuming its survival at time T is formulated as follows,

$$\begin{aligned} P_{ccp,i} \equiv P_i(A) &= P\left((\sum_{j \neq i} U_j Y_j - E - DF_{s,i})^+ > \tilde{DF}_i^s + \sum_{j \neq i} \tilde{DF}_j \right) \\ &= P\left(\sum_{j \neq i} C_j Y_j > E + DF + \tilde{DF}_i^s + \sum_{j \neq i} \tilde{DF}_j \right) \leq 1 - \alpha, \end{aligned} \tag{30}$$

where $DF_{s,i}$, denoting the prefunded default fund of surviving members assuming CM_i's survival, is defined in (14), and $\tilde{DF}_i \equiv L_i$ and $\tilde{DF}_i^s \equiv L_i^s$; $i = 1, ..., n$, are defined in (12) and (15), respectively. Note that, clearly, in formulating the CCP's default probability from the CM_i's perspective, $P_{ccp,i}$, other clearing members are not assumed to survive at time T. The last equality on the right side above is derived based on simple algebraic manipulations used to derive (10). The upper bound, $1 - \alpha$, on CCP's default probability is derived by using our proposed specifications of the total prefunded default fund contributions given in (6)-(7).

C Total Losses to the Clearing Members: The Unconditional Case

As illustrated in Appendix B, the CCP may default in the uncapped case only when all clearing members default at time T. So, when the unfunded default funds are uncapped, the total loss to CM_i will be only due to the default of other clearing members. That is,

$$L_i^{t,uc} = L_i^{df} + L_i^{uc},$$

where L_i^{uc} is given by (11) and L_i^{df}, denoting the losses to the prefunded default fund of the CM_i, is given by

$$L_i^{df} \equiv \min\left\{\frac{DF_i(1-Y_i)}{DF_s}\left(\sum_j (C_j - DF_j)Y_j - E\right)^+, DF_i(1-Y_i)\right\}.$$

where $DF_s \equiv \sum_{j=1}^n DF_j(1-Y_j)$. The derive L_i^{df} above, one can use arguments similar to the ones led to the derivation of (16).

Let $\tilde{Y} \equiv \mathbf{1}\{L^{(2)} > \tilde{DF}\}$ denote the CCP's default indicator when CM_i's unfunded default fund is capped by a multiple of DF_i. Then, the total potential loss in the capped case becomes,

$$L_i^t = L_i^{df} + L_i + \tilde{U}_i(1-Y_i)\tilde{Y},$$

where $E[\tilde{Y}]$ is given in (27). Also, $\tilde{U}_i$ denotes the CM_i's loan-equivalent time-T exposure at default to the CCP and is defined in (19).

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